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ESTIMATES OF M–TERM APPROXIMATIONS OF FUNCTIONS OF SEVERAL VARIABLES IN THE LORENTZ SPACE BY A CONSTRUCTIVE METHOD

G. Akishev

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Key words: Lorentz space, Nikol'skii–Besov class, best M–term approximation, constructive method.

AMS Mathematics Subject Classification: 41A10, 41A25, 42A05.

Abstract. In the paper, the Lorentz space $L_{q,\tau}(\mathbb{T}^m)$ of periodic functions of several variables, the Nikol'skii–Besov class $S^{\overline{r}}_{q,\tau,\theta}B$ and the associated class $W^{a,b,\overline{r}}_{q,\tau}$ for $1 < q, \tau < \infty$, $1 \leqslant \theta \leqslant \infty$ are considered. Estimates are established for the best M-term trigonometric approximations of functions of the classes $W^{a,b,\overline{r}}_{q,\tau_1}$ and $S^{\overline{r}}_{q,\tau_1,\theta}B$ in the norm of the space $L_{p,\tau_2}(\mathbb{T}^m)$ for different relations between the parameters $q, \tau_1, p, \tau_2, a, \theta$. The proofs of the theorems are based on the constructive method developed by V.N. Temlyakov.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$– be the sets of all natural, integer, real numbers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}^m$—m-dimensional Euclidean point space $\overline{x} = (x_1, \ldots, x_m)$ with real coordinates; $\mathbb{T}^m = [0, 2\pi)^m$ and $\mathbb{I}^m = [0, 1)^m$ — m-dimensional cube.

$L_{p,\tau}(\mathbb{T}^m)$ will denote the Lorentz space of all real-valued Lebesgue-measurable functions f that have a 2π-period in each variable and for which the quantity

$$\|f\|_{p,\tau} = \left\{ \frac{\tau}{p} \int\limits_0^1 \left(f^*(t)\right)^\tau t^{\frac{\tau}{p}-1} dt \right\}^{\frac{1}{\tau}}, \ 1 < p < \infty, 1 \leqslant \tau < \infty,$$

is finite, where $f^*(t)$ is the non-increasing rearrangement of the function $|f(2\pi\overline{x})|$, $\overline{x} \in \mathbb{I}^m$ (see [34], pp. 213–216).

In the case $\tau = p$, the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ coincides with the Lebesgue space $L_p(\mathbb{T}^m)$ with the norm (see for example, [26, Chapter 1, Section 1.1, p. 11])

$$\|f\|_p = \left[\int_0^{2\pi} \ldots \int_0^{2\pi} |f(x_1, \ldots, x_m)|^p dx_1 \ldots dx_m \right]^{\frac{1}{p}}, \ 1 \leq p < \infty.$$

We will introduce the notation $a_{\overline{n}}(f)$-Fourier coefficients of the function $f \in L_1(\mathbb{T}^m)$ by system $\{e^{i\langle \overline{n}, \overline{x}\rangle}\}_{\overline{n} \in \mathbb{Z}^m}$ and $\langle \overline{y}, \overline{x} \rangle = \sum\limits_{j=1}^m y_j x_j$;

$$\delta_{\overline{s}}(f, \overline{x}) = \sum_{\overline{n} \in \rho(\overline{s})} a_{\overline{n}}(f) \, e^{i\langle \overline{n}, \overline{x}\rangle},$$

where

$$\rho(\bar{s}) = \left\{\bar{k} = (k_1, ..., k_m) \in \mathbb{Z}^m : \quad [2^{s_j-1}] \le |k_j| < 2^{s_j}, j = 1, ..., m\right\},$$

$[a]$ is the integer part of a real number a, $\bar{s} = (s_1, ..., s_m)$, $s_j \in \mathbb{Z}_+$.

For a given $p \in [1, \infty)$, a numerical sequence $\{a_{\bar{n}}\}_{\bar{n}\in\mathbb{Z}^m}$ belongs to the space l_p if

$$\left\|\{a_{\bar{n}}\}_{\bar{n}\in\mathbb{Z}^m}\right\|_{l_p} = \left[\sum_{\bar{n}\in\mathbb{Z}^m} |a_{\bar{n}}|^p\right]^{\frac{1}{p}} < \infty.$$

Further, for a vector $\bar{r} = (r_1, ..., r_m)$ and the zero vector $\bar{0} = (0, ..., 0)$, the inequality $\bar{r} > \bar{0}$ means that $r_j > 0$ for all $j = 1, 2, ..., m$. Let $1 \leqslant \theta \leqslant \infty$. We will consider an analogue of the Nikol'skii-Besov class

$$\mathbb{S}^{\bar{r}}_{p,\tau,\theta}B = \left\{f \in \mathring{L}_{p,\tau}(\mathbb{T}^m) : \left\|\left\{2^{\langle\bar{s},\bar{r}\rangle} \|\delta_{\bar{s}}(f)\|_{p,\tau}\right\}_{\bar{s}\in\mathbb{Z}^m_+}\right\|_{l_\theta} \leqslant 1\right\}.$$

In the case $\tau = p$, the class $\mathbb{S}^{\bar{r}}_{p,\tau,\theta}B$ coincides with the well-known Nikol'skii-Besov class $S^{\bar{r}}_{p,\theta}B$ in the space $L_p(\mathbb{T}^m)$ (see for example [8], [23]). Currently, there are various generalizations of the Nikol'skii–Besov spaces and their further applications in the theory of approximation of functions, harmonic analysis and in other branches of mathematics (see, for example, [9], [15], [16], [18], [36], [40]).

For a given vector $\bar{r} = (r_1, ..., r_m) > \bar{0} = (0, \ldots, 0)$ put $\bar{\gamma} = \frac{\bar{r}}{r_1}$ and

$$Q^{(\bar{\gamma})}_n = \cup_{\langle\bar{s},\bar{\gamma}\rangle<n}\rho(\bar{s}),$$

$S^{(\bar{\gamma})}_{Q_{n,\bar{\gamma}}}(f, \bar{x}) = \sum_{\bar{k}\in Q^{(\bar{\gamma})}_n} a_{\bar{k}}(f)e^{i\langle\bar{k},\bar{x}\rangle}$ will denote a partial sum of the Fourier series of a function f.

Let $\bar{k}^{(j)} \in \mathbb{Z}^m$. The quantity

$$e_M(f)_{p,\tau} = \inf_{\bar{k}^{(j)}, b_j} \left\|f - \sum_{j=1}^{M} b_j e^{\langle i\bar{k}^{(j)},\bar{x}\rangle}\right\|_{p,\tau}$$

is called the best M–term trigonometric approximation of a function $f \in L_{p,\tau}(\mathbb{T}^m)$, $M \in \mathbb{N}$, $\bar{k}^{(j)} \in \mathbb{Z}^m$. If $F \subset L_{p,\tau}(\mathbb{T}^m)$ is some functional class, then we put

$$e_M(F)_{p,\tau} = \sup_{f\in F} e_M(f)_{p,\tau}.$$

In the case $\tau = p$ instead of $e_M(F)_{p,\tau}$ we will write $e_M(F)_p$.

The best M–term approximation of a function $f \in L_2[0, 1]$ by polynomials via an orthonormal system was first defined by S.B. Stechkin [33] who established a criterion for the absolute convergence of the Fourier series via this system. Further, important results on estimating M-term approximations of functions for various classes of Sobolev, Nikol'skii–Besov, Lizorkin–Triebel were obtained by R.S. Ismagilov [21], Yu. Makovoz [25], V.E. Mayorov [24], E.S. Belinsky [12] – [14], B.S. Kashin [22], R. DeVore [16], V. N. Temlyakov [35] – [39], A.S. Romanyuk [27], [28], Dinh Dung [17], Wang Heping and Sun Yongsheng [41], M. Hansen and W.Sickel [19], [20] , S.A. Stasyuk [30] – [32], A.L. Shidlich [29].

