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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
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The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

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TIME OPTIMAL CONTROL PROBLEM WITH INTEGRAL CONSTRAINT FOR THE HEAT TRANSFER PROCESS

Sh.A. Alimov, G.I. Ibragimov

Communicated by K.N. Ospanov

Key words: heat transfer process, control function, integral constraint, optimal control, optimal time.

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Abstract. In the present paper a mathematical model of thermocontrol processes is studied. Several convectors are installed on the disjoint subsets Γ_k of the wall $\partial\Omega$ of a volume Ω and each convector produces a hot or cold flow with magnitude equal to $\mu_k(t)$, which are control functions, and on the surface $\partial\Omega \setminus \Gamma$, $\Gamma = \cup\Gamma_k$, a heat exchange occurs by the Newton law. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. A necessary and sufficient condition is found for solvability of this problem. An equation for the optimal transfer time is found, and an optimal control function is constructed explicitly.

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1 Introduction

The control problem for evolution equations is a classical problem. The controllability in finite-dimensional linear systems can be described in terms of the rank of a matrix generated by the coefficient matrix and the matrix of the control action.

Controlled systems described by PDEs are typically infinite-dimensional. There are many works on controllability/observability of systems governed by PDEs. The works of Russell [16] and Lions [15] are classical in this area. However, compared with Kalman's classical theory, the theories on controllability of systems governed by PDEs are not very mature. Important researches in this area can be found in the works [6, 3, 20, 19]. For other related works in this direction, we refer to [7, 9, 18, 17, 2, 1].

The time-optimal control problem for PDFs of parabolic type was first concerned in [10]. More detailed information on the optimal control problems for the systems governed by PDEs is given in the monograph [11].

The decomposition method is widely used in studying control and differential game problems for the systems in distributed parameters. This method leads us to a control problem described by an infinite system of ordinary differential equations (see, for example, [12, 6, 8, 9, 18, 5]). The paper [4] is devoted to the control problem for an infinite system of differential equations.

In the present paper, we study a mathematical model of thermocontrol processes. Several convectors are installed on the disjoint subsets Γ_k of the wall $\partial\Omega$ of a volume Ω and each convector produces a hot or cold flow with magnitude equal to $\mu_k(t)$, which are the control functions, and on the surface $\partial\Omega \setminus \Gamma$, $\Gamma = \cup\Gamma_k$, a heat exchange occurs by the Newton law. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. We obtain a necessary and sufficient condition of solvability of the problem. We find an equation for the optimal transfer time, and construct an optimal control function explicitly.

2 Statement of problem

We study the following heat equation [2]

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - p(x)u(x,t), \quad p(x) \geq 0, t > 0, \tag{2.1}$$

with the boundary conditions

$$\frac{\partial u(x,t)}{\partial n} = \mu_k(t)a_k(x), \quad x \in \Gamma_k, \quad t > 0, \tag{2.2}$$

and

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) = 0, \quad x \in \partial\Omega \setminus \Gamma_k, \quad t > 0, \tag{2.3}$$

and the initial condition

$$u(x,0) = 0, \tag{2.4}$$

where Ω is a subset of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is piecewise smooth, Γ_k are disjoint subsets of $\partial\Omega$ which are convectors (heaters or coolers). It is assumed that the boundaries $\partial\Gamma_k$ of Γ_k are piecewise smooth, $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$. The functions $h(x)$ (the thermal conductivity of the walls), $a_k(x)$ (the power density of the k-th convector) and $p(x)$ are given, $h(x)$ and $a_k(x)$ are assumed to be given piecewise smooth non-negative non-trivial functions, $p(x)$ is a sufficiently smooth function in $\bar{\Omega} = \Omega \cup \partial\Omega$.

The meaning of boundary conditions (2.2) and (2.3) is that each convector produces a hot or cold flow with magnitude of output given by a measurable real-valued function $\mu_k(t)$, and on the surface $\partial\Omega \setminus \Gamma$ a heat exchange occurs by the Newton law.

Let

$$\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_m(t)), \quad \mu : [0,\infty) \to \mathbb{R}^m, \quad \mu(\cdot) \in L_2[0,\infty). \tag{2.5}$$

Definition 1. We call a function $\mu : [0,\infty) \to \mathbb{R}^m$ with measurable coordinates $\mu_i(t)$, $t \geq 0$, $i = 1, ..., m$, an admissible control if it satisfies the following integral constraint

$$\int_0^\infty |\mu(t)|^2 dt \leq \rho^2, \tag{2.6}$$

where ρ is a given positive number.

We extend the functions $h(x)$ and $a(x)$ to the whole boundary $\partial\Omega$ by setting $h(x) = 0$ for $x \in \Gamma$ and $a_k(x) = 0$ for $x \in \partial\Omega \setminus \Gamma_k$ [2].

