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KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ON THE HADAMARD AND MARCHAUD-HADAMARD-TYPES MIXED FRACTIONAL INTEGRO-DIFFERENTIATION

M.U. Yakhshiboev

Communicated by V.I. Burenkov

Key words: Hadamard-type fractional integration, mixed Lebesgue spaces, dilation operator, the Hadamard and Marchaud-Hadamard-type fractional derivatives.

AMS Mathematics Subject Classification: 26A33, 41A35, 46E30.

Abstract. The paper is devoted to the integral representations of the Marchaud-Hadamard and Marchaud-Hadamard-type truncated mixed fractional derivatives in weighted mixed Lebesgue spaces. Inversion theorems and characterization of tne Hadamard and Hadamard-type mixed fractional integrals of functions in weighted mixed Lebesgue spaces are proven.

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1 Introduction

It is known that the Riemann-Liouville fractional integro-differentiation is formally a fractional power of $\left(\frac{d}{dx}\right)^{\alpha}$ and is invariant relative to translation [19, 20]. J. Hadamard [10] suggested a construction of fractional integro-differentiation, which is a fractional power of the type $\left(x\frac{d}{dx}\right)^{\alpha}$. This construction is well suited to the case of the half-axis and is invariant relative to dilation.

We consider the Hadamard and Hadamard-type fractional integro-differentiation of functions of several variables in mixed Lebesgue spaces. Lebesgue spaces with a mixed norm were introduced and studied in [2]. The boundedness of operators on mixed norm spaces was studied in [1, 3, 17, 23]. A number of properties of mixed Lebesgue spaces can be found in [5]. Since the function spaces with mixed norm have finer structures than the corresponding classical function spaces, they naturally arise in studies of solutions of partial differential equations used to model physical processes involving spatial and time variables, such as thermal or wave equations [9, 11, 16].

The one-dimensional Hadamard and Hadamard-type fractional integro-differentiation has been studied by many researchers [6-8], [12-15], [21-22], [25]. A number of properties of the Hadamard fractional integration can be found in [20, 19]. In this paper, we extended the operation of the Hadamard and Hadamard-type fractional integro-differentiation to the case of multivariable functions, when these operators, applied to each variable or to some of them, give the so-called partial and mixed fractional integrals and derivatives in the framework of spaces $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ with a mixed norm.

Partial and mixed Marchaud fractional derivatives in the case of two variables were considered in [20]. In [13], [14], the conditions were obtained for the existence of unique solutions to problems of Cauchy type for nonlinear differential equations with fractional Hadamard and Marchaud-Hadamard-type derivatives in spaces of summable functions and for the solutions in a closed form of Cauchy type problems for linear differential equations of fractional order.

In [4], the properties of some integro-differential operators that generalize the fractional differentiation operators in the Hadamard and Hadamard-Marchaud sense in the class of harmonic functions were considered. As an application of the obtained properties, the solvability of nonlocal problems for the Laplace equation in a ball was studied.

In this paper, we obtain integral representations for the Marchaud-Hadamard and Marchaud-Hadamard-type of truncated fractional derivatives. In addition, the inversion theorems and characterization of ordinary Hadamard-type fractional integrals of functions from $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ are proven.

The consideration is conducted in the framework of spaces with a mixed norm

$$\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\left(\mathbb{R}^n_+,\ \frac{dx}{x}\right)=$$

$$=\left\{f:\left\|f;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|=\left\{\int\limits_0^\infty[\ldots(\int\limits_0^\infty|f(x)|^{p_1}x_1^{-\gamma_1}\frac{dx_1}{x_1})^{\frac{p_2}{p_1}}\ldots]^{\frac{p_n}{p_{n-1}}}x_n^{-\gamma_n}\frac{dx_n}{x_n}\right\}^{\frac{1}{p_n}}<\infty\right\},$$

$$C_{\overline{\gamma}}\left(\mathbb{R}^n_+\right)=\left\{f:\|f;\ C_{\overline{\gamma}}\|=\sup_{x\in\mathbb{R}^n_+}\left|x^{-\overline{\gamma}}f(x)\right|<\infty,\ \lim_{|x|\to 0}x^{-\overline{\gamma}}f(x)=\lim_{|x|\to\infty}x^{-\overline{\gamma}}f(x)\right\},$$

where $\gamma_i\geq 0$, $i=\overline{1,n}$. Norm in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ is determined by the formula

$$\|f\|_{\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}}=\left\|f;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|=\left\|x^{-\overline{\gamma}*}f;\ \mathfrak{L}^{\overline{p}}\right\|,1\leq\overline{p}\leq\infty, \tag{1.1}$$

where $x^{-\overline{\gamma}^*}=x_1^{-\gamma_1^*}\cdot\ldots\cdot x_n^{-\gamma_n^*}$,

$$\gamma_i^*=\left\{\begin{array}{l}\frac{\gamma_i}{p_i},\quad 1\leq p_i<\infty,\\ \gamma_i,\ p_i=\infty,\ i=\overline{1,n}.\end{array}\right. \tag{1.2}$$