To estimate M–term approximations of functions of the Nikol'skii–Besov class $S^{\bar{r}}_{p,\theta}B$ in the space $L_q(\mathbb{T}^m)$ two methods were used: non-constructive and constructive. The first method is based on Lemma 2.3 [14] (also see [25], [24]) which is proved by probabilistic reasoning. The second method was developed by V.N. Temlyakov [37], [38] and is based on greedy algorithms (see [36], [39]). Further, a constructive method of n–term approximations for the trigonometric system was developed by D.B. Bazarkhanov and V.N. Temlyakov in [10] and in [11]. A survey of the results on this theory can be found in [18]. Estimates for n–term approximations of functions of the Nikol'skii–Besov class in the Lorentz space are investigated in [1] – [3].

For a constructive method for estimating n - term approximations of functions of the Nikol'skii–Besov class $S^{\overline{r}}_{p,\theta}B$ V.N. Temlyakov [37], [38] introduced the class $W^{a,b,\overline{r}}_q$. In this article, we will consider an analogue of this class in the Lorentz space.

For a function $f \in L_1(\mathbb{T}^m)$ put

$$f_{l,\overline{r}}(\overline{x}) = \sum_{l \leqslant \langle \overline{s}, \overline{\gamma} \rangle < l+1} \delta_{\overline{s}}(f, \overline{x}), \ l \in \mathbb{Z}_+,$$

where $\overline{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \leq \ldots \leq \gamma_m$, $\gamma_j = \frac{r_j}{r_1}$, $r_j > 0$, $j = 1, \ldots, m$.

We will consider the following class defined in [37], [38]

$$W^{a,b,\overline{r}}_A = \{f \in L_1(\mathbb{T}^m) : \ \|f_{l,\overline{r}}\|_A \leqslant 2^{-la} l_0^{(\nu-1)b}\},$$

where $l_0 = \max\{1, l\}$, $l \in \mathbb{Z}_+$ and

$$\|f_{l,\overline{r}}\|_A = \sum_{l \leqslant \langle \overline{s}, \overline{\gamma} \rangle < l+1} \sum_{\overline{n} \in \rho(\overline{s})} |a_{\overline{n}}(f)|.$$

We also define the class

$$W^{a,b,\overline{r}}_{q,\tau} = \{f \in L_1(\mathbb{T}^m) : \ \|f_{l,\overline{r}}\|_{q,\tau} \leqslant 2^{-la} l_0^{(\nu-1)b}\},$$

where $a > 0$, $b \in \mathbb{R}$, $l_0 = \max\{1, l\}$.

We will introduce the following notation

$$\|f\|_{W^{a,b,\overline{r}}_{q,\tau}} = \sup_{l \in \mathbb{Z}_+} \|f_{l,\overline{r}}\|_{q,\tau} 2^{la} l_0^{-(\nu-1)b}, 1 < q, \tau < \infty.$$

In the case $\tau = q$, the class $W^{a,b,\overline{r}}_{q,\tau}$ is defined by V.N. Temlyakov [37], [38] and in this case, instead of $W^{a,b,\overline{r}}_{q,q}$ we will write $W^{a,b,\overline{r}}_q$.

For the class $W^{a,b,\overline{r}}_{q,\tau_1}$ we put

$$e_n(W^{a,b,\overline{r}}_{q,\tau_1})_{p,\tau_2} = \sup_{f \in W^{a,b,\overline{r}}_{q,\tau_1}} e_n(f)_{p,\tau_2}, \ 1 < q, p, \tau_1, \tau_2 < \infty.$$

In the case $\tau = q$, the order-sharp estimates for the best n-th trigonometric approximations of functions belonging to the class $W^{a,b,\overline{r}}_q$ in the space $L_p(\mathbb{T}^m)$, $1 < q \leqslant p < \infty$ were established by V.N. Temlyakov [37], [38]. In particular, he proved

Theorem 1.1 ([38, Theorem 3.2]). Let $1 < q \leq 2 < p < \infty$ and $(\frac{1}{q} - \frac{1}{p})p' < a < \frac{1}{q}$, $p' = \frac{p}{p-1}$, then

$$e_n(W^{a,b,\overline{r}}_q)_p \asymp n^{-\frac{p}{2}(a + \frac{1}{p} - \frac{1}{q})} (\log_2 n)^{(\nu-1)(b + a(p-1) - (\frac{1}{q} - \frac{1}{p})p)}.$$

Here and in what follows, the notation $A_n \asymp B_n$ means that there exist positive numbers C_1, C_2 independent of $n \in \mathbb{N}$ such that $C_1 A_n \leqslant B_n \leqslant C_2 A_n$ for $n \in \mathbb{N}$.

In [38], the problem of finding order-sharp estimates for $e_n(W^{a,b,\overline{r}}_q)_p$ by the constructive method, in the case of $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p})p'$, $1 < q \leqslant 2 < p < \infty$ remains open.

We will consider the problem of estimating the best M–term trigonometric approximations for the Lorentz space. The main results of the article are formulated and proved in the third section (see Theorem 3.1 and Theorem 3.2). In the second section, we formulate some auxiliary assertions required for proving the main results. In the fourth section, as an application of Theorem 3.1, we establish an upper bound for the best M - term approximations of functions of the Nikol'skii-Besov class in the Lorentz space (see Theorem 4.1).

2 Auxiliary statements

Theorem 2.1. (see [5]). *Let $1 < q < \lambda < \infty$, $1 < \tau, \theta < \infty$. If a function $f \in L_{q,\tau}(\mathbb{T}^m)$, then*

$$\|f\|_{q,\tau} \geqslant C\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{l=1}^{m} 2^{s_l(1/\lambda-1/q)\tau}\|\delta_{\bar{s}}(f)\|_{\lambda,\theta}^{\tau}\Big)^{1/\tau},$$

where $C > 0$ is independent of f.

Theorem 2.2. (see [5]). *Let $1 < p < q < \infty$, $1 < \tau_1, \tau_2 < \infty$. If the function $f \in L_{p,\tau_1}(\mathbb{T}^m)$ satisfies the condition*

$$\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{j=1}^{m} 2^{s_j\tau_2(1/p-1/q)}\|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{q,\tau_2}(\mathbb{T}^m)$ and the following inequality holds

$$\|f\|_{q,\tau_2} \leqslant C\left(\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{j=1}^{m} 2^{s_j\tau_2(1/p-1/q)}\|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2}\right)^{1/\tau_2},$$

where $C > 0$ is independent of f.

Let $A(\mathbb{T}^m)$ be the space $f \in L(\mathbb{T}^m)$ with absolutely converging Fourier series with the norm (see [11], [37], [38])

$$\|f\|_A = \sum_{\bar{k}\in\mathbb{Z}^m} |a_{\bar{k}}(f)|.$$

As a corollary of Theorem 1.1 [38], the following statement is true, which we will often use in the proofs of theorems.

Lemma 2.1. *Let $2 \leqslant p < \infty$ and $1 < \tau < \infty$. There exist constructive approximation methods $G_M(f)$ based on greedy-type algorithms that lead to M–term polynomials with respect the system $\{e^{i\langle\bar{k},\bar{x}\rangle}\}_{\bar{k}\in\mathbb{Z}^m}$ with the following property:*

$$\|f - G_M(f)\|_{p,\tau} \leq CM^{-\frac{1}{2}}p^{\frac{1}{2}}\|f\|_A,$$

for all $f \in A(\mathbb{T}^m)$, where $C > 0$ is independent of $M \in \mathbb{N}$ and of f.