Next, consider the following vector-functions

$$a(x) = (a_1(x), a_2(x), ..., a_m(x)), \quad a : \partial\Omega \to \mathbb{R}^m, \tag{2.7}$$

Using (2.5) and (2.7) we can combine conditions (2.2) and (2.3) as follows

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) = \mu(t) \cdot a(x), \quad x \in \partial\Omega, \quad t > 0, \tag{2.8}$$

We define a generalized solution of the initial-boundary value problem (2.1), (2.4), (2.8) as a function $u(x,t)$ that satisfies the equation

$$\int_0^t ds \int_\Omega [\nabla u(x,s)\nabla\eta(x,s) + p(x)u(x,s)\eta(x,s)]dx$$
$$- \int_0^t ds \int_\Omega [u(x,s)\frac{\partial \eta(x,s)}{\partial s}dx + \int_\Omega u(x,s)\eta(x,s)]dx$$
$$= \int_0^t ds \int_{\partial\Omega} [\mu(s)\cdot a(x)]\eta(x,s)d\sigma(x) - \int_0^t ds \int_{\partial\Omega} h(x)u(x,s)\eta(x,s)d\sigma(x) \tag{2.9}$$

for $0 < t \le T$, for any number $T > 0$ and any function $\eta(x,t) \in W_2^{1,1}(\Omega \times [0,T])$ (see formula (5.5) and Theorem 5.1 in [14], III.5).

Next, we define generalized solution of the eigenvalue problem for the Laplace operator [2]

$$-\Delta v(x) + p(x)v(x) = \lambda v(x), \quad x \in \Omega, \tag{2.10}$$

with the boundary condition

$$\frac{\partial v(x)}{\partial n} + h(x)v(x) = 0, \quad x \in \partial\Omega, \tag{2.11}$$

as a function $v(x)$ in the Sobolev space $W_2^1(\Omega)$ which satisfies the equation

$$\int_\Omega [\nabla v(x)\nabla\eta(x) + p(x)v(x)\eta(x)]dx + \int_{\partial\Omega} h(x)v(x)\eta(x)d\sigma(x) = \lambda \int_\Omega v(x)\eta(x)dx \tag{2.12}$$

for any function $\eta \in W_2^1(\Omega)$ (see [13], Sec. III.6, formula (6.3)).

We consider this problem in the Hilbert space $L_2(\Omega)$ with the inner product $(u,v) = \int_\Omega u(x)v(x)dx$ and norm $||u|| = \sqrt{(u,u)}$. It is well known that under the above assumptions there exists a sequence of positive eigenvalues $\{\lambda_i\}_{i=1}^\infty$ such that

$$0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_i \le ..., \quad \lambda_i \to \infty, \quad i \to \infty,$$

and the corresponding eigenfunctions $v_i(x)$ form an orthonormal basis $\{v_i\}_{i=1}^\infty$ in $L_2(\Omega)$ (see, for example, [13], Sec. III.6).

We will investigate the following problem: Let $u_0(x) \in L_2(\Omega)$. Find a time θ and an admissible control $\mu(t)$, $t \ge 0$, such that the solution $u(x,t)$ of the initial-boundary value problem (2.1), (2.4), (2.6), (2.8) exists, is unique and satisfies the following condition

$$(u(x,\theta), v_i(x)) = (u_0(x), v_i(x)), \quad i = 1, 2, ..., m. \tag{2.13}$$

3 Main result

3.1 Integral equation for $\mu(t)$

We use some properties of the Green function G [2] defined by the following equation:

$$G(x,y,t) = \sum_{i=1}^\infty e^{-\lambda_i t} v_i(x)v_i(y), \quad x, y \in \Omega \cup \partial\Omega, \quad t > 0. \tag{3.1}$$

Since $h(x) \geq 0$, $x \in \partial\Omega$, and $h(x)$ is not identically 0. Then

$$G(x, y, t) \geq 0, \quad (x, y) \in \bar{\Omega} \times \bar{\Omega}, \quad t > 0,$$

and the solution of boundary-value problem (2.1), (2.4), (2.8) can be represented by the Green function as follows

$$u(x, t) = \int_0^t ds \int_{\partial\Omega} G(x, y, t - s)\mu(s) \cdot a(y) d\sigma(y), \tag{3.2}$$

where $a(y)$, $y \in \partial\Omega$, and $\mu(s)$, $s \geq 0$, are defined by (2.5) and (2.7). Since $\mu(t) \cdot a(x) = \sum_{j=1}^m \mu_j(t) a_j(x)$, we obtain

$$u(x, t) = \sum_{j=1}^m \int_0^t \mu_j(s) ds \int_{\partial\Omega} G(x, y, t - s) a_j(y) d\sigma(y). \tag{3.3}$$

By the condition (2.13) we have

$$\int_\Omega u(x, \theta) v_i(x) dx = \int_\Omega u_0(x) v_i(x) dx \doteq c_i, \quad c_i \in \mathbb{R}, \quad i = 1, 2, ..., m. \tag{3.4}$$

To evaluate the integral in the left-hand side of (3.4), we substitute (3.3) into (3.4) to obtain

$$\int_\Omega u(x, \theta) v_i(x) dx = \int_\Omega v_i(x) dx \sum_{j=1}^m \int_0^\theta \mu_j(s) ds \int_{\partial\Omega} G(x, y, \theta - s) a_j(y) d\sigma(y) \tag{3.5}$$

