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KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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MAPS BETWEEN FRÉCHET ALGEBRAS WHICH STRONGLY PRESERVES DISTANCE ONE

A. Zivari-Kazempour

Communicated by E. Kissin

Key words: Mazur-Ulam Theorem, Fréchet algebras, strictly convex, isometry.

AMS Mathematics Subject Classification: 46H40; 47A10.

Abstract. We prove that if $T : X \longrightarrow Y$ is a 2-isometry between real linear 2-normed spaces, then T is affine whenever Y is strictly convex. Also under some conditions we show that every surjective mapping $T : A \longrightarrow B$ between real Fréchet algebras, which strongly preserves distance one, is affine.

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1 Introduction and preliminaries

An algebra A over the complex field $\mathbb{C}$, is called a *Fréchet algebra* if it is a complete metrizable topological linear space. The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms, i.e.,

$$p_k(xy) \leq p_k(x)p_k(y),$$

for all $k \in \mathbb{N}$ and $x, y \in A$, such that $p_k(x) \leq p_{k+1}(x)$, whenever $k \in \mathbb{N}$ and $x \in A$, [6, 7]. A Fréchet algebra A with the above generating seminorms (p_k) is denoted by (A, p_k).

A map $f : X \longrightarrow Y$ between real normed spaces is an *isometry* if $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$, and f is *affine* if

$$f(ta + (1 - t)b) = tf(a) + (1 - t)f(b),$$

for all $a, b \in X$ and $0 \leq t \leq 1$.

An isometry need not be affine [12]. There are two important results describing cases in which every isometry is affine. The first basic result is due to Baker.

Theorem 1.1. [1] *Let X and Y be two real normed linear spaces and suppose that Y is strictly convex. If $T : X \longrightarrow Y$ is an isometry, then T is affine.*

The second result is the Mazur-Ulam theorem.

Theorem 1.2. [8] *Every bijective isometry $T : X \longrightarrow Y$ between normed spaces is affine.*

This result was proved by S. Mazur and S. Ulam [8] in 1932, and their proof is also given in the book [2, p. 166]. See also [9] and [11] for different proofs. Theorem 1.2 was improved by relaxing the subjectivity condition in [5], and is generalized for Fréchet algebras by the author in [12].

Recently, Chu in [3] proved that Theorem 1.2 holds when X and Y are linear 2-normed spaces, i.e., he proved that every 2-isometry between two linear 2-normed spaces is affine.

A mapping $T : X \longrightarrow Y$ between real normed spaces X and Y is said to strongly preserve distance n if for all $x, y \in X$ with $\|x - y\| = n$ it follows that $\|T(x) - T(y)\| = n$ and conversely. In particular, T strongly preserves distance one if $n = 1$.

A different kind of generalization of the Mazur-Ulam theorem was given by Rassias and Semrl in [10]. They proved under a special hypotheses that every surjective mapping $T : X \longrightarrow Y$ between real normed linear spaces X and Y which strongly preserves distance one, is affine [10, Theorem 5].

Also it is shown that the Rassias's theorem holds when X is a linear 2-normed space under some conditions [4].

In this paper, we prove that Theorem 1.1 holds when X and Y are linear 2-normed spaces. We also give an extension of the Rassias's theorem for Fréchet algebras.

2 The Baker result for linear 2-normed spaces

Definition 1. Let X be a real linear space with $dim X \geq 2$ and let $p(\cdot,\cdot) : X^2 \longrightarrow \mathbb{R}^+$ be function. Then $(X, p(\cdot,\cdot))$ is called a linear 2-seminormed space if

(a) $p(x, y) = 0 \Leftrightarrow x, y$ are linearly dependent,

(b) $p(x, y) = p(y, x)$,

(c) $p(\lambda x, y) = |\lambda| p(x, y)$,

(d) $p(x, y + z) \leq p(x, y) + p(x, z)$,

(e) $p(x, z) \leq p(x, y) + p(y, z)$,

for all $x, y, z \in X$, $\lambda \in \mathbb{R}$.

In particular, if we define $p(x, y) = \|x, y\|$, then we obtain a real linear 2-normed space in the sense of [3, Definition 2.1]. For example, define $p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $p\big((a, b), (x, y)\big) = |ay - bx|$. Then $(\mathbb{R}^2, p)$ is a linear 2-normed space.

Let $(X, p(\cdot,\cdot))$ be a linear 2-seminormed space. Then we say that X is strictly convex if the equality

$$p(a, x) + p(a, y) = p(a, x + y), \quad a, x, y \in X,$$

implies that $x = \lambda y$ for some $\lambda > 0$.