Proof. . We will choose a number $p_0 \in (p, \infty)$. It is known that $L_{p_0}(\mathbb{T}^m) \subset L_{p,\tau}(\mathbb{T}^m)$ and $\|g\|_{p,\tau} \leqslant C\|g\|_{p_0}$ for a function $g \in L_{p_0}(\mathbb{T}^m)$ (see [34, Theorem 3.11]). Now, according to Theorem 1.1 [38] or Theorem 2.6 [37], it is easy to verify that the assertion of Lemma 2.1 is true. □

3 Main results

Theorem 3.1. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \leq \ldots r_m$, $1 < q < 2 < p < \infty$, $1 < \tau_1, \tau_2 < \infty$, $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$, $\tau_2^{'} = \frac{\tau_2}{\tau_2 - 1}$ and $b \in \mathbb{R}$.*

If $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$, $\tau_2^{'} = \frac{\tau_2}{\tau_2 - 1}$, then

$$e_M(W_{q,\tau_1}^{a,b,\bar{r}})_{p,\tau_2} \asymp M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log_2 M)^{(\nu-1)b},\ M > 1.$$

If $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, *then*

$$e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2} \leqslant C \begin{cases} M^{-\frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log_2 M)^{(\nu-1)b}(\log_2\log_2 M)^{1/\tau_2}, & \text{if } (\nu-1)b\tau_2+1 \neq 0, \\ M^{-\frac{\tau_2'}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}, & \text{if } (\nu-1)b\tau_2+1 = 0 \end{cases}$$

for $M \geqslant 4$, *where* $C > 0$ *is independent of* M *and* f.

Proof. For a natural number M, there is a number $n \in \mathbb{N}$ such that $M \asymp 2^n n^{\nu-1}$.

Let $\nu \geqslant 2$ be a natural number. We put

$$n_1 = \frac{p}{2}n - p\Big(\frac{1}{2} - \frac{1}{\tau_2}\Big)(\nu-1)\log n,$$

$$n_2 = \frac{p}{2}n + \frac{p}{2}(\nu-1)\log n.$$

We will introduce the notation

$$S_l = \Big(2^{la\tau_1}\bar{l}^{-(\nu-1)b\tau_1} \sum_{l \leq \langle \bar{s},\bar{\gamma}\rangle < l+1} 2^{\langle \bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}$$

and

$$m_l = \Big[2^{-l\frac{\tau_2'}{p}} S_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1, \ l \in \mathbb{Z}_+,$$

where $\langle \bar{s}, \bar{1}\rangle = \sum_{j=1}^{m} s_j$, $p' = \frac{p}{p-1}$ and $[y]$ is an integer part of the number y.

By $G(l)$ we denote the set of indices $\bar{s}$, $l \leq \langle \bar{s},\bar{\gamma}\rangle < l+1$, with the largest $\|\delta_{\bar{s}}(f)\|_2$ and $m_l = |G(l)|$ is the number of elements in the set $G(l)$.

Let us consider the functions

$$F_1(\overline{x}) = \sum_{n \leqslant l < n_1} f_l(\overline{x}),$$

$$F_2(\overline{x}) = \sum_{n_1 \leqslant l < n_2} \sum_{\bar{s} \notin G(l)} \delta_{\bar{s}}(f, \overline{x}),$$

$$F_3(\overline{x}) = \sum_{n_1 \leqslant l < n_2} \sum_{\bar{s} \in G(l)} \delta_{\bar{s}}(f, \overline{x}).$$

We will estimate $\|F_1\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_1\|_A = \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s},\bar{\gamma}\rangle < l+1} \sum_{\bar{k} \in \rho(\bar{s})} |a_{\bar{k}}(f)| \leqslant$$

$$2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s},\bar{\gamma}\rangle < l+1} 2^{\langle \bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$= 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s},\bar{\gamma}\rangle < l+1} 2^{\langle \bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2 2^{\langle \bar{s},\bar{1}\rangle\frac{1}{q}}. \quad (3.1)$$

Now, to the inner sum on the right side of inequality (3.1), applying Hölder's inequality to the inner sum for $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$ and $1<\tau_1<\infty$, we get

$$\|F_1\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{q}}\Big)^{1/\tau_1'}. \tag{3.2}$$

We will choose numbers δ_j such that $\delta_j=\gamma_j$ for $j=1,\ldots,\nu$ and $1<\delta_j<\gamma_j$ for $j=\nu+1,\ldots,m$. Then, by Lemma G [35], we have

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1'}{q}}\Big)^{1/\tau_1'}\leqslant C2^{\frac{l}{q}}l^{\frac{\nu-1}{\tau_1'}}, \tag{3.3}$$

where $C>0$ is independent of l. According to Theorem 2.1, for $1<q<2$ and $\lambda=\theta=2$, we have

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\leqslant C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}, \tag{3.4}$$

where here and in the rest of the proof C denotes a positive number which depends only on numerical parameters, and may be different on different occurrences.

Now, taking into account that the function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$, $\frac{1}{q}-a>0$, from inequalities (3.2), (3.3) and (3.4), we obtain

$$\|F_1\|_A\leqslant C\sum_{l=n}^{n_1-1}2^{\frac{l}{q}}l^{(\nu-1)/\tau_1'}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}$$

$$\leqslant C\sum_{l=n}^{n_1-1}2^{\frac{l}{q}}l^{(\nu-1)/\tau_1'}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}$$

$$\leqslant C\sum_{l=n}^{n_1-1}2^{l(\frac{1}{q}-a)}l^{(\nu-1)(b+\frac{1}{\tau_1'})}\leqslant C2^{n_1(\frac{1}{q}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}.$$

Thus,

$$\|F_1\|_A\leqslant C2^{n_1(\frac{1}{q}-a)}n_1^{(\nu-1)(b+\frac{1}{\tau_1'})} \tag{3.5}$$

for a function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$ and $\frac{1}{q}-a>0$, $1<q<2$ and $1<\tau_1<\infty$. By Lemma 2.1 for the function F_1, using a constructive method, one can find an M–term trigonometric polynomial $G_M(F_1,\overline{x})$ such that

$$\|F_1-G_M(F_1)\|_{p,\tau_2}\leqslant CM^{-1/2}\|F_1\|_A,\ 2<p<\infty. \tag{3.6}$$

Now, taking into account the definition of the number n_1 and the condition $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$

from estimates (3.5) and (3.6) we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-1/2} 2^{n_1(\frac{1}{q}-a)} n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$= CM^{-1/2} 2^{n\frac{p}{2}(\frac{1}{q}-a)} n^{-(\nu-1)p(\frac{1}{2}-1)(\frac{1}{\tau_1}-a)} n_1^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$\leqslant CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)} n^{(\nu-1)p(1-\frac{1}{\tau_2})(a-\frac{1}{q})} n^{(\nu-1)(b+\frac{1}{\tau_1'})}$$
$$= CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)} n^{(\nu-1)(\frac{p}{\tau_2'}(a-\frac{1}{q})+\frac{1}{\tau_1'})} n^{(\nu-1)b}$$
$$= CM^{-1/2}(2^n n^{\nu-1})^{\frac{p}{2}(\frac{1}{q}-a)} n^{(\nu-1)(\frac{p}{\tau_2'}(a-\tau_2'(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2}))} n^{(\nu-1)b}$$
$$\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}, \quad (3.7)$$

in the case $a \leqslant (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $1 < q < 2 < p < \infty$, $1 < \tau_1, \tau_2 < \infty$.