By (3.1)

$$\int_\Omega G(x, y, t) v_i(x) dx = e^{-\lambda_i t} v_i(y), \quad y \in \Omega \cup \partial\Omega. \tag{3.6}$$

Then equation (3.4) takes the form

$$\sum_{j=1}^m \int_0^\theta e^{-\lambda_i(\theta - s)} \mu_j(s) ds \int_{\partial\Omega} v_i(y) a_j(y) d\sigma(y) = c_i, \quad t > 0. \tag{3.7}$$

Denote

$$\int_{\partial\Omega} v_i(y) a_j(y) d\sigma(y) = a_{ij}, \quad i, j = 1, 2, ..., m. \tag{3.8}$$

We obtain then the following equations

$$\sum_{j=1}^m \int_0^\theta e^{-\lambda_i(\theta - s)} a_{ij} \mu_j(s) ds = c_i, \quad i, j = 1, 2, ..., m, \tag{3.9}$$

which can be written as

$$\int_0^\theta A(\theta - s)\mu(s) ds = c, \quad c^T = (c_1, c_2, ..., c_m), \quad \mu^T(s) = (\mu_1(s), ..., \mu_m(s)), \tag{3.10}$$

(see (2.5)) where

$$A(\theta - s) = \begin{bmatrix} a_{11}e^{-\lambda_1(\theta-s)} & a_{12}e^{-\lambda_1(\theta-s)} & \dots & a_{1m}e^{-\lambda_1(\theta-s)} \\ a_{21}e^{-\lambda_2(\theta-s)} & a_{22}e^{-\lambda_2(\theta-s)} & \dots & a_{2m}e^{-\lambda_2(\theta-s)} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}e^{-\lambda_m(\theta-s)} & a_{m2}e^{-\lambda_m(\theta-s)} & \dots & a_{mm}e^{-\lambda_m(\theta-s)} \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1(\theta-s)}a_1^T \\ e^{-\lambda_2(\theta-s)}a_2^T \\ \vdots \\ e^{-\lambda_m(\theta-s)}a_m^T \end{bmatrix} \tag{3.11}$$

is a $m \times m$ matrix, where $a_i^T = (a_{i1}, a_{i2}, ..., a_{im})$, $i = 1, 2, ..., m$, are row vectors of the matrix

$$A_0 = A(0) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}.$$

3.2 Important subspaces

Next, we study the problem of finding an admissible control $\mu(t)$ that satisfies equation (3.9) for some time θ. To this end we consider the following operator $L : L_2[0, \infty) \to \mathbb{R}^m$ defined by the equation

$$L\mu = L(\theta)\mu = \int_0^\theta A(\theta - s)\mu(s)ds, \quad \mu(\cdot) \in L_2[0, \infty), \tag{3.12}$$

where $\mu(t)$ does not need to satisfy (2.6), and the Gram matrix

$$W = W(\theta) = \int_0^\theta A(\theta - s)A^T(\theta - s)ds \tag{3.13}$$

where A^T is the transpose of A. Clearly, by (3.11)

$$A^T(\theta - s) = \left[e^{-\lambda_1(\theta-s)}a_1,\ e^{-\lambda_2(\theta-s)}a_2,\ \dots,\ e^{-\lambda_m(\theta-s)}a_m\right].$$

We have

$$A(\theta - s)A^T(\theta - s) = \begin{bmatrix} e^{-\lambda_1(\theta-s)}a_1^T \\ e^{-\lambda_2(\theta-s)}a_2^T \\ \vdots \\ e^{-\lambda_m(\theta-s)}a_m^T \end{bmatrix} \left[\ e^{-\lambda_1(\theta-s)}a_1,\quad e^{-\lambda_2(\theta-s)}a_2,\quad \dots,\quad e^{-\lambda_m(\theta-s)}a_m\ \right]$$

$$= \begin{bmatrix} e^{-2\lambda_1(\theta-s)}a_1^Ta_1 & e^{-(\lambda_1+\lambda_2)(\theta-s)}a_1^Ta_2 & \dots & e^{-(\lambda_1+\lambda_m)(\theta-s)}a_1^Ta_m \\ e^{-(\lambda_2+\lambda_1)(\theta-s)}a_2^Ta_1 & e^{-2\lambda_2(\theta-s)}a_2^Ta_2 & \dots & e^{-(\lambda_2+\lambda_m)(\theta-s)}a_2^Ta_m \\ \dots & \dots & \dots & \dots \\ e^{-(\lambda_m+\lambda_1)(\theta-s)}a_m^Ta_1 & e^{-(\lambda_m+\lambda_2)(\theta-s)}a_m^Ta_2 & \dots & e^{-2\lambda_m(\theta-s)}a_m^Ta_m \end{bmatrix}, \tag{3.14}$$