The paper has the following structure. In Sections 2, 3, 4, we give definitions and various auxiliary features of multiple integro-differentiation of Hadamard and Hadamard-type for multivariable functions (in terms of tensor products), and the auxiliary lemmas for spaces $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ are given in Section 5. Sections 6, 7, 8, 9 contain the proofs of basic results: the boundedness of the fractional integration of Hadamard and Hadamard type in spaces with mixed norms is proven in Section 6; in Section 7 we describe the integral representations of truncated mixed fractional derivatives of Marchaud-Hadamard and Marchaud-Hadamard-type in weighted mixed Lebesgue spaces. Sections 8 and 9 contain the inversion theorem and characterization the Hadamard and Hadamard-type mixed fractional integrals on functions from $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$.

Notations. $\mathbb{N},\mathbb{R}=\mathbb{R}^1,\mathbb{C}$ are the sets of all positive integers, real numbers and complex numbers respectively; $\mathbb{R}^1_+=(0;+\infty)$ is the semi-axis; $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x=(x_1,x_2,\ldots,x_n)$; $\dot{\mathbb{R}}^n-$ compactification of $\mathbb{R}^n$ by one infinitely remote point. $\mathbb{R}^n_+=\{x\in\mathbb{R}^n: x_1>0,\ x_2>0,\ldots,x_n>0\}$. Everywhere below: E is the identity operator; $(\Pi_\delta f)(x)=f(x\circ\delta)$, $x,\delta\in\mathbb{R}^n_+$ is the dilation operator. Introduce mixed finite difference of function f of vector order $l=(l_1,l_2,\ldots,l_n)$, $l_k\in\mathbb{N}$ with a "multiplicative" vector step of $t\in\mathbb{R}^n_+$:

$$(\tilde{\Delta}^l_t f)(x)=\tilde{\Delta}^{l_1}_{\xi_1}[\tilde{\Delta}^{l_2}_{\xi_2}\ldots(\tilde{\Delta}^{l_n}_{\xi_n}f)](x)=\sum_{0\leq|k|\leq l}(-1)^{|k|}\tbinom{l}{k}f\left(x\circ t^k\right), \tag{1.3}$$

here $x\circ t^k=\left(x_i\cdot t_1^{k_1},\ldots,x_n\cdot t_n^{k_n}\right)$ and $\binom{l}{k}=\prod\limits_{i=1}^n\binom{l_i}{k_i}$, $\binom{l_i}{k_i}$ are the binomial coefficients, k is a multi-index. Let us agree that the record $1\leq\overline{p}<\infty$ and $\overline{p}=\overline{\infty}$, where $\overline{p}=(p_1,,\ldots,p_n)$, $\overline{\infty}=(\infty,\ldots,\infty)$ means that, $1\leq p_i<\infty$,$p_i=\infty$, $i=\overline{1,n}$. $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\left(\mathbb{R}^n_+,\ \frac{dx}{x}\right)$, $1\leq\overline{p}<\infty$; $C\left(\dot{\mathbb{R}}^n_+\right)=\{\,f\ :\ f\in C\left(\dot{\mathbb{R}}^n_+\right),f(0)=f(\infty)\,\},\overline{p}=\overline{\infty}$. Let $\omega=(\omega_1,\ldots,\omega_n)$, then $\rho^\omega=(\rho_1^{\omega_1},\ldots,\rho_n^{\omega_n})$,

$x \circ \rho^{\omega} = (x_1 \cdot \rho_1^{\omega_1}, \ldots, x_n \cdot \rho_n^{\omega_n})$, $(x : \rho^{\omega}) = (x \cdot \rho^{-\omega}) = \left(\frac{x_1}{\rho_1^{\omega_1}}, \ldots, \frac{x_n}{\rho_n^{\omega_n}}\right)$. If $u = (u_1, u_2, \ldots, u_n)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $u_+^{\alpha} = \prod_{i=1}^{n} (u_i)_+^{\alpha_i}$, $(u_i)_+^{\alpha_i} = \begin{cases} u_i^{\alpha_i}, & u_i > 0, \\ 0, & u_i < 0. \end{cases}$ We use $\aleph(\alpha, l) = \prod_{i=0}^{n} \aleph(\alpha_i, l_i)$, $\aleph(\alpha_i, l_i) = \int\limits_0^{\infty} t^{-1-\alpha_i} (1 - e^{-t})^{l_i} \, dt$ as the normalization constant, known in the theory of fractional differentiation; $C_0^{\infty}(\mathbb{R}_+^n)$ is the class of all infinitely continuously differentiable functions with compact support in $\mathbb{R}_+^n$.