Let us estimate $\|F_2\|_{p,\tau_2}$. By Theorem 2.2, for $p = \tau_1 = 2$ and replacing q by p, taking into account that

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant m_l^{-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b} 2^{-l(\frac{1}{2}-\frac{1}{q})} S_l,$$

for $\bar{s} \notin G(l)$, for $\tau_2 - \tau_1 \geqslant 0$ we have

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2}\Big)^{1/\tau_2}$$
$$= C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}$$
$$\leqslant C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b} 2^{-l(\frac{1}{2}-\frac{1}{q})} S_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2}$$
$$= C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la} l^{(\nu-1)b} 2^{-l(\frac{1}{2}-\frac{1}{q})}\Big)^{\tau_2-\tau_1} m_l^{-\frac{\tau_2-\tau_1}{\tau_1}} S_l^{\tau_2-\tau_1}$$
$$\times \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}. \quad (3.8)$$

Since $1 < q < 2 < p$, then $(\frac{1}{2} - \frac{1}{p})\tau_2 - (\frac{1}{2} - \frac{1}{q})\tau_1 > 0$. Therefore, taking into account that $1 \leqslant \gamma_j$, $j = 1, ..., m$, it is easy to verify that

$$\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}$$
$$\times \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}\Big(2^{-la} l^{(\nu-1)b}\Big)^{\tau_1} S_l^{\tau_1}.$$

Therefore,

$$S_l^{\tau_2-\tau_1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1} \leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1}\Big(2^{-la} l^{(\nu-1)b}\Big)^{\tau_1} S_l^{\tau_2}.$$

Hence, from inequality (3.8) we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la}l^{(\nu-1)b}2^{-l(\frac{1}{2}-\frac{1}{q})}\Big)^{\tau_2-\tau_1} m_l^{-\frac{\tau_2-\tau_1}{\tau_1}} 2^{(l+1)((\frac{1}{2}-\frac{1}{p})\tau_2-(\frac{1}{2}-\frac{1}{q})\tau_1)}$$

$$\times\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_1} S_l^{\tau_2}\Big)^{1/\tau_2} = C\Big(\sum_{l=n_1}^{n_2-1}\Big(2^{-la}l^{(\nu-1)b}\Big)^{\tau_2} m_l^{-\frac{\tau_2-\tau_1}{\tau_1}} 2^{l(\frac{1}{\tau_1}-\frac{1}{\tau_2})\tau_2} S_l^{\tau_2}\Big)^{1/\tau_2}.$$

Now, substituting the values of the numbers m_l, from here we get

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a+\frac{1}{p}-\frac{1}{q})\tau_2} l^{(\nu-1)b\tau_2}\Big(2^{-l\frac{\tau_2'}{p}} S_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1}} S_l^{\tau_2}\Big)^{1/\tau_2}$$

$$= C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} S_l^{\tau_1}\Big)^{1/\tau_2}. \quad (3.9)$$

Further, using inequality (3.4) and taking into account that the function $f \in W^{a,b,\bar r}_{q,\tau_1}$ and $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$ we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} S_l^{\tau_1}$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2}\Big(2^{la}l^{-(\nu-1)b}\Big\|\sum_{l\leq\langle\bar s,\bar\gamma\rangle<l+1}\delta_{\bar s}(f)\Big\|_{q,\tau_1}\Big)^{\tau_1}$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} \leqslant C2^{-n_2(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} n_2^{(\nu-1)b\tau_2}. \quad (3.10)$$

It is easy to verify that if $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, then

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')\tau_2} l^{(\nu-1)b\tau_2} \leqslant C\begin{cases} n^{(\nu-1)b\tau_2}\log n, & \text{if } (\nu-1)b\tau_2+1\neq 0,\\ 1, & \text{if } (\nu-1)b\tau_2+1=0.\end{cases} \quad (3.11)$$

If $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, then from (3.9) and (3.10) we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} 2^{-n_2(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2')} n_2^{(\nu-1)b}.$$

Now, by the definition of the number n_2 and taking into account that $M \asymp 2^n n^{\nu-1}$, from this formula, we obtain that

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{\tau_2-\tau_1}{\tau_1\tau_2}} (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'))} n^{(\nu-1)b}$$

$$= C(2^n n^{(\nu-1)})^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} n^{(\nu-1)b} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b} \quad (3.12)$$

for the function $f \in W^{a,b,\bar r}_{q,\tau_1}$ in the case of $a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2'$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

If $a=(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2'$, then from (3.9) and (3.11) we obtain that

$$\|F_2\|_{p,\tau_2}\leqslant C\Big(2^n n^{(\nu-1)}\Big)^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\begin{cases} n^{(\nu-1)b}(\log n)^{\frac{1}{\tau_2}}, & \text{if } (\nu-1)b\tau_2+1\neq 0,\\ 1, & \text{if } (\nu-1)b\tau_2+1=0.\end{cases} \quad (3.13)$$

Next, we estimate $\|F_3\|_A$. By applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_3\|_A=\sum_{l=n_1}^{n_2-1}\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|$$
$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$
$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2. \quad (3.14)$$

Now, to the inner sum on the right side of inequality (3.14) applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$ and $1<\tau_1<\infty$, we get

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$
$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}1\Big)^{1/\tau_1'}$$
$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}m_l^{1/\tau_1'}.$$

Further, substituting the values of the numbers m_l, from this formula, we obtain that

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$
$$\leqslant\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}\Big(2^{-l\frac{\tau_2'}{p}}S_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}+1\Big)^{1/\tau_1'}$$
$$=C\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big(2^{-l\frac{\tau_2'}{p}}S_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}+1\Big)^{1/\tau_1'}$$
$$\leqslant C\Big\{\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big(2^{-l\frac{\tau_2'}{p}}S_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{1/\tau_1'}+\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big\}$$
$$=C\Big\{2^{n\frac{\tau_2'}{2\tau_1'}}n^{(\nu-1)\frac{\tau_2'}{2\tau_1'}}\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q}+\frac{\tau_2'}{p\tau_1'})}l^{(\nu-1)b}S_lS_l^{\frac{\tau_1}{\tau_1'}}+\sum_{l=n_1}^{n_2-1}2^{-l(a-\frac{1}{q})}l^{(\nu-1)b}S_l\Big\}. \quad (3.15)$$

Since

$$\frac{\tau_2'}{p\tau_1'}-\frac{1}{q}=\tau_2'(\frac{1}{p\tau_1'}-\frac{1}{q\tau_2'}),\ \ S_lS_l^{\frac{\tau_1}{\tau_1'}}=S_l^{\tau_1},$$

then

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1'}} = \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} l^{(\nu-1)b} S_l^{\tau_1}. \quad (3.16)$$

Now, by using inequality (3.4) and taking into account that the function $f \in W_q^{a,b,\bar{r}}$ in the case $a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'})<0$ from equality (3.16), we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1'}}$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1}\Big)^{\tau_1}$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} l^{(\nu-1)b} \leqslant C 2^{-n_2(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} n_2^{(\nu-1)b}. \quad (3.17)$$

and if $a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'})=0$, then according to (3.11)

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b} S_l S_l^{\frac{\tau_1}{\tau_1'}} \leqslant C \left\{ \begin{array}{rl} n^{(\nu-1)b}\log n, & \text{if } (\nu-1)b+1\neq 0, \\ 1, & \text{if } (\nu-1)b+1=0. \end{array} \right. \quad (3.18)$$

Since $a-\frac{1}{q}<0$, then again using Theorem 2. 1 for $\lambda=\theta=2$ and taking into account that the function $f\in W_{q,\tau_1}^{a,b,\bar{r}}$, we get

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} S_l \leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1}\Big)$$

$$\leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{q})} l^{(\nu-1)b} \leqslant C 2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b}. \quad (3.19)$$

Now from inequalities (3.15), (3.17) and (3.19), it follows that

$$\sum_{l=n_1}^{n_2-1} 2^{\frac{l+1}{q}} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\in G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})} \|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C \Big\{ 2^{n\frac{\tau_2'}{2\tau_1'}} n^{(\nu-1)\frac{\tau_2'}{2\tau_1'}} 2^{-n_2(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} n_2^{(\nu-1)b} + 2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b} \Big\}, \quad (3.20)$$

in the case $a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'})<0$. By the definition of the number n_2, we have

$$2^{-n_2(a-\frac{1}{q})} n_2^{(\nu-1)b} = (2^{n\frac{p}{2}} n^{(\nu-1)\frac{p}{2}})^{-(a-\frac{1}{q})} n_2^{(\nu-1)b} \leqslant C (2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})} n^{(\nu-1)b}$$

and

$$(2^n n^{\nu-1})^{\frac{\tau_2'}{2\tau_1'}} 2^{-n_2(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))} = (2^n n^{\nu-1})^{\frac{\tau_2'}{2\tau_1'}} (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-\tau_2'(\frac{1}{q\tau_2'}-\frac{1}{p\tau_1'}))}$$

$$= (2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})} (2^n n^{\nu-1})^{-(\frac{p}{2}(\frac{1}{q}-p'(\frac{1}{q}-\frac{1}{p}))-\frac{p'}{2q'})}.$$

Now, taking into account that $\frac{p}{2}(a-\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}}))-\frac{\tau_2^{'}}{2\tau_1^{'}}=\frac{p}{2}(a-\frac{1}{q})$ according to these relations from formula (3.20), we obtain

$$\sum_{l=n_1}^{n_2-1}2^{\frac{l+1}{q}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b},$$

for $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}=\tau_2^{'}(\frac{1}{q\tau_2^{'}}-\frac{1}{p\tau_1^{'}})$, $b\in\mathbb{R}$. Therefore, inequality (3.14) implies that

$$\|F_3\|_A\leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b} \tag{3.21}$$

for $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$, $b\in\mathbb{R}$.