$$W = W(\theta) = \int_0^\theta A(\theta-s)A^T(\theta-s)ds$$

$$= \begin{bmatrix} a_1^Ta_1\int_0^\theta e^{-2\lambda_1(\theta-s)}ds & a_1^Ta_2\int_0^\theta e^{-(\lambda_1+\lambda_2)(\theta-s)}ds & \dots & a_1^Ta_m\int_0^\theta e^{-(\lambda_1+\lambda_m)(\theta-s)}ds \\ a_2^Ta_1\int_0^\theta e^{-(\lambda_2+\lambda_1)(\theta-s)}ds & a_2^Ta_2\int_0^\theta e^{-2\lambda_2(\theta-s)}ds & \dots & a_2^Ta_m\int_0^\theta e^{-(\lambda_2+\lambda_m)(\theta-s)}ds \\ \dots & \dots & \dots & \dots \\ a_m^Ta_1\int_0^\theta e^{-(\lambda_m+\lambda_1)(\theta-s)}ds & a_m^Ta_2\int_0^\theta e^{-(\lambda_m+\lambda_2)(\theta-s)}ds & \dots & a_m^Ta_m\int_0^\theta e^{-2\lambda_m(\theta-s)}ds \end{bmatrix} \tag{3.15}$$

Thus, $W(\theta)$ is an $m \times m$ symmetric matrix.

Denote by

$$R(L) = \left\{ x \mid x = \int_0^\theta A(\theta-s)\mu(s)ds, \quad \mu(\cdot) \in L_2[0,\infty) \right\} \tag{3.16}$$

the range of the operator L, and by

$$R(W(\theta)) = \{x \mid x = W(\theta)\eta, \quad \eta^T = (\eta_1, \eta_2, ..., \eta_m) \in \mathbb{R}^m\} \tag{3.17}$$

the range of the Gram matrix $W(\theta)$. Since the matrix $W(\theta)$ is symmetric, therefore $R(W(\theta))$ is a row space as well as a column space of the matrix $W(\theta)$. Note that $R(L)$ and $R(W(\theta))$ are subspaces of $\mathbb{R}^m$. We prove the following statement.

Lemma 3.1. $R(L) = R(W(\theta))$ *for any* $\theta > 0$.

Proof. 1) First, show that $R(W(\theta)) \subset R(L)$ where $\theta > 0$ is any fixed number. Let $x \in R(W)$. Then $x = W\eta$ for some $\eta \in \mathbb{R}^m$. Choose the control

$$\mu(t) = A^T(\theta - t)\eta, \quad 0 \le t \le \theta. \tag{3.18}$$

For this control,

$$L(\theta)\mu = \int_0^\theta A(\theta-s)\mu(s)ds = \int_0^\theta A(\theta-s)A^T(\theta-s)\eta ds = W(\theta)\eta = x. \tag{3.19}$$

Thus, $x \in R(L)$.

2) We show now that $R(L) \subset R(W)$. Let $x \in R(L)$. Then there exists $\mu(\cdot) \in L_2[0,\infty)$ such that

$$x = \int_0^\theta A(\theta-s)\mu(s)ds. \tag{3.20}$$

Assume the contrary, $x \notin R(W)$. Then by the fact that the subspace $ker(W) = \{x \in \mathbb{R}^m \mid Wx = 0\}$ is orthogonal to the row space of W and, hence, to $R(W)$, the vector x can be represented as follows

$$x = x_1 + x_2, \quad x_1 \in R(W), \quad x_2 \in ker(W), \quad x_2 \neq 0. \tag{3.21}$$

Note that $x_2 \neq 0$ since otherwise $x = x_1 \in R(W)$ which contradicts our assumption.

Since $Wx_2 = 0$, we have $x_2^T W x_2 = 0$, hence,

$$x_2^T W x_2 = \int_0^\theta |x_2^T A(\theta - s)|^2 ds = 0, \tag{3.22}$$

and so

$$x_2^T A(\theta - s) = 0, \quad 0 \le s \le \theta.$$

Consequently,

$$x_2^T \cdot x = x_2^T \int_0^\theta A(\theta - s)\mu(s) ds = \int_0^\theta x_2^T A(\theta - s)\mu(s) ds = 0.$$

However,

$$x_2^T \cdot x = x_2^T \cdot x_1 + x_2^T \cdot x_2 = |x_2|^2 \neq 0.$$

Contradiction. The proof of the lemma is complete. □

Corollary 3.1. *Equation* (3.10) *is satisfied for some* θ *and* $\mu(t)$, $0 \le t \le \theta$, *if and only if* $c \in R(W)$.

Next, we denote $B = [A_0, \Lambda A_0, \Lambda^2 A_0, ..., \Lambda^{m-1} A_0]$ which is an $m \times m^2$ matrix, where

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda_m^k \end{bmatrix}, \quad k = 1, 2, ..., m-1. \tag{3.23}$$

The following lemma shows that the subspace $R(W(\theta))$ does not depend on θ.

Lemma 3.2. $R(B) = R(W(\theta))$ *for any* $\theta > 0$.