2 Partial and mixed Hadamard and Hadamard-type fractional integrals and derivatives

We start with defining the partial and mixed Hadamard and Hadamard-type fractional integrals and derivatives.

Definition 1. Let $x \in \mathbb{R}_+^n$. The left and the right partial Hadamard-type fractional integrals of order $\alpha_k \in \mathbb{R}$ $(\alpha_k > 0)$ of a function φ with respect to the variable x_k are defined by

$$\left(J_{+,\mu_k}^{\alpha_k} \varphi\right)(x) := \frac{1}{\Gamma(\alpha_k)} \int\limits_0^{x_k} \left(\frac{t}{x_k}\right)^{\mu_k} \left(\ln \frac{x_k}{t}\right)^{\alpha_k - 1} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{1}{\Gamma(\alpha_k)} \int_0^1 u^{\mu_k} \left(\ln \frac{1}{u}\right)^{\alpha_k - 1} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u},$$

and

$$\left(J_{-,\mu_k}^{\alpha_k} \varphi\right)(x) := \frac{1}{\Gamma(\alpha_k)} \int\limits_{x_k}^{\infty} \left(\frac{x_k}{t}\right)^{\mu_k} \left(\ln \frac{t}{x_k}\right)^{\alpha_k - 1} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, ..., x_n) \frac{dt}{t}$$
$$= \frac{1}{\Gamma(\alpha_k)} \int\limits_1^{\infty} u^{-\mu_k} (\ln u)^{\alpha_k - 1} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u},$$

respectively, where $\mathbf{e}_k = (\underbrace{0, \ldots, 0,}_{k-1} 1, 0, \ldots, 0)$, $x \circ u\mathbf{e}_k = (x_1, \ldots, x_{k-1}, x_k \cdot u, x_{k+1}, \ldots, x_n)$.

Definition 2. Let $x \in \mathbb{R}_+^n$. The left and the right partial Hadamard-type fractional derivatives of order α_k $(0 < \alpha_k < 1)$ of a function φ with respect to the variable x_k are defined by

$$\left(\mathfrak{D}_{+,\mu_k}^{\alpha_k} \varphi\right)(x) = \frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_0^{x_k} \left(\frac{t}{x_k}\right)^{\mu_k} \left(\ln \frac{x_k}{t}\right)^{-\alpha_k} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_0^1 u^{\mu_k} \left(\ln \frac{1}{u}\right)^{-\alpha_k} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u} = x_k^{1-\mu_k} \frac{\partial}{\partial x_k} \left(J_{+,\mu_k}^{1-\alpha_k} \varphi\right)(x), \tag{2.1}$$

$$\left(\mathfrak{D}_{-,\mu_k}^{\alpha_k} \varphi\right) = \frac{-x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_{x_k}^{\infty} \left(\frac{x_k}{t}\right)^{\mu_k} \left(\ln \frac{t}{x_k}\right)^{-\alpha_k} \varphi(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \frac{dt}{t}$$
$$= \frac{-x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int\limits_1^{\infty} u^{-\mu_k} (\ln u)^{-\alpha_k} \varphi(x \circ u\mathbf{e}_k) \frac{du}{u} = -x_k^{1+\mu_k} \frac{\partial}{\partial x_k} \left(J_{-,\mu_k}^{1-\alpha_k} \varphi\right)(x) \tag{2.2}$$

respectively.

Definition 3. For a function $\varphi(x)$, defined on $\mathbb{R}^n_+$, the following integrals

$$\left(J^{\alpha}_{+\cdots+}\varphi\right)(x)=\int\limits_0^{x_1}\ldots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{x_i}{t_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\cdots\frac{dt_1}{t_1},\tag{2.3}$$

$$\left(J^{\alpha}_{-\cdots-}\varphi\right)(x)=\int\limits_{x_1}^{\infty}\ldots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\cdots\frac{dt_1}{t_1}\tag{2.4}$$

are called the integrals of fractional order α ($\alpha_i>0,\ i=\overline{1,n}$) in the sense of Hadamard (left and right, respectively).