Let $(X, p(\cdot,\cdot))$ and $(Y, q(\cdot,\cdot))$ be linear 2-seminormed spaces. A map $T : X \longrightarrow Y$ is called 2-*isometry* if

$$p(x - z, y - z) = q(T(x) - T(z), T(y) - T(z)),$$

for all $x, y, z \in X$. Moreover, T is called affine if

$$T(\lambda x + (1 - \lambda) y) = \lambda T(x) + (1 - \lambda) T(y),$$

for all $\lambda \in [0, 1]$ and every $x, y \in X$.

Lemma 2.1. *Let $(X, p(\cdot,\cdot))$ be linear 2-seminormed space. Then for all $x, y \in X$ and $\lambda \in \mathbb{R}$, we have $p(x, y) = p(x, y + \lambda x)$.*

Proof. Since $p(x,\lambda x)=0$, we have $p(x,y+\lambda x)\le p(x,y)+p(x,\lambda x)=p(x,y)$. On the other hand, let $a=y+\lambda x$. Then $y=a-\lambda x$ and so

$$p(x,y)=p(x,a-\lambda x)\le p(x,a)+p(x,\lambda x)=p(x,y+\lambda x),$$

for all $x,y\in X$. □

Lemma 2.2. *Suppose that X is a strictly convex 2-seminormed space and $a,b\in X$. Then $u=\frac{1}{2}(a+b)$ is the unique element of X satisfying*

$$2p(a-c,a-u)=2p(b-u,b-c)=p(a-c,b-c), \tag{2.1}$$

for some $c\in X$ with $p(a-c,b-c)\neq 0$.

Proof. The result is clear if $a=b$. By Lemma 2.1, we have that for all $c\in X$,

$$\begin{aligned}2p(a-c,a-u)&=2p(a-c,a-\frac{1}{2}(a+b))\\&=p(a-c,a-b)\\&=p(a-c,a-b+(-1)(a-c))\\&=p(a-c,b-c).\end{aligned}$$

Similarly, $2p(b-u,b-c)=p(a-c,b-c)$. Therefore it suffices to prove the uniqueness of u. Suppose that $w\in X$ is such that

$$2p(a-c,a-w)=2p(b-w,b-c)=p(a-c,b-c), \tag{2.2}$$

for some $c\in X$ with $p(a-c,b-c)\neq 0$. By (2.1) and (2.2), we get

$$p(a-c,a-\frac{1}{2}(u+w))\le\frac{1}{2}p(a-c,a-u)+\frac{1}{2}p(a-c,a-w)=\frac{1}{2}p(a-c,b-c). \tag{2.3}$$

Similarly,

$$p(b-\frac{1}{2}(u+w),b-c)\le\frac{1}{2}p(b-u,b-c)+\frac{1}{2}p(b-w,b-c)=\frac{1}{2}p(a-c,b-c). \tag{2.4}$$

If either of these inequalities were strict, then by using Lemma 2.1, we obtain

$$\begin{aligned}p(a-c,b-c)&\le p(a-c,c-\frac{1}{2}(u+w))+p(c-\frac{1}{2}(u+w),b-c)\\&=p(a-c,a-\frac{1}{2}(u+w))+p(b-\frac{1}{2}(u+w),b-c)\\&<p(a-c,b-c),\end{aligned}$$

which is a contradiction. Thus, the equality holds in (2.3) and (2.4). Therefore

$$p(a-c,a-\frac{1}{2}(u+w))=\frac{1}{2}p(a-c,a-u)+\frac{1}{2}p(a-c,a-w).$$

Since X is strictly convex, we conclude that $a-u=\lambda(a-w)$ for some $\lambda>0$. On the other hand, by (2.1) and (2.2), $p(a-c,a-u)=p(a-c,a-w)$. Hence $\lambda=1$ and $u=w$. □

Now we prove our main theorem. The idea of the proof can be found in [1].

Theorem 2.1. *Let $(X, p(\cdot,\cdot))$ and $(Y, q(\cdot,\cdot))$ be two real 2-seminormed spaces, where Y is strictly convex. If $T : X \longrightarrow Y$ is a 2-isometry, then T is affine.*

Proof. We may assume without loss of generality that $T(0) = 0$. Indeed, if $T(0) \neq 0$, then $\phi(x) = T(x) - T(0)$ is an isometry and $\phi(0) = 0$. Since T is 2-isometry, from Lemma 2.1 we have

$$\begin{aligned} 2q(T(a) - T(c), T(a) - T(\frac{a+b}{2})) &= 2p(a - c, a - \frac{a+b}{2}) \\ &= p(a - c, b - c) \\ &= q(T(a) - T(c), T(b) - T(c)). \end{aligned}$$