Proof. 1) Show that $R(W(\theta)) \subset R(B)$. Indeed, let $x \in R(W(\theta))$ for some θ. By (3.11) we have

$$\begin{aligned} A(\theta - s) &= \begin{bmatrix} a_1^T \left(1 - \frac{\lambda_1(\theta-s)}{1!} + \frac{\lambda_1^2(\theta-s)^2}{2!} - \frac{\lambda_1^3(\theta-s)^3}{3!} + ...\right) \\ a_2^T \left(1 - \frac{\lambda_2(\theta-s)}{1!} + \frac{\lambda_2^2(\theta-s)^2}{2!} - \frac{\lambda_2^3(\theta-s)^3}{3!} + ...\right) \\ \vdots \\ a_m^T \left(1 - \frac{\lambda_m(\theta-s)}{1!} + \frac{\lambda_m^2(\theta-s)^2}{2!} - \frac{\lambda_m^3(\theta-s)^3}{3!} + ...\right) \end{bmatrix} \\ &= \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} - \frac{(\theta - s)}{1!} \begin{bmatrix} \lambda_1 a_1^T \\ \lambda_2 a_2^T \\ \vdots \\ \lambda_m a_m^T \end{bmatrix} + \frac{(\theta - s)^2}{2!} \begin{bmatrix} \lambda_1^2 a_1^T \\ \lambda_2^2 a_2^T \\ \vdots \\ \lambda_m^2 a_m^T \end{bmatrix} - \frac{(\theta - s)^3}{3!} \begin{bmatrix} \lambda_1^3 a_1^T \\ \lambda_2^3 a_2^T \\ \vdots \\ \lambda_m^3 a_m^T \end{bmatrix} + ... \\ &= A_0 - \frac{(\theta - s)}{1!}\Lambda^1 A_0 + \frac{(\theta - s)^2}{2!}\Lambda^2 A_0 - \frac{(\theta - s)^3}{3!}\Lambda^3 A_0 + ... \end{aligned} \tag{3.24}$$

By the Cayley-Hamilton theorem every square matrix satisfies its characteristic equation, therefore Λ^k for $k \ge m$, can be represented as a linear combination of the matrices I, Λ, Λ^2,..., Λ^{m-1}. Using this fact and (3.24) we obtain

$$A(\theta - s) = \sum_{k=0}^{m-1} \beta_k(\theta - s)\Lambda^k A_0$$

for some scalar functions $\beta_k(\theta-s)$. By Lemma 3.1 $R(W(\theta)) = R(L)$, therefore $x \in R(L)$, and, hence, there exists $\mu(t)$, $0 \le t \le \theta$, such that

$$x = \int_0^\theta A(\theta-s)\mu(s)ds = \sum_{k=0}^{m-1} \Lambda^k A_0 \int_0^\theta \beta_k(\theta-s)\mu(s)ds$$
$$= \sum_{k=0}^{m-1} \Lambda^k A_0 \eta_k = B\eta \in R(B),$$

where $\eta^T = [\eta_0^T, \eta_1^T, ..., \eta_{m-1}^T] \in \mathbb{R}^{m^2}$,

$$\eta_k = \int_0^\theta \beta_k(\theta-s)\mu(s)ds \in \mathbb{R}^m, \quad k = 0, 1, 2, ..., m-1.$$

Thus, $x \in R(B)$.

2) Show that $R(B) \subset R(W(\theta))$. Let $x \in R(B)$. Then $x = B\eta$ for some $\eta \in \mathbb{R}^{m^2}$. We show that $x \in R(W(\theta))$. Assume the contrary, $x \notin R(W(\theta))$ for some $\theta > 0$. Then $x = x_1 + x_2$ with $x_1 \in R(W(\theta))$, $x_2 \in ker(W(\theta))$, where $x_2 \neq 0$ since otherwise $x = x_1 \in R(W(\theta))$. Note that

$$x^T \cdot x_2 = x_1^T x_2 + x_2^T \cdot x_2 = |x_2|^2 \neq 0.$$

From the inclusion $x_2 \in ker(R(W(\theta)))$ we obtain $W(\theta)x_2 = 0$ and so

$$x_2^T W(\theta) x_2 = \int_0^\theta |x_2^T A(\theta-s)|^2 ds = 0.$$

This implies that $x_2^T A(\theta-s) = 0$, $0 \le s \le \theta$. Differentiating this equation k times for $k = 0, 1, ..., m-1$ and letting $t = \theta$ we obtain

$$x_2^T A_0 = 0, \quad x_2^T \Lambda A_0 = 0, \quad ..., \quad x_2^T \Lambda^{m-1} A_0 = 0.$$

Thus, $x_2^T B = 0$. Then, $x_2^T x = x_2^T B\eta = 0$. This contradicts the condition $x_2^T x \neq 0$. Therefore, $x \in R(W(\theta))$. □

It should be noted that Lemma 3.2 shows that $R(W(\theta))$, the subspace of $\mathbb{R}^m$, does not depend on θ. Also, this lemma implies that $rank(W(\theta)) = rank(B)$.