Definition 4. For a function $\varphi(x)$, defined on $\mathbb{R}^n_+$, the integrals

$$\left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)=\int\limits_0^{x_1}\ldots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{t_i}{x_i}\right)^{\mu_i}\left(\ln\frac{x_i}{t_i}\right)^{\alpha-1}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},\tag{2.5}$$

$$\left(J^{\alpha}_{-\cdots-,\mu}\varphi\right)(x)=\int\limits_{x_1}^{\infty}\ldots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{x_i}{t_i}\right)^{\mu_i}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},\tag{2.6}$$

$$\left(\Im^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)=\int\limits_0^{x_1}\ldots\int\limits_0^{x_n}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{t_i}{x_i}\right)^{\mu_i}\left(\ln\frac{x_i}{t_i}\right)^{\alpha-1}\frac{dt_1}{x_1}\cdots\frac{dt_n}{x_n},$$

$$\left(\Im^{\alpha}_{-\cdots-,\mu}\varphi\right)(x)=\int\limits_{x_1}^{\infty}\ldots\int\limits_{x_n}^{\infty}\varphi(t)\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}\left(\frac{x_i}{t_i}\right)^{\mu_i}\left(\ln\frac{t_i}{x_i}\right)^{\alpha_i-1}\frac{dt_1}{x_1}\cdots\frac{dt_n}{x_n}$$

are called the mixed integrals of fractional order α ($\alpha_i>0,\ i=\overline{1,n}$) of the Hadamard type (left and right, respectively).

Operators (2.3)-(2.6) commute with the dilation operator $\Pi_\rho J^{\alpha}_{\pm\cdots\pm}=J^{\alpha}_{\pm\cdots\pm}\Pi_\rho$, $\Pi_\rho J^{\alpha}_{\pm\cdots\pm,\mu}=J^{\alpha}_{\pm\cdots\pm,\mu}\Pi_\rho$, and are related to the Riemann-Liouville operator $I^{\alpha}_{\pm\cdots\pm}$ by the following equalities

$$J^{\alpha}_{\pm\cdots\pm}\varphi=Q^{-1}I^{\alpha}_{\pm\cdots\pm}Q\varphi,\ \left(J^{\alpha}_{\pm\cdots\pm,\mu}\varphi\right)(x)=\left(M_{\mp\mu}Q^{-1}I^{\alpha}_{\pm\cdots\pm}QM_{\pm\mu}\varphi\right)(x),$$

where $(Q\varphi)(x)=\varphi(e^x)=\varphi(e^{x_1},\ldots,e^{x_n})$, $(Q^{-1}\varphi)(x)=\varphi(\ln x)=\varphi(\ln x_1,\ldots,\ln x_n)$, $(M_{\pm\mu}\varphi)(x)=x_1^{\pm\mu_1}\ldots x_n^{\pm\mu_n}\varphi(x_1,\ldots,x_n)$ (see [20], p. 251 and [8], p. 11).

The operators $J^{\alpha}_{\pm\cdots\pm}$ and $J^{\alpha}_{\pm\cdots\pm,\mu}$ have semi-group properties:

$$J^{\alpha}_{\pm\cdots\pm}J^{\beta}_{\pm\cdots\pm}\varphi=J^{\alpha+\beta}_{\pm\cdots\pm}\varphi\,(\alpha\ge0,\ \beta\ge0),$$

$$J^{\alpha}_{\pm\cdots\pm,\mu}J^{\beta}_{\pm\cdots\pm,\mu}\varphi=J^{\alpha+\beta}_{\pm\cdots\pm,\mu}\varphi\,(\alpha\ge0,\ \beta\ge0).$$

The expressions

$$\left(\mathfrak{D}^{\alpha}_{+\ldots+,\mu}f\right)(x)$$

$$=\prod_{k=1}^{n}\frac{x_k^{1-\mu_k}}{\Gamma(1-\alpha_k)}\frac{\partial^n}{\partial x_1\ldots\partial x_n}\int\limits_0^{x_1}\ldots\int\limits_0^{x_n}\prod_{k=1}^{n}t_k^{\mu_k}\left(\ln\frac{x_k}{t_k}\right)^{-\alpha_k}f(t)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

$$\left(\mathfrak{D}^{\alpha}_{-\ldots-,\mu}f\right)(x)$$

$$= \prod_{k=1}^{n} \frac{(-1)^n x_k^{1+\mu_k}}{\Gamma(1-\alpha_k)} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{k=1}^{n} t_k^{-\mu_k} \left(\ln \frac{t_k}{x_k}\right)^{-\alpha_k} f(t) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$

are called the mixed fractional derivatives of the Hadamard-type of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(0 < \alpha_k < 1,\ k = \overline{1,n})$.