Since $a-\frac{1}{q}<0$, then from inequalities (3.13), (3.18) and (3.19) it follows that inequality (3.21) is also true in the case $a=(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$.

By Lemma 2.1 for the function F_3 there exists an M-term polynomial $G_M(F_3,\overline{x})$ such that

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-1/2}\|F_3\|_A.$$

Therefore, according to inequality (3.21) from this formula, we obtain that

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\leqslant CM^{-1/2}(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{q})}n^{(\nu-1)b}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b} \tag{3.22}$$

in the case $a\leqslant(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$.

We represent the function $f\in W^{a,b,\bar{r}}_{q,\tau_1}$ as a sum

$$f(\overline{x})=S_{Q_{n,\bar{\gamma}}}(f,\overline{x})+F_1(\overline{x})+F_2(\overline{x})+F_3(\overline{x})+\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f,\bar{x}).$$

Therefore, from estimates (3.7), (3.12), (3.22), it follows that

$$\begin{aligned}&\|f-(S_{Q_{n,\bar{\gamma}}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\\&\qquad\leqslant\|F_1-G_M(F_1)\|_{p,\tau_2}+\|F_3-G_M(F_3)\|_{p,\tau_2}+\|F_2\|_{p,\tau_2}\\&\qquad+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}+\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2},\end{aligned} \tag{3.23}$$

in the case $a<(\frac{1}{q}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{q\tau_2})\tau_2^{'}$, $b\in\mathbb{R}$. Since $1<q<p<\infty$, then by Theorem 2.2, inequality (3.4) and the definition of the class $W^{a,b,\bar{r}}_{q,\tau_1}$ and taking into account such that $a+\frac{1}{p}-\frac{1}{q}>0$ and $1<\tau_1\leqslant\tau_2<\infty$, we have

$$\begin{aligned}&\Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}=\Big\|\sum_{l=n_2}^{\infty}\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}\\&\qquad\leqslant C\Big(\sum_{l=n_2}^{\infty}2^{l(\frac{1}{q}-\frac{1}{p})\tau_2}\Big\|\sum_{l\leqslant\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}}\leqslant C\Big(\sum_{l=n_2}^{\infty}2^{-l(a+\frac{1}{p}-\frac{1}{q})p}l^{(\nu-1)bp}\Big)^{\frac{1}{p}}\\&\qquad\leqslant C2^{-n_2(a+\frac{1}{p}-\frac{1}{q})}n_2^{(\nu-1)b}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}.\end{aligned} \tag{3.24}$$

Now from inequalities (3.23) and (3.24), it follows that

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$ for $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$, $b \in \mathbb{R}$ and $1 < q < 2 < p < \infty$ and $\nu \geqslant 2$.

If $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$, then

$$\frac{p}{2}(a + \frac{1}{p} - \frac{1}{q}) = \frac{\tau_2^{'}}{2}(\frac{1}{\tau_1} - \frac{1}{\tau_2}).$$

Therefore, in the case $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$ and $(\nu-1)b\tau_2 + 1 \neq 0$ from inequalities (3.13), (3.22) and (3.7), we obtain

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{\tau_2^{'}}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log M)^{(\nu-1)b}(\log\log M)^{1/\tau_2}.$$

Hence

$$e_M(f)_{p,\tau_2} \leqslant CM^{-\frac{\tau_2^{'}}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}(\log M)^{(\nu-1)b}(\log\log M)^{1/\tau_2}$$

for the function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < 2 < p < \infty$, $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$ and $(\nu-1)b\tau_2 + 1 \neq 0$, $\nu \geqslant 2$.

If $a = (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$ and $(\nu-1)b\tau_2 + 1 = 0$, then from inequalities (3.7), (3.13) and (3.21), it follows that

$$e_M(f)_{p,\tau_2} \leqslant CM^{-\frac{\tau_2^{'}}{2}(\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < 2 < p < \infty$.

Let $\nu = 1$ i.e. $r_1 < r_{\nu+1} \leqslant ... \leqslant r_m$. For $M \asymp 2^n$, there is a natural number n such that $M \asymp 2^n$. In this case, put $n_1 = n\frac{p}{2}$ and consider the function

$$F_1(\overline{x}) = \sum_{l=n}^{n_1-1} f_l(\overline{x}).$$

Now, repeating the arguments in the proof of inequality (3.5) for the function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, we obtain

$$\|F_1\|_A \leqslant C2^{n_1(\frac{1}{q}-a)}, \tag{3.25}$$

in the case $\frac{1}{q} - a > 0$, for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$. Hence, from inequalities (3.25) and (3.6), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leqslant CM^{-1/2}\|F_1\|_A \leqslant CM^{-1/2}2^{n_1(\frac{1}{q}-a)} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} \tag{3.26}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $\frac{1}{q} - a > 0$. By the property of the norm and according to (3.26), we have

$$\begin{aligned}\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1))\|_{p,\tau_2} &\leqslant \|F_1 - G_M(F_1)\|_{p,\tau_2} + \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \\ &\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})} + \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f)\Big\|_{p,\tau_2},\end{aligned} \tag{3.27}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $a < \frac{1}{q}$. Further, repeating the proof of inequality (3.24) with n_2 replaced by n_1, we have

$$\Big\| \sum_{\langle \bar{s},\bar{\gamma}\rangle \geqslant n_1} \delta_{\bar{s}}(f) \Big\|_{p,\tau_2} \leqslant C 2^{-n_1(a+\frac{1}{p}-\frac{1}{q})} \leqslant C M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}, \tag{3.28}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case of $\frac{1}{p} - \frac{1}{q} < a$, $1 < q < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

Now from inequalities (3.27) and (3.28), it follows that

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1))\|_{p,\tau_2} \leqslant C M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})},$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, in the case $\frac{1}{p} - \frac{1}{q} < a < \frac{1}{q}$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$. Hence

$$e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2} \leqslant C M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})},$$

in the case $\nu = 1$ and $\frac{1}{p} - \frac{1}{q} < a < \frac{1}{q}$, $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant \tau_2 < \infty$.

Lower bound for $e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2}$. Let $M \in \mathbb{N}$ and $N = [\frac{p}{2}\log_2 M]$ is an integer part of the number $\frac{p}{2}\log_2 M$.

Let $\bar{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m_+$ such that $\prod_{j=1}^m 2^{s_j} = 2^N$. Consider the function

$$f_0(\bar{x}) = 2^{-N(1-\frac{1}{q})} 2^{-Na} N^{(\nu-1)b} \sum_{\bar{k}\in\rho(\bar{s})} e^{\langle \bar{k},\bar{x}\rangle}.$$

Then

$$\|f_{0,l}\|_{q,\tau_1} = \Big\| \sum_{l \leqslant \langle \bar{s},\bar{\gamma}\rangle < l+1} \delta_{\bar{s}}(f_0) \Big\|_{q,\tau_1} = 0$$

for $l \neq N$. If $l = N$, then by virtue of the estimate for the norm of the Dirichlet kernel in the Lorentz space (see [5, p. 13]), we have

$$\|f_{0,l}\|_{q,\tau_1} = \|f_0\|_{q,\tau_1} \leqslant C 2^{-Na} N^{(\nu-1)b}.$$

Thus, the function $f_0 \in W^{a,b,\bar{r}}_{q,\tau_1}$, $1 < q < \infty$, $1 < \tau_1, \tau_2 < \infty$, $a > 0$, $b \in \mathbb{R}$.