Lemma 3.3. *If $rank(B) = m$, then for any $\theta > 0$, the matrix $W(\theta)$ is positive definite.*

Proof. For any $x \in \mathbb{R}^m$, $x \neq 0$, we have

$$x^T W(\theta) x = \int_0^\theta x^T A(\theta-s) A^T(\theta-s) x ds = \int_0^\theta |A^T(\theta-s)x|^2 ds \ge 0,$$

and if we assume that $x^T W(\theta) x = 0$ for some $x \neq 0$, then

$$\int_0^\theta |x^T A(\theta-s)|^2 ds = 0,$$

and so $x^T A(\theta - s)x = 0$ for all $0 \le s \le \theta$. Taking derivatives of this equation with respect to s and evaluating at $s = \theta$ we have

$$x^T \Lambda^k A_0 = 0, \quad k = 0, 1, 2, ...$$

This implies that $x^T B = 0$, and so $rank(B) < m$. Contradiction, since $rank(B) = rank(W(\theta)) = m$. Thus, $x^T W(\theta)x$ cannot equal to 0 for any $x \neq 0$. Hence, $W(\theta)$ is positive definite. □

Further, we assume that $rank(B) = m$. Then $det(W(t)) \neq 0$ for any $t > 0$ and the equation $W(\theta)x = c$ has the unique solution $x = W^{-1}(\theta)c$.

Lemma 3.4. *For any $c \in \mathbb{R}^m$, $c \neq 0$, the function $g(t) = c^T W^{-1}(t)c$, $t > 0$, is non-increasing and $\lim\limits_{t \to +0} g(t) = +\infty$.*

Proof. We show first that $g(t)$, $t > 0$, is non-increasing. Since $det(W(t)) \neq 0$, differentiating the equation $W^{-1}(t)W(t) = I$, where I is the $m \times m$ identity matrix, we obtain

$$\frac{d}{dt}\left(W^{-1}(t)\right) W(t) + W^{-1}(t)\frac{d}{dt}\left(W(t)\right) = 0.$$

Hence, the derivative of the inverse matrix is

$$\frac{d}{dt}\left(W^{-1}(t)\right) = -W^{-1}(t)\frac{d}{dt}\left(W(t)\right) W^{-1}(t).$$

Then,

$$\frac{d}{dt}g(t) = c^T \frac{d}{dt}\left(W^{-1}(t)\right) c = -c^T W^{-1}(t)\frac{d}{dt}\left(W(t)\right) W^{-1}(t)c.$$

It is not difficult to verify that

$$\begin{aligned} \frac{d}{dt}\left(W(t)\right) &= \begin{bmatrix} a_1^T a_1 e^{-2\lambda_1 t} & a_1^T a_2 e^{-(\lambda_1+\lambda_2)t} & \dots & a_1^T a_m e^{-(\lambda_1+\lambda_m)t} \\ a_2^T a_1 e^{-(\lambda_2+\lambda_1)t} & a_2^T a_2 e^{-2\lambda_2 t} & \dots & a_2^T a_m e^{-(\lambda_2+\lambda_m)t} \\ \vdots & \vdots & \vdots & \vdots \\ a_m^T a_1 e^{-(\lambda_m+\lambda_1)t} & a_m^T a_2 e^{-(\lambda_m+\lambda_2)t} & \dots & a_m^T a_m e^{-(\lambda_m+\lambda_m)t} \end{bmatrix} \\ &= A(t)A^T(t), \quad A^T(t) = \left[e^{-\lambda_1 t}a_1,\ e^{-\lambda_2 t}a_2,\ \dots,\ e^{-\lambda_m t}a_m\right], \end{aligned} \tag{3.25}$$

and so

$$\frac{d}{dt}g(t) = -c^T W^{-1}(t)A(t)A^T(t)W^{-1}(t)c = -|A^T(t)W^{-1}(t)c|^2 \le 0.$$

Thus, $g(t)$ is non-increasing.

Next, we show that $\lim\limits_{t \to +0} g(t) = +\infty$. Since $W(t)$ is symmetric, there exists an orthogonal matrix $Q(t)$ (by definition $Q^{-1}(t) = Q^T(t)$) such that

$$W(t) = Q(t)D(t)Q^T(t), \quad D(t) = \begin{bmatrix} \nu_1(t) & 0 & 0 & \dots & 0 \\ 0 & \nu_2(t) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \nu_m(t) \end{bmatrix},$$

where $\nu_1(t), ..., \nu_m(t)$ are eigenvalues of the matrix $W(t)$. Since by Lemma 3.3 the matrix $W(t)$ is positive definite, therefore the eigenvalues of $W(t)$ are positive, that is, $\nu_i(t) > 0$ for all $i = 1, ..., m$.

Recall, $Q(t)$ as an orthogonal matrix has the following properties $Q(t)Q^T(t) = Q^T(t)Q(t) = I$, and for any $x \in \mathbb{R}^m$, $|Q(t)x| = |Q^T(t)x| = |x|$. Then,

$$W^{-1}(t) = Q(t)D^{-1}(t)Q^T(t), \quad D^{-1}(t) = \begin{bmatrix} \frac{1}{\nu_1(t)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\nu_2(t)} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{\nu_m(t)} \end{bmatrix},$$

Letting $\xi(t) = Q^T(t)c$ we have $|\xi(t)| = |Q^T(t)c| = |c|$ and

$$\begin{aligned} c^T W(t)c &= c^T Q(t)D(t)Q^T(t)c = \xi^T(t)D(t)\xi(t) \\ &= \nu_1(t)\xi_1^2(t) + ... + \nu_m(t)\xi_m^2(t), \end{aligned} \tag{3.26}$$

where $\xi(t) = (\xi_1(t), ..., \xi_m(t))$, $|\xi(t)|^2 = \xi_1^2(t) + ... + \xi_m^2(t) = |c|^2$.