Similarly,

$$\begin{aligned} 2q(T(b) - T(\frac{a+b}{2}), T(b) - T(c)) &= 2p(b - \frac{a+b}{2}, b - c) \\ &= p(a - c, b - c) \\ &= q(T(a) - T(c), T(b) - T(c)), \end{aligned}$$

for every $a, b, c \in X$. As Y is strictly convex, replacing in (2.1), a by $T(a)$, b by $T(b)$, c by $T(c)$ and u by $T(\frac{a+b}{2})$ and using the uniqueness of u, we get that

$$T(\frac{a+b}{2}) = \frac{1}{2}(T(a) + T(b)).$$

Thus, T preserves the midpoints of line segments, and hence T is affine by Lemma 2.2 of [12]. □

Corollary 2.1. *Let X and Y be two real linear 2-normed spaces, and suppose that Y is strictly convex. If $T : X \longrightarrow Y$ is a 2-isometry, then T is affine.*

3 Maps that preserve distance one

Let A and B be two linear spaces equipped the seminorms p and q, respectively. A map $T : A \longrightarrow B$ is called *isometry* if

$$p(x - y) = q(T(x) - T(y)),$$

for all $x, y \in A$.

Theorem 3.1. *Let A and B be two linear spaces equipped the seminorms p and q, respectively. Suppose that $dimB \geq 2$ and $T : A \longrightarrow B$ is a surjective mapping that strongly preserves distance one. Then T strongly preserves distance n.*

Proof. We first prove that $dimA \geq 2$ and T is one to one. Since $dimB \geq 2$, there exist elements $x, y, z \in B$ such that

$$q(x - y) = q(x - z) = q(y - z) = 1.$$

The mapping T is given to be surjective and strongly preserving distance one, so there exist elements $a, b, c \in A$ such that $T(a) = x$, $T(b) = y$, $T(c) = z$ and

$$p(a - b) = p(a - c) = p(b - c) = 1.$$

Therefore $dimA \geq 2$. Now let $a, b \in A$ such that $a \neq b$ and $T(a) = T(b)$. We can fined $c \in A$ such that $p(a - c) = 1$ and $p(b - c) \neq 1$. Hence

$$q(T(b) - T(c)) = q(T(a) - T(c)) = 1.$$

This implies that $p(b-c)=1$, which is not possible. Thus, T is one to one. Hence T is bijective and both T and T^{-1} preserve distance one.

Next we prove by induction that T strongly preserves distance n for all $n \in \mathbb{N}$. Suppose that T strongly preserves distance n and $a, b \in A$ with $p(a-b)=n+1$. Similar to the proof of [10, Theorem 1], we have that

$$|q\big(T(a)-T(b)\big)-p(a-b)|<1,$$

for all $a, b \in A$. Therefore $q(T(a)-T(b)) \leq n+1$. Define

$$u=T(a)+\frac{1}{k}(T(b)-T(a)),$$

where $k=q(T(b)-T(a))$. There exists $v \in A$ such that $u=T(v)$. From $q(u-T(a))=1$ we deduce $p(v-a)=1$. If $q(u-T(b))<n$, then we have $p(v-b)<n$. So

$$p(a-b) \leq p(a-v)+p(v-b)<n+1,$$

which is a contradiction. Therefore, $q(u-T(b)) \geq n$. This implies that

$$n \leq q(u-T(b))=q\big((T(a)-T(b))(1-\frac{1}{k})\big)=|1-\frac{1}{k}|\ k=|k-1|.$$

Thus, $q(T(a)-T(b))=k=n+1$. Similarly, T^{-1} strongly preserves distance $n+1$.

Conversely, let $a, b \in A$ such that $q(T(a)-T(b))=n+1$. Assume that $x=T(a)$ and $y=T(b)$. Then $q(x-y)=n+1$. Since T^{-1} strongly preserves distance $n+1$, hence

$$n+1=q(T^{-1}(x)-T^{-1}(y))=p(a-b).$$

□

Let A and B be two linear spaces equipped with seminorms p and q, respectively. We call $T: A \longrightarrow B$ a *Lipschitz mapping* if there is a $L \geq 0$ such that

$$q\big(T(a)-T(b)\big) \leq L\ p(a-b),$$

for all $a, b \in A$. In this case, the constant L is called the Lipschitz constant.