For $\alpha_k \geq 1,\ k = \overline{1,n}$, the mixed fractional derivatives of the Hadamard-type are introduced in the following way

$$(\mathfrak{D}^{\alpha}_{+\cdots+,\mu} f)(x) = \prod_{k=1}^{n} \frac{x_k^{[\alpha_k]+1-\mu_k}}{\Gamma([\alpha_k]+1-\alpha_k)} \times$$

$$\times \frac{\partial^{[\alpha_1]+\cdots+[\alpha_n]+n}}{\partial x_1^{[\alpha_1]+1} \ldots \partial x_n^{[\alpha_n]+1}} \int_0^{x_1} \cdots \int_0^{x_n} \prod_{k=1}^{n} t_k^{\mu_k} \left(\ln \frac{x_k}{t_k}\right)^{[\alpha_k]-\alpha_k} f(t) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \tag{2.7}$$

$$(\mathfrak{D}^{\alpha}_{-\cdots-,\mu} f)(x) = \prod_{k=1}^{n} \frac{(-1)^{[\alpha_1]+\cdots+[\alpha_n]+n} x_k^{[\alpha_k]+1+\mu_k}}{\Gamma([\alpha_k]+1-\alpha_k)} \times$$

$$\times \frac{\partial^{[\alpha_1]+\cdots+[\alpha_n]+n}}{\partial x_1^{[\alpha_1]+1} \ldots \partial x_n^{[\alpha_n]+1}} \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{k=1}^{n} t_k^{-\mu_k} \left(\ln \frac{t_k}{x_k}\right)^{[\alpha_k]-\alpha_k} f(t) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \tag{2.8}$$

where $\alpha_k > 0,\ k = \overline{1,n}$ and $[\alpha_k],\ k = \overline{1,n}$ are the integral parts of $\alpha_k,\ k = \overline{1,n}$. Substituting $t_i = x_i \cdot y_i,\ t_i = x_i \cdot y_i^{-1}, i = \overline{1,n}$, integrals (2.5), (2.6) can be written in the following way:

$$(J^{\alpha}_{+\cdots+,\mu}\varphi)(x) = \int_0^{\infty} \cdots \int_0^{\infty} \varphi(x \circ y) \prod_{i=1}^{n} k^{+}_{\mu_i,\alpha_i}(y_i) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n},$$

$$(J^{\alpha}_{-\cdots-,\mu}\varphi)(x) = \int_0^{\infty} \cdots \int_0^{\infty} \varphi\left(x \circ y^{-1}\right) \prod_{i=1}^{n} k^{+}_{\mu_i,\alpha_i}(y_i) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n},$$

where $x \circ y = (x_1 \cdot y_1, \ldots, x_n \cdot y_n)$, $x \circ y^{-1} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right)$,

$$k^{+}_{\mu_i,\alpha_i}(y_i) = \begin{cases} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu} \left(\ln \frac{1}{y_i}\right)^{\alpha_i - 1}, & 0 < y_i < 1, \\ 0, & y_i > 1, \end{cases} ,\ i = \overline{1,n}.$$

Next we introduce a modification of mixed fractional integrals with a kernel "improved" at infinity:

$$(I^{\alpha,l}_{+\cdots+,\mu;\tau}\varphi)(x) = \int_0^{\infty} \cdots \int_0^{\infty} \left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y)\, \varphi(x \circ y) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}, \tag{2.9}$$

$$(I^{\alpha,l}_{-\cdots-,\mu;\tau}\varphi)(x) = \int_0^{\infty} \cdots \int_0^{\infty} \left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y)\, \varphi\left(x \circ y^{-1}\right) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}, \tag{2.10}$$

where $\tau \in \mathbb{R}^n_+$, $l_i > \alpha_i > 0$, $i = \overline{1,n}$,

$$\left(\tilde{\Delta}^{l}_{\tau^{-1}} k^{+}_{\mu,\alpha}\right)(y) = \tilde{\Delta}^{l_1}_{\tau_1^{-1}} \tilde{\Delta}^{l_2}_{\tau_2^{-1}} \cdots (\tilde{\Delta}^{l_n}_{\tau_n^{-1}} k^{+}_{\mu,\alpha})(y),\ k^{+}_{\mu,\alpha}(y) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu_i} \left(\ln \frac{1}{y_i}\right)_+^{\alpha_i - 1}.$$

It is obvious that $I^{\alpha,l}_{\pm\cdots\pm,\mu;\tau}\varphi = \tilde{\Delta}^l_\tau I^{\alpha}_{\pm\cdots\pm,\mu}\varphi$ on sufficiently good functions $\varphi(x)$, i.e. operators (2.9)-(2.10) are obtained by applying the definition in (1.3) of the difference operators $\tilde{\Delta}^{(l_1,\dots,l_n)}_{(\tau_1,\dots,\tau_n)}$ with a "multiplicative" step to the operators $J^{\alpha}_{\pm\cdots\pm,\mu}\varphi$. They have the advantage over $J^{\alpha}_{\pm\cdots\pm,\mu}\varphi$, at $l_i > \alpha_i > 0,\ i=\overline{1,n}$, as they are limited in the space $L_{\bar{p},\bar{\gamma}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$ for all $1 \le p_i < \infty,\ \gamma_i > 0,\ i=\overline{1,n}$ (i.e., including the case of $\gamma_i = 0,\ i=\overline{1,n}$).