Let K_M be an arbitrary set of M harmonics $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m_+$ and $\mathbb{T}(K_M)$ is the set of trigonometric polynomials with harmonics from K_M. Consider an additional function

$$h(\bar{x}) = \sum_{\bar{k}\in\rho(\bar{s})\setminus K_M} e^{\langle \bar{k},\bar{x}\rangle}.$$

Then, by the property of the norm, the estimate for the norm of the Dirichlet kernel, and Parseval's equality, we have

$$\|h\|_{p',\tau_2'} \leqslant \|g_0\|_{p',\tau_2'} + \|g_0 - h\|_{p',\tau_2'} \leqslant \|g_0\|_{p',\tau_2'} + C\|g_0 - h\|_2 \leqslant C\{2^{\frac{N}{p}} + \sqrt{M}\} \leqslant C_0\sqrt{M},$$

where $g_0(\bar{x}) = \sum_{\bar{k}\in\rho(\bar{s})} e^{\langle \bar{k},\bar{x}\rangle}$, $2 < p < \infty$, $1 < \tau_2 < \infty$, $\beta' = \frac{\beta}{\beta-1}$. Therefore, for any polynomial $T \in \mathbb{T}(K_M)$, due to Hölder's inequality in the Lorentz space, we have

$$\int_{\mathbb{T}^m} (f_0(\bar{x}) - T(\bar{x}))h(\bar{x})d\bar{x} \leqslant \|f_0 - T\|_{p,\tau_2}\|h\|_{p',\tau_2'} \leqslant C\sqrt{M}\|f_0 - T\|_{p,\tau_2}, \tag{3.29}$$

for $2 < p < \infty$, $1 < \tau_2 < \infty$.

On the other hand, taking into account the orthogonality of the trigonometric system, we have

$$\int\limits_{\mathbb{T}^m} (f_0(\overline{x}) - T(\overline{x}))h(\overline{x})d\overline{x} = \int\limits_{\mathbb{T}^m} f_0(\overline{x})h(\overline{x})d\overline{x} = 2^{-N(1-\frac{1}{q})}2^{-Na}N^{(\nu-1)b}\sum_{\overline{k}\in\rho(\overline{s})} 1$$
$$= 2^{-N(1-\frac{1}{q})}2^{-Na}N^{(\nu-1)b}(|\rho(\overline{s}) \setminus K_M| - M) \geqslant 2^{-N(a+1-\frac{1}{q})}N^{(\nu-1)b}(2M - M)$$
$$= 2^{-N(a+1-\frac{1}{q})}N^{(\nu-1)b}2^N.$$

Therefore, from inequality (3.29), we obtain

$$\|f_0 - T\|_{p,\tau_2} \geqslant C2^{-N(a-\frac{1}{q})}N^{(\nu-1)b}2^N M^{-\frac{1}{2}} \geqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b},$$

for any polynomial $T \in \mathbb{T}(K_M)$, $2 < p < \infty$, $1 < \tau_2 < \infty$. Hence

$$e_M(W^{a,b,\overline{r}}_{q,\tau_1})_{p,\tau_2} \geqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b},$$

in the case $\frac{1}{q} - \frac{1}{p} < a < (\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})\tau_2^{'}$, $1 < q < 2 < p < \infty$, $1 < \tau_2 < \infty$. □

Theorem 3.2. *Let* $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \leq \ldots r_m$, $2 < p < \infty$, $1 < \max\{\tau_1, 2\} \leqslant \tau_2 < \infty$, $\frac{1}{2} - \frac{1}{p} < a < (\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})\tau_2^{'}$, $\tau_2^{'} = \frac{\tau_2}{\tau_2-1}$ *and* $b \in \mathbb{R}$, *then*

$$e_M(W^{a,b,\overline{r}}_{2,\tau_1})_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log_2 M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

where $C > 0$ *is independent of* $M > 1$.

Proof. As in the proof of Theorem 3.1, consider the functions F_j, $j = 1, 2, 3$. By formula(3.1), we have

$$\|F_1\|_A = \sum_{l=n}^{n_1-1}\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}\sum_{\overline{k}\in\rho(\overline{s})}|a_{\overline{k}}(f)| \leqslant 2^{-\frac{m}{2}}\sum_{l=n}^{n_1-1}\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1} 2^{\langle\overline{s},\overline{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2. \tag{3.30}$$

If $2 < \tau_1 < \infty$, then according to the inequality of different metrics for trigonometric polynomials in the Lorentz space [4] we have

$$\|\delta_{\overline{s}}(f)\|_2 \leqslant C\Big(\sum_{j=1}^{m}(s_j + 1)\Big)^{\frac{1}{2}-\frac{1}{\tau_1}}\|\delta_{\overline{s}}(f)\|_{2,\tau_1},$$

where here and in the rest of the proof C denotes a positive number which depends only on numerical parameters, and may be different on different occurrences.
Therefore, from Lemma 1.6 [5] for $p = 2$ and $2 < \tau_1 < \infty$ we obtain

$$\left(\sum_{\overline{s}\in\mathbb{Z}_+}\Big(\sum_{j=1}^{m}(s_j + 1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}} \leqslant C\left(\sum_{\overline{s}\in\mathbb{Z}_+}\|\delta_{\overline{s}}(f)\|_{2,\tau_1}^{\tau_1}\right)^{\frac{1}{\tau_1}} \leqslant C\|f\|_{2,\tau_1}. \tag{3.31}$$

According to inequality (3.31) and Hölder's inequality, we obtain

$$\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1} 2^{\langle\overline{s},\overline{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2 \leqslant \left(\sum_{l\leq\langle\overline{s},\overline{\gamma}\rangle<l+1}\Big(\sum_{j=1}^{m}(s_j + 1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}}$$

$$\times \left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1^{'}}{2}}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1^{'}}\right)^{\frac{1}{\tau_1^{'}}}$$

$$\leqslant C\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{\tau_1^{'}}{2}}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1^{'}}\right)^{\frac{1}{\tau_1^{'}}}, \quad (3.32)$$

where $\tau_1^{'}=\frac{\tau_1}{\tau_1-1}$, $1<\tau_1<\infty$. We will choose numbers δ_j such that $\delta_j=\gamma_j$ for $j=1,\ldots,\nu$ and $1<\delta_j<\gamma_j$ for $j=\nu+1,\ldots,m$. Then, by Lemma G [35], from inequality (3.32) we have

$$\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{\delta}\rangle\frac{\tau_1^{'}}{2}}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1^{'}}\right)^{\frac{1}{\tau_1^{'}}},$$

$$\leqslant C\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1} 2^{\frac{l}{2}} l^{(\nu-1)\frac{1}{\tau_1^{'}}} l^{\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.33)$$

in the case $2<\tau_1<\infty$. Therefore, taking into account that the function $f\in W^{a,b,\bar{r}}_{2,\tau_1}$ and $a<\frac{1}{2}$ from (3.30) and (3.33) we get

$$\|F_1\|_A\leqslant C\sum_{l=n}^{n_1-1} 2^{\frac{l}{2}} l^{(\nu-1)\frac{1}{\tau_1^{'}}} l^{\frac{1}{2}-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b}\leqslant C2^{-n_1(a-\frac{l}{2})} n_1^{(\nu-1)(b+\frac{1}{\tau_1^{'}})} n_1^{\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.34)$$

in the case $q=2<p<\infty$, $2<\tau_1<\infty$, $a<\frac{1}{2}$. Since $2<p<\infty$, then by Lemma 2.1 for the function F_1 there exists a M-term polynomial $G_M(F_1,\overline{x})$ such that

$$\|F_1-G_M(F_1)\|_{p,\tau_2}\leqslant CM^{-\frac{1}{2}}\|F_1\|_A.$$

Therefore, according to inequality (3.34) and taking into account the definition of the number n_1 and the relation $M\asymp 2^n n^{\nu-1}$ from this formula, we obtain that

$$\|F_1-G_M(F_1)\|_{p,\tau_2}\leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}}, \quad (3.35)$$

in the case $q=2<p<\infty$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $a<\frac{1}{2}$.