Theorem 3.2. *Let (A, p) and (B, q) be two real linear spaces with $dim B \geq 2$. Suppose that $T: A \longrightarrow B$ is a Lipschitz mapping with $L=1$ and let T be a surjective strongly preserves distance one. Then T is an isometry.*

Proof. By Theorem 3.1, T is one to one and strongly preserves distance n for all $n \in \mathbb{N}$. Let $a, b \in A$ and N be a positive integer satisfying $p(a-b)<N$. Let

$$q(T(a)-T(b))<p(a-b). \tag{3.1}$$

Take

$$c=a+\frac{N}{p(a-b)}(b-a).$$

Then $p(c-a)=N$ and since $p(a-b)<N$, we get

$$p(c-b)=p\big((a-b)(1-\frac{N}{p(b-a)})\big)=(\frac{N}{p(b-a)}-1)p(b-a)=N-p(b-a).$$

Therefore, we obtain

$$\begin{aligned} N = q(T(c) - T(a)) \leq & q(T(c) - T(b)) + q(T(b) - T(a)) \\ < & q(T(c) - T(b)) + p(a - b) \\ < & N - p(b - a) + p(a - b) \\ = & N, \end{aligned}$$

which is not possible. Hence the equality holds in (3.1), and T is an isometry. □

Combining Theorem 3.2 and Theorem 2.3 of [12], we get the following result.

Corollary 3.1. *Let (A, p) and (B, q) be two real linear spaces with $dimB \geq 2$. Suppose that $T : A \longrightarrow B$ is a Lipschitz mapping with $L = 1$ and let T be a surjective strongly preserves distance one. Then T is affine.*

Corollary 3.2. *Let (A, p) and (B, q) be two real linear spaces, where B is strictly convex and $dimB \geq 2$. Suppose that $T : A \longrightarrow B$ is a surjective mapping strongly preserving distance one. Then T is affine.*

Proof. By Theorem 3.1, T is one to one and strongly preserves distance n for all $n \in \mathbb{N}$. Let $a, b \in A$ and $p(a - b) = \frac{1}{n}$. We can find $c \in A$ such that $p(a - c) = p(b - c) = 1$. Let $u = c + n(b - c)$. Then

$$p(u - c) = n, \quad \textit{and} \quad p(b - u) = n - 1.$$

Therefore,

$$q(T(b) - T(c)) = 1, \qquad q(T(u) - T(c)) = n, \qquad q(T(b) - T(u)) = n - 1.$$

Since B is strictly convex, $T(u) - T(b) = (n - 1)\big(T(b) - T(c)\big)$ and so we have

$$T(b) = \frac{1}{n}T(u) + \frac{n-1}{n}T(c).$$

Similarly, if we take $v = c + n(a - c)$, then we obtain

$$T(a) = \frac{1}{n}T(v) + \frac{n-1}{n}T(c).$$

Using $p(u - v) = 1$, we get

$$q(T(a) - T(b)) = \frac{1}{n}q(T(u) - T(v)) = \frac{1}{n}.$$

Thus, T preserves distance $\frac{1}{n}$ for all $n \in \mathbb{N}$.

Now let $p(a - b) \leq \frac{m}{n}$, where $m \in \mathbb{N}$ with $m \geq 2$. As $dimA \geq 2$, we can find a finite sequence $x_i \in A$ for $i = 0, 1, ..., m$ with $x_0 = a$ and $x_m = b$ such that $p(x_i - x_{i+1}) = \frac{1}{n}$. Hence

$$q(T(a) - T(b)) \leq \sum_{i=0}^{m-1} q(T(x_i) - T(x_{i+1})) = \frac{m}{n}.$$

Thus, for all $a, b \in A$,

$$q(T(a) - T(b)) \leq p(a - b).$$

Consequently, T is a Lipschitz mapping with $L = 1$ and by Theorem 3.2, it is affine. □

Next we give an example of an isometry between Fréchet algebras which is not affine. This example shows that the strict convexity of Fréchet algebra B in Theorem 3.2 of [12] is essential.

Example 1. Let $I_k = [-k, k]$ for $k \in \mathbb{N}$, and consider the Fréchet algebra $B = C(\mathbb{R})$, the algebra of continuous functions on $\mathbb{R}$ with a compact open topology, and the sequence

$$p_k(f) = sup\{|f(x)| : \quad x \in I_k\},$$

of submultiplicative seminorms. It is easy to check that B is not strictly convex. Choose $f, g \in B$ such that $p_k(f) = p_k(g) = 1$, $p_k(f + g) = p_k(f) + p_k(g)$ and f, g is linearly independent. For all $\lambda \in \mathbb{R}$, define $T : \mathbb{R} \longrightarrow B$ by

$$T(\lambda) = \begin{cases} \lambda f & \lambda \leq 1 \\ f + (\lambda - 1)g & \lambda > 1. \end{cases}$$

Then by the same method as in [1] we conclude that T is an isometry, but is not affine.

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