At $\mu = 0$ the partial and mixed Hadamard fractional integrals and derivatives are obtained.

3 Mixed fractional integro-differentiation in terms of tensor products

It is convenient to use the concept of the tensor product of operators, introduced by the following definition.

Definition 5. Let $A_1u_1, A_2u_2, \dots, A_nu_n$ be the linear operators defined on functions $u_1(x), u_2(x), \dots, u_n(x)$ of one variable. The tensor product of operators $A_1, A_2, \dots, A_n$ is an operator$A_1 \otimes A_2 \otimes \dots \otimes A_n$ which is defined on functions of the form

$$\varphi(x_1, x_2, \dots, x_n) = \sum_i u^i_1(x_1) \cdot \ldots \cdot u^i_n(x_n) \tag{3.1}$$

by the relation

$$(A_1 \otimes A_2 \otimes \dots \otimes A_n)\varphi(x_1, x_2, \dots, x_n) = \sum_i A_1\, u^i_1(x_1) \cdot \ldots \cdot A_n u^i_n(x_n).$$

From Definition 5, it follows that the operators of mixed fractional integro-differentiation $J^{\alpha}_{\pm\cdots\pm}\varphi$, $J^{\alpha}_{\pm\cdots\pm,\mu}\varphi$, $\mathfrak{D}^{\alpha}_{\pm\cdots\pm}f$, $\mathfrak{D}^{\alpha}_{\pm\cdots\pm,\mu}f$, $\alpha = (\alpha_1, \dots, \alpha_n)$ are the tensor products of the corresponding one-dimensional operators

$$J^{\alpha}_{\pm\cdots\pm}\varphi = J^{\alpha_1}_{\pm} \otimes \dots \otimes J^{\alpha_n}_{\pm}\varphi, \tag{3.2}$$

$$J^{\alpha}_{\pm\cdots\pm,\mu}\varphi = J^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes J^{\alpha_n}_{\pm,\mu_n}\varphi, \tag{3.3}$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\pm}f = \mathfrak{D}^{\alpha_1}_{\pm} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm}f, \tag{3.4}$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\pm,\mu}f = \mathfrak{D}^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f. \tag{3.5}$$

The following operators are also considered

$$J^{\alpha}_{\pm\cdots\mp\cdots\pm}\varphi = J^{\alpha_1}_{\pm} \otimes \dots \otimes J^{\alpha_i}_{\mp} \otimes \dots \otimes J^{\alpha_n}_{\pm}\varphi,$$

$$J^{\alpha}_{\pm\cdots\mp\cdots\pm,\mu}\varphi = J^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes J^{\alpha_i}_{\mp,\mu_i} \otimes \dots \otimes J^{\alpha_n}_{\pm,\mu_n}\varphi,$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\mp\cdots\pm}f = \mathfrak{D}^{\alpha_1}_{\pm} \otimes \dots \otimes \mathfrak{D}^{\alpha_i}_{\mp} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm}f,$$

$$\mathfrak{D}^{\alpha}_{\pm\cdots\mp\cdots\pm,\mu}f = D^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes \mathfrak{D}^{\alpha_i}_{\mp,\mu_i} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f,$$

with the appropriate choice of signs. The case $\alpha_i = 0$ for some i means the absence of integro-differentiation in (3.2) - (3.5) in the i-th variable

$$J^{(\alpha_1,\dots,0,\dots,\alpha_n)}_{\pm\cdots\pm\cdots\pm}\varphi = J^{\alpha_1}_{\pm} \otimes \dots \otimes J^{\alpha_{i-1}}_{\pm} \otimes E \otimes J^{\alpha_{i+1}}_{\pm} \otimes \dots \otimes J^{\alpha_n}_{\pm}\varphi,$$

$$J^{(\alpha_1,\dots,0,\dots,\alpha_n)}_{\pm\cdots\pm\cdots\pm,\mu}\varphi = J^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes J^{\alpha_{i-1}}_{\pm,\mu_{i-1}} \otimes E \otimes J^{\alpha_{i+1}}_{\pm,\mu_{i+1}} \otimes \dots \otimes J^{\alpha_n}_{\pm,\mu_n}\varphi,$$

$$\mathfrak{D}^{(\alpha_1,\dots,0,\dots,\alpha_n)}_{\pm\cdots\pm\cdots\pm}f = \mathfrak{D}^{\alpha_1}_{\pm} \otimes \dots \otimes \mathfrak{D}^{\alpha_{i-1}}_{\pm} \otimes E \otimes D^{\alpha_{i+1}}_{\pm} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm}f,$$

$$\mathfrak{D}^{(\alpha_1,\dots,0,\dots,\alpha_n)}_{\pm\cdots\pm\cdots\pm,\mu}f = \mathfrak{D}^{\alpha_1}_{\pm,\mu_1} \otimes \dots \otimes \mathfrak{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}} \otimes E \otimes \mathfrak{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}} \otimes \dots \otimes \mathfrak{D}^{\alpha_n}_{\pm,\mu_n}f.$$