$$\begin{aligned} g(t) &= c^T W^{-1}(t)c = c^T Q(t)D^{-1}(t)Q^T(t)c \\ &= \xi^T(t)D^{-1}(t)\xi(t) = \frac{\xi_1^2(t)}{\nu_1(t)} + ... + \frac{\xi_m^2(t)}{\nu_m(t)}. \end{aligned} \tag{3.27}$$

For the entries $w_{ij}(t)$, $i, j \in I$, of the matrix $W(t)$ we have

$$w_{ij}(t) = a_i^T a_j \int_0^t e^{-(\lambda_i+\lambda_j)(t-s)} ds \to 0 \text{ as } t \to +0. \tag{3.28}$$

Therefore, $c^T W(t)c \to 0$ as $t \to +0$, and hence by (3.26)

$$\nu_1(t)\xi_1^2(t) + ... + \nu_m(t)\xi_m^2(t) \to 0$$

as $t \to +0$. Consequently, $\nu_i(t)\xi_i^2(t) \to 0$ for all $i = 1, 2, ..., m$ as $t \to +0$.

Since the sphere $\xi_1^2(t) + ... + \xi_m^2(t) = |c|^2 \neq 0$ is a compact set, the sequence $\xi(t_n)$, $n = 1, 2, ...$, where $t_n \to 0$ as $n \to \infty$, contains a convergent subsequence. Without restriction of generality, we assume that the sequence $\xi(t_n)$, $n = 1, 2, ...$, is convergent and

$$\xi(t_n) \to \xi_0 = (\xi_{10}, ..., \xi_{m0}), \quad |\xi_0| = |c| \text{ as } n \to \infty.$$

Let $\xi_{s0} \neq 0$ for some $1 \le s \le m$. We obtain then from $\nu_s(t_n)\xi_s^2(t_n) \to 0$ and $\xi_s(t_n) \to \xi_{s0} \neq 0$ that $\nu_s(t_n) \to 0$ as $n \to \infty$. Next, letting $t = t_n$ in (3.27) and passing to the limit as $n \to \infty$ we obtain

$$g(t_n) = \frac{\xi_1^2(t_n)}{\nu_1(t_n)} + ... + \frac{\xi_s^2(t_n)}{\nu_s(t_n)} + ... + \frac{\xi_m^2(t_n)}{\nu_m(t_n)} \to +\infty$$

since $\xi_s(t_n) \to \xi_{s0} \neq 0$ and $0 < \nu_s(t_n) \to 0$. Hence, $g(t) \to +\infty$ as $t \to +0$. □

Lemma 3.4 implies that $\inf_{t>0} g(t) = \lim_{t\to\infty} g(t)$. Let

$$\rho_0 \doteq \left(\lim_{t\to\infty} g(t)\right)^{1/2} = \left(\lim_{t\to\infty} c^T W^{-1}(t)c\right)^{1/2}.$$

3.3 Necessary and sufficient condition for solvability of Problem 1

Now, we turn to Problem 1, or equivalently, to the problem of finding a control $\mu(t)$ and time θ such that

$$\int_0^\theta A(\theta - s)\mu(s)ds = c, \quad \int_0^\theta |\mu(s)|^2 ds \le \rho^2. \tag{3.29}$$

Theorem 3.1. *Let* $rank(B) = m$. *(i) If* $\rho > \rho_0$, *then Problem 1 is solvable; (ii) if Problem 1 is solvable, then* $\rho \ge \rho_0$.

Proof. (i) Let $\rho > \rho_0$. Since the function $g(t) = c^T W^{-1}(t)c$ is continuous and decreases on $t \in (0, \infty)$ from $+\infty$ to ρ_0^2, therefore, there exists a time $\theta > 0$ such that $c^T W^{-1}(\theta)c = \rho^2$. Assume that θ is the first time that satisfies this equation. Set

$$\mu(t) = A^T(\theta - t)\eta, \quad 0 \le t \le \theta, \quad \eta = W^{-1}(\theta)c.$$

Then $\mu(t)$ is admissible since $W^{-1}(\theta)$ is symmetric and

$$\int_0^\theta |\mu(t)|^2 dt = \int_0^\theta |A^T(\theta - s)\eta|^2 ds = \eta^T W(\theta)\eta = c^T W^{-1}(\theta)c = \rho^2. \tag{3.30}$$

Also, we have

$$\int_0^\theta A(\theta - s)\mu(s)ds \quad = \quad \int_0^\theta A(\theta - s)A^T(\theta - s)\eta ds = W(\theta)\eta = c. \tag{3.31}$$

Hence, Problem 1 is solvable.