For the estimate $\|F_3\|_A$ by applying Hölder's inequality for the sum and Parseval's equality, we obtain

$$\|F_3\|_A=\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\sum_{\overline{k}\in\rho(\overline{s})}|a_{\overline{k}}(f)|\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)} 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})}\|\delta_{\bar{s}}(f)\|_2. \quad (3.36)$$

Now, to the inner sum on the right side of inequality (3.36), by applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1^{'}}=1$ and $1<\tau_1<\infty$ we will have

$$\|F_3\|_A\leqslant C\sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}|G(l)|^{\frac{1}{\tau_1^{'}}}. \quad (3.37)$$

We will put

$$\tilde{S}_l = \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1} \|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}$$

and

$$m_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1.$$

Then from (3.37), it follows that

$$\|F_3\|_A \leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l m_l^{\frac{1}{\tau_1'}}$$
$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l \Big\{2^{-l\frac{\tau_2'}{p}} \tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}} + 1\Big\}^{\frac{1}{\tau_1'}}$$
$$\leqslant C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} + \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l\Big\}. \quad (3.38)$$

Since $\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} = \tilde{S}_l^{\tau_1}$ and $-\frac{1}{2} + \frac{\tau_2'}{p\tau_1'} = \tau_2'(-\frac{1}{2} + \frac{1}{p} - \frac{1}{p\tau_1} + \frac{1}{2\tau_2})$, then according to (3.31), we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} = \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{\tau_1}$$
$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\Big)^{\tau_1}$$
$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}.$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$. Since $a - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0$, then taking into account the definition of the number n_2 from this formula, we obtain that

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} \leqslant C 2^{-n_2(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}$$
$$\leqslant C 2^{-n\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{-(\nu-1)\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.39)$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$, $2 < \tau_1 < \infty$.

Further, according to inequality (3.31), taking into account the function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ and $a - \frac{1}{2} < 0$ we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l \leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}$$
$$\times \Big(2^{la\tau_1} l^{-(\nu-1)b\tau_1} \Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\Big)$$
$$\leqslant C\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2})} l^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \leqslant C 2^{-n_2(a-\frac{1}{2})} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}$$
$$\leqslant C 2^{-n\frac{p}{2}(a-\frac{1}{2})} n^{-(\nu-1)\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}. \quad (3.40)$$

Now from inequalities (3.38), (3.39) and (3.40), it follows that

$$\|F_3\|_A \leqslant C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} 2^{-n\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{-(\nu-1)\frac{p}{2}(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} + (2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}\Big\}$$

for a function $f \in W^{a,b,\bar{r}}_{q,\tau_1}$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$, $a - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0$.

Since $\frac{p}{2}(a - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})) - \frac{\tau_2'}{2\tau_1'} = \frac{p}{2}(a - \frac{1}{2})$, then it follows that

$$\|F_3\|_A \leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}. \tag{3.41}$$

Since $2 < p < \infty$, then by Lemma 2.1 for the function F_3 by a constructive method there is a M–term polynomial $G_M(F_3, \overline{x})$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leqslant CM^{-\frac{l}{2}}\|F_3\|_A.$$

Therefore, according to (3.41), we have

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leqslant CM^{-\frac{l}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \tag{3.42}$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$, $2 < p < \infty$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. In formula (3.8), the inequality is proved

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.$$

Now, taking into account that

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant m_l^{-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b} l^{\frac{1}{2}-\frac{1}{\tau_1}} \tilde{S}_l$$

for $\overline{s} \notin G(l)$ and substituting the values of the numbers m_l, for $\tau_2 - \tau_1 \geqslant 0$, hence we have

$$\begin{aligned}
\|F_2\|_{p,\tau_2} &\leqslant C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b} l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2} \\
&= C\Big(\sum_{l=n_1}^{n_2-1}\Big(\Big(2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{1}{\tau_1}} 2^{-la} l^{(\nu-1)b}\tilde{S}_l l^{\frac{1}{2}-\frac{1}{\tau_1}}\Big)^{\tau_2-\tau_1} \\
&\quad\times \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2} \\
&= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)} l^{(\nu-1)b(\tau_2-\tau_1)} l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)} \\
&\quad\times \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.
\end{aligned} \tag{3.43}$$

Further, taking into account that $1 \leqslant \gamma_j,\ j = 1, ..., m$ and using inequality (3.31), it is easy to verify that

$$\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}$$
$$\leqslant 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} (\sum_{j=1}^{m}(s_j+1))^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}$$
$$\leqslant C2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}^{\tau_1}$$
$$\leqslant C2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1} \Big(2^{-la} l^{(\nu-1)b}\Big)^{\tau_1} \quad (3.44)$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$, $2 < \tau_1 \leqslant \tau_2 < \infty$.

Now from inequalities (3.43) and (3.44), it follows that

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}$$
$$\times\Big(\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)} l^{(\nu-1)b(\tau_2-\tau_1)} l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)} 2^{(l+1)(\frac{1}{2}-\frac{1}{p})\tau_2} l^{-(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}\Big(2^{-la} l^{(\nu-1)b}\Big)^{\tau_1}\Big)^{1/\tau_2}$$
$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1} 2^{-l\tau_2(a-\frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}))} l^{(\nu-1)b\tau_2} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2}.$$

Since

$$a - \frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1) - (\frac{1}{2}-\frac{1}{p}) = a - \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}),$$

then taking into account the definition of the number n_2, from this formula, we get

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} 2^{-n_2(a-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}$$
$$\leqslant C2^{-n\frac{p}{2}(a-\frac{1}{p}-\frac{1}{2})} n^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}}, \quad (3.45)$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $2 < \tau_1 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Now from inequalities (3.35), (3.42) and (3.45), it follows that

$$\|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2}$$
$$\leqslant \|F_1 - G_M(F_1)\|_{p,\tau_2} + \|F_3 - G_M(F_3)\|_{p,\tau_2} + \|F_2\|_{p,\tau_2}$$
$$+ \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2} \delta_{\bar{s}}(f,\bar{x})\Big\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}} + \Big\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geqslant n_2} \delta_{\bar{s}}(f,\bar{x})\Big\|_{p,\tau_2}$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $2 < \tau_1 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Further, using inequality (3.24) for $q = 2$ and taking into account that $\frac{1}{2}-\frac{1}{\tau_1} \geqslant 0$ from this formula, we obtain

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $2 < \tau_1 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Let $1 < \tau_1 \leqslant 2$. Then, by Lemma 1.5 [5], the following inequality holds

$$\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{2,\tau_1}^2\Big)^{1/2} \leqslant C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}. \tag{3.46}$$

Since $1 < \tau_1 \leqslant 2$, then (see [34, p. 217, Theorem 3.11])

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant C\|\delta_{\bar{s}}(f)\|_{2,\tau_1}. \tag{3.47}$$

From inequalities (3.30), (3.47) and (3.46), it follows that

$$\|F_1\|_A \leqslant C\sum_{l=n}^{n_1-1}2^{l/2}\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{2,\tau_1}.$$

Now taking into account that the function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ and the choice of the number n_1 from this formula, we get that

$$\|F_1\|_A \leqslant CM^{-\frac{p}{2}(a-\frac{1}{2})}(\log M)^{(\nu-1)(b+\frac{p}{\tau_2'}(a-\frac{1}{2})}, \tag{3.48}$$

for $a < 1/2$. Further, arguing as in the proof of inequality (3.35), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \ll M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}} \ll M^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}, \tag{3.49}$$

in the case $q = 2 < p < \infty$, $1 < \tau_1 \leqslant 2$, $1 < \tau_2 < \infty$, $a < \frac{1}{2}$.