4 Mixed Marchaud-Hadamard and Marchaud-Hadamard-type fractional differentiation

Derivatives (2.7), (2.8) can be easily reduced on sufficiently good functions $f(x)$ to a form similar to the fractional Marchaud derivative.

Definition 6. For a function $f(x)$ defined on $\mathbb{R}^n_+$, the expression

$$\left(D^{\alpha}_{\pm\cdots\pm}f\right)(x)=\frac{1}{\prod\limits_{k=1}^{n}\aleph(\alpha_k,l_k)}\int\limits_0^1\cdots\int\limits_0^1\prod_{k=1}^{n}\left(\ln\frac{1}{t_k}\right)^{-1-\alpha_k}\left(\tilde{\Delta}^l_{t^{\pm1}}f\right)(x)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

is called the mixed fractional Marchaud-Hadamard derivative of order $\alpha=(\alpha_1,\ldots,\alpha_n),\alpha_i>0,\ i=\overline{1,n}$.

The mixed fractional Marchaud-Hadamard derivatives $D^{\alpha}_{\pm\cdots\pm}f$ are related to the fractional Marchaud derivatives $\mathbb{D}^{\alpha}_{\pm\cdots\pm}f$ by the equalities

$$D^{\alpha}_{\pm\cdots\pm}f=Q^{-1}\mathbb{D}^{\alpha}_{\pm\cdots\pm}Qf,$$

where $(Qf)(x)=f(e^{x_1},\ldots,e^{x_n})$, $(Q^{-1}f)(x)=f(\ln x_1,\ldots,\ln x_n)$.

The partial fractional derivatives of the Hadamard-type (2.1)-(2.2) can be written (on sufficiently good functions) in the Marchaud form

$$\left(D^{\alpha_k}_{\pm,\mu_k}f\right)=\frac{\alpha_k}{\Gamma(1-\alpha_k)}\int\limits_0^1 t^{\mu_k}\left(\ln\frac{1}{t}\right)^{-\alpha_k-1}\left[f(x)-f\left(x\circ t^{\pm1}\mathbf{e}_k\right)\right]\frac{dt}{t}+\mu_k^{\alpha_k}f(x)$$

$$=\frac{\alpha_k}{\Gamma(1-\alpha_k)}\int\limits_0^\infty e^{-\mu_k t}\frac{f(x)-f\left(x\circ e^{\pm t\mathbf{e}_k}\right)}{t^{\alpha_k+1}}dt+\mu_k^{\alpha_k}f(x),\tag{4.1}$$

where $\mathbf{e}_k=(\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0)$, $x\circ t^{\pm1}\mathbf{e}_k=(x_1,\ldots,x_{k-1},x_k\cdot t^{\pm1},x_{k+1},\ldots,x_n)$. Hence it is easy to see that for the mixed fractional derivatives of the Marchaud-Hadamard-type, instead of (4.1) we obtain

$$D^{\alpha}_{\pm\cdots\pm,\mu}f=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\left(\tilde{D}^{\alpha_2}_{\pm,\mu_2}+\mu_2^{\alpha_2}E\right)\otimes\ldots\otimes\left(\tilde{D}^{\alpha_n}_{\pm,\mu_n}+\mu_n^{\alpha_n}E\right)f$$

$$=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}f+\sum_{i=1}^{n}\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_i^{\alpha_i}E}f$$

$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}f+\cdots+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\left(\mu_1^{\alpha_1}E\otimes\ldots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_{ij}}_{\pm,\mu_{ij}}}f$$

$$+\sum_{i=1}^{n}\left(\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_i}_{\pm,\mu_i}}f+\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}Ef,$$

where $0<\alpha_k<1,\ k=\overline{1,n}$,

$$\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_i^{\alpha_i}E}=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n},$$

$$\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}$$

$$=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_{j-1}}_{\pm,\mu_{j-1}}\otimes\mu_j^{\alpha_j}E\otimes\tilde{D}^{\alpha_{j+1}}_{\pm,\mu_{j+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n},$$

$$\left(\tilde{D}^{\alpha_i}_{\pm,\mu_i}+\mu_i^{\alpha_i}E\right)g\left(x\right)=\frac{\alpha_i}{\Gamma\left(1-\alpha_i\right)}\int\limits_0^1 u_i^{\mu_i}\frac{(\tilde{\Delta}^1_{u_i^{\pm1}}g)\left(x\right)}{\left(\ln\frac{1}{u_i}\right)^{\alpha_i+1}}\frac{du_i}{u_i}+\mu_i^{\alpha_i}g\left(x\right).$$