We turn to the part (ii) of the theorem. Let Problem 1 be solvable. Then there exists a time τ and a control $\mu(t)$, $0 \le t \le \tau$, such that

$$\int_0^\tau A(\tau - s)\mu(s)ds = c, \quad \int_0^\tau |\mu(s)|^2 ds \le \rho^2. \tag{3.32}$$

We show that $\rho \ge \rho_0$. Clearly, for the control

$$\mu_0(t) = A^T(\tau - t)\eta_0, \quad 0 \le t \le \tau, \quad \eta_0 = W^{-1}(\tau)c,$$

we have

$$\int_0^\tau A(\tau - s)\mu_0(s)ds = c, \quad \int_0^\tau |\mu_0(s)|^2 ds = \eta_0^T W(\tau)\eta_0. \tag{3.33}$$

If we show the inequality

$$\int_0^\tau |\mu(s)|^2 ds \ge \eta_0^T W(\tau)\eta_0, \tag{3.34}$$

then in view of (3.32) and (3.33) we obtain the inequality $\int_0^\tau |\mu_0(s)|^2 ds \le \rho^2$.

To show (3.34), we multiply by η_0 the both sides of equation in (3.32) to obtain

$$\int_0^\tau \eta_0^T A(\tau - s)\mu(s)ds = \eta_0^T c = \eta_0^T W(\tau)\eta_0.$$

Using the Cauchy-Schwartz inequality in the left-hand side yields

$$\begin{aligned} \eta_0^T W(\tau)\eta_0 &= \int_0^\tau \eta_0^T A(\tau - s)\mu(s)ds \\ &\le \left(\int_0^\tau |\eta_0^T A(\tau - s)|^2 ds\right)^{1/2} \left(\int_0^\tau |\mu(s)|^2 ds\right)^{1/2} \\ &\le \left(\eta_0^T W \eta_0\right)^{1/2} \left(\int_0^\tau |\mu(s)|^2 ds\right)^{1/2}. \end{aligned} \tag{3.35}$$

This implies (3.34). Hence,

$$\eta_0^T W(\tau)\eta_0 = \int_0^\tau |\mu_0(s)|^2 ds \le \int_0^\tau |\mu(s)|^2 ds \le \rho^2.$$

Consequently, we have

$$\rho_0^2 = \inf_{t>0} \eta_0^T W(t)\eta_0 \le c^T W^{-1}(\tau)c = \eta_0^T W(\tau)\eta_0 \le \rho^2,$$

which is the desired result. □

3.4 Optimal transfer time and optimal control

Let $\rho > \rho_0$. As denoted above that $t = \theta$ is the minimum root of the equation

$$c^T W^{-1}(t)c = \rho^2. \tag{3.36}$$

Theorem 3.2. *The number θ, the root of equation (3.36), is the optimal transfer time of the state $u(x,t)$ from the state $u(x,0) = 0$ to the state for which $(u(x,\theta), v_i(x)) = (u_0(x), v_i(x))$, $i = 1, ..., m$.*

Proof. We show that the control

$$\mu(t) = A^T(\theta - t)\eta, \quad 0 \le t \le \theta, \quad \eta = W^{-1}(\theta)c,$$

which satisfies equation (3.10), is optimal. Assume the contrary, let for some control $\bar{\mu}(t)$, $0 \le t \le \theta_0$, $\theta_0 < \theta$,

$$\int_0^{\theta_0} A(t - s)\bar{\mu}(s)ds = c, \quad \int_0^{\theta_0} |\bar{\mu}(s)|^2 ds \le \rho^2. \tag{3.37}$$

Then, it is not difficult to verify that the control

$$\mu_0(t) = A^T(\theta_0 - t)\eta_0, \quad 0 \le t \le \theta_0, \quad \eta_0 = W^{-1}(\theta_0)c,$$

satisfies the relations

$$\int_0^{\theta_0} A(t-s)\mu_0(s)ds = c, \quad \int_0^{\theta_0} |\mu_0(s)|^2 ds \le \rho^2. \tag{3.38}$$

Thus,

$$\rho^2 \ge \int_0^{\theta_0} |\mu_0(s)|^2 ds = \eta_0^T W(\theta_0)\eta_0 = c^T W^{-1}(\theta_0)c \ge c^T W^{-1}(\theta)c = \rho^2.$$

Hence,

$$c^T W^{-1}(\theta_0)c = \rho^2,$$

which contradicts the fact that θ is the smallest root of equation (3.36). Thus, θ is the optimal transfer time. □

4 Conclusions

In the present paper, we have studied a mathematical model of thermocontrol processes. The control functions $\mu_k(t)$ are subjected to an integral constraint. The problem is to find control functions to transfer the state of the process to a given state. We have found a necessary and sufficient condition for existence of a control function which transfers the state of the system to a given state. Also, we have found an equation for the optimal transfer time, and constructed an optimal control function that transfers the state of the system to a given state.

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Shavkat Arifjanovich Alimov
Department of Mathematics
National University of Uzbekistan
100174 Tashkent, Uzbekistan
Email: alimovsh@gmail.com

Gafurjan Ismailovich Ibragimov
Institute of Mathematics,
Uzbekistan Academy of Science
Student town
100174 Tashkent, Uzbekistan
and
Tashkent State University of Economics,
49 Islom Karimov St,
100066, Tashkent, Uzbekistan
Email: gofurjon.ibragimov@tsue.uz

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