In order to estimate $\|F_3\|_A$, we put

$$\tilde{S}_l = \Big(2^{la\tau_1}l^{-(\nu-1)b\tau_1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2}$$

and

$$\tilde{m}_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^2 2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1.$$

In inequality (3.36), it was proved that

$$\|F_3\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\bar{s}}(f)\|_2$$

$$\leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{(l+1)/2}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\|\delta_{\bar{s}}(f)\|_2. \tag{3.50}$$

By to the inner sum on the right side of inequality (3.50) applying Hölder's inequality and substituting the value of the number $\tilde{m}_l := |G(l)|$ from (3.50), we obtain

$$\|F_3\|_A \leqslant 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}2^{(l+1)/2}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2}|G(l)|^{1/2}$$

$$\ll 2^{-\frac{m-1}{2}}\Big\{\sum_{l=n_1}^{n_2-1}2^{l(\frac{1}{2}-a)}l^{(\nu-1)b}2^{-l\frac{\tau_2'}{2p}}\tilde{S}_l^2(2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}} + \sum_{l=n_1}^{n_2-1}2^{l(\frac{1}{2}-a)}l^{(\nu-1)b}\tilde{S}_l\Big\}. \tag{3.51}$$

Now, using inequalities (3.46) and (3.47) and taking into account the value of the numbers $\tilde{S}_l$, we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \tilde{S}_l^2 \leqslant \sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \Big(2^{la} l^{-(\nu-1)b} \Big\| \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f) \Big\|_{2,\tau_1} \Big). \tag{3.52}$$

Since the function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ and

$$a - \frac{1}{2} + \frac{\tau_2^{'}}{2p} = a - \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2}) \leqslant a - \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0,$$

then from inequality (3.52) we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(a-\frac{1}{2}+\frac{\tau_2^{'}}{2p})} l^{(\nu-1)b} \tilde{S}_l^2 \leqslant C \sum_{l=n_1}^{n_2-1} 2^{-l(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} l^{(\nu-1)b}$$
$$\leqslant C 2^{-n_2(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b}. \tag{3.53}$$

Since the function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ and $a - \frac{1}{2} < 0$, then arguing similarly we can prove that

$$\sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-a)} l^{(\nu-1)b} \tilde{S}_l \leqslant C 2^{n_2(\frac{1}{2}-a)} n_2^{(\nu-1)b}. \tag{3.54}$$

Now from inequalities (3.51), (3.53) and (3.54), it follows that

$$\|F_3\|_A \leqslant C\Big\{ (2^n n^{(\nu-1)})^{\frac{\tau_2^{'}}{4}} 2^{-n_2(a-\tau_2^{'}(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} n_2^{(\nu-1)b} + 2^{n_2(\frac{1}{2}-a)} n_2^{(\nu-1)b} \Big\}$$
$$\leqslant C (2^n n^{(\nu-1)})^{-\frac{p}{2}(a-\frac{1}{2})} n^{(\nu-1)b}, \tag{3.55}$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2$ and $1 < \tau_2 < \infty$, $a < \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Therefore, according to Lemma 2.1 for the function F_3, by a constructive method there is a M–term polynomial $G_M(F_3, \overline{x})$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leqslant CM^{-\frac{l}{2}} \|F_3\|_A. \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})} (\log M)^{(\nu-1)b}, \tag{3.56}$$

for a function $f \in W_{2,\tau_1}^{a,b,\bar{r}}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2$, $1 < \tau_2 < \infty$, $a < \tau_2^{'}(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Let us estimate $\|F_2\|_{p,\tau_2}$. To do this, note that if $\bar{s} \notin G(l)$, then

$$\|\delta_{\bar{s}}(f)\|_2 \leqslant \tilde{m}_l^{-\frac{1}{2}} 2^{-la} l^{(\nu-1)b} \tilde{S}_l \tag{3.57}$$

and (see formula (3.8))

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\bar{s}}(f)\|_2^{\tau_2} \Big)^{1/\tau_2}$$
$$= C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1, \bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\bar{s}}(f)\|_2^{\tau_2-2} \|\delta_{\bar{s}}(f)\|_2^2 \Big)^{1/\tau_2}. \tag{3.58}$$

Further, if $\tau_2 - 2 \geqslant 0$, then using inequality (3.57) and repeating the reasoning in the proof (3.45), we obtain

$$\|F_2\|_{p,\tau_2} \leqslant C(2^n n^{\nu-1})^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})} n^{(\nu-1)b} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{2})}(\log M)^{(\nu-1)b}, \qquad (3.59)$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $q = 2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$.

Now from inequalities (3.49), (3.56), (3.59), it follows that

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G^p_M(F_1) + G^p_M(F_3))\|_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(a+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)b+\frac{1}{2}-\frac{1}{\tau_1}},$$

for a function $f \in W^{a,b,\bar{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$, $a < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$, $b \in \mathbb{R}$. □

Remark 1. In the case $\tau_1 = q$ and $\tau_2 = p$ Theorem 3.1 and Theorem 3.2 complement Theorem 3.2 [38].

Remark 2. Estimates for the quantity $e_M(W^{a,b,\bar{r}}_{q,\tau_1})_{p,\tau_2}$ for other values of the parameters q, p, τ_1, τ_2, a are announced in [6].

4 Conclusion

Now, using Theorem 3.1, we can obtain estimates for M–term approximations of a function in the Nikol'skii–Besov class.

Theorem 4.1. *Let* $1 < q < 2 < p < \infty$, $1 < \tau_1 \leqslant 2 \leqslant \tau_2 < \infty$ *and* $\frac{1}{q} - \frac{1}{p} < r_1 = ... = r_{\nu-1} < r_{\nu+1} \leqslant r_m$.

1. If $1 \leqslant \theta \leqslant \tau_1$ *and* $\frac{1}{q} - \frac{1}{p} < r_1 < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$, *then*

$$e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})},$$

where $C > 0$ *is independent of* M.

Proof. Let $f \in S^{\bar{r}}_{q,\tau_1,\theta}B$. Since $1 < \tau_1 \leq 2$ and $1 < q < \infty$, then

$$\|f_l\|_{q,\tau_1} = \Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f)\Big\|_{q,\tau_1} \leqslant C\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \|\delta_{\bar{s}}(f)\|^{\tau_1}_{q,\tau_1}\Bigg)^{1/\tau_1},$$

where $C > 0$ is independent of l and f. If $1 \leqslant \theta \leqslant \tau_1$, then according to Jensen's inequality [26, Lemma 3.3.3] from this formula, we obtain

$$\|f_l\|_{q,\tau_1} \leqslant C\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} \|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta}$$

$$\leqslant C2^{-lr_1}\Bigg(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{r}\rangle\theta}\|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta} \leqslant C2^{-lr_1}\Bigg(\sum_{\bar{s}\in\mathbb{Z}_+} 2^{\langle\bar{s},\bar{r}\rangle\theta}\|\delta_{\bar{s}}(f)\|^{\theta}_{q,\tau_1}\Bigg)^{1/\theta}.$$

Hence $\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B \subset W^{r_1,0,\bar{r}}_{q,\tau_1}$ in the case $1 \leqslant \theta \leqslant \tau_1 \leqslant 2$ and $1 < q < \infty$. Therefore, according to Theorem 3.1, for $a = r_1$ and $b = 0$, we have the estimate

$$e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2} \leqslant CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})},$$

in the case $\frac{1}{q} - \frac{1}{p} < r_1 < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$, where $C > 0$ is independent of M.

Note that if $1 \leqslant \theta \leqslant \tau_1$, then $\tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2}) \leqslant \frac{1}{q} - \frac{\tau_2'}{p\theta'}$. □

Remark 3. If $1 < \tau_1 < \theta \leqslant \tau_2 < \infty$, then $\frac{1}{q} - \frac{\tau_2'}{p\theta'} < \tau_2'(\frac{1}{q} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{q\tau_2})$. In this case, estimates of the quantity $e_M(\mathbb{S}^{\bar{r}}_{q,\tau_1,\theta}B)_{p,\tau_2}$ are given in [7].

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