In particular, at $n=2$

$$D^{\alpha}_{\pm\ldots\pm,\mu}f=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\left(\tilde{D}^{\alpha_2}_{\pm,\mu_2}+\mu_2^{\alpha_2}E\right)f$$

$$=\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\tilde{D}^{\alpha_2}_{\pm,\mu_2}\right)f+\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\mu_2^{\alpha_2}E\right)f+\left(\mu_1^{\alpha_1}E\otimes\tilde{D}^{\alpha_2}_{\pm,\mu_2}\right)f+\left(\mu_1^{\alpha_1}E\otimes\mu_2^{\alpha_2}E\right)f$$

$$=\frac{\alpha_1\alpha_2}{\Gamma\left(1-\alpha_1\right)\Gamma\left(1-\alpha_2\right)}\int\limits_0^1\int\limits_0^1 u_1^{\mu_1}u_2^{\mu_2}\frac{[\tilde{\Delta}^1_{u_2^{\pm1}}(\tilde{\Delta}^1_{u_1^{\pm1}}f)]\left(x\right)}{\left(\ln\frac{1}{u_1}\right)^{\alpha_1+1}\left(\ln\frac{1}{u_2}\right)^{\alpha_2+1}}\frac{du_1}{u_1}\frac{du_2}{u_2}$$

$$+\mu_2^{\alpha_2}\frac{\alpha_1}{\Gamma\left(1-\alpha_1\right)}\int\limits_0^1 u_1^{\mu_1}\frac{(\tilde{\Delta}^1_{u_1^{\pm1}}f)\left(x\right)}{\left(\ln\frac{1}{u_1}\right)^{\alpha_1+1}}\frac{du_1}{u_1}$$

$$+\mu_1^{\alpha_1}\frac{\alpha_2}{\Gamma\left(1-\alpha_2\right)}\int\limits_0^1 u_2^{\mu_2}\frac{(\tilde{\Delta}^1_{u_2^{\pm1}}f)\left(x\right)}{\left(\ln\frac{1}{u_2}\right)^{\alpha_2+1}}\frac{du_2}{u_2}+\mu_1^{\alpha_1}\mu_2^{\alpha_2}f\left(x_1,x_2\right),$$

where $0<\alpha_k<1,\ k=1,2.$

Definition 7. The expression

$$\left(D^{\alpha}_{\pm\cdots\pm;\rho}f\right)\left(x\right)=\frac{1}{\prod\limits_{k=1}^{n}\aleph\left(\alpha_k,l_k\right)}\int\limits_0^{\rho_1}\cdots\int\limits_0^{\rho_n}\prod_{k=1}^{n}\left(\ln\frac{1}{t_k}\right)^{-1-\alpha_k}\left(\tilde{\Delta}^l_{t^{\pm1}}f\right)\left(x\right)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

$0<\rho_i<1$, $i=\overline{1,n}$, is called the "truncated" mixed fractional Marchaud-Hadamard derivative of order $\alpha=\left(\alpha_1,\ldots,\alpha_n\right),\alpha_i>0,\ i=\overline{1,n}$.

In the sequel, we assume by definition that

$$D^{\alpha}_{\pm\cdots\pm}f=\lim_{\rho\to1}D^{\alpha}_{\pm\cdots\pm,\rho}f(\alpha_i>0,\ i=\overline{1,n}),$$

$$D^{\alpha}_{\pm\cdots\pm,\mu}f=\lim_{\rho\to1}D^{\alpha}_{\pm\cdots\pm,\mu;\rho}f,(0<\alpha_i<1,\ i=\overline{1,n}),$$

where the limit is taken in the space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$.

5 Auxiliary lemmas for spaces $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$

Lemma 5.1. *The space $C_0^\infty\left(\mathbb{R}^n_+\right)$ is dense in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$, $1 \le \overline{p} < \infty$, and in*

$$C_{\overline{\gamma},0}\left(\dot{\mathbb{R}}^n_+\right) = \left\{f : f(x) = x^{\overline{\gamma}} g(x),\ g(x) \in C\left(\dot{\mathbb{R}}^n_+\right),\ \lim_{|x|\to 0} g(x) = \lim_{|x|\to\infty} g(x) = 0\right\},$$

for any $-\infty < \gamma_i < \infty$, $i = \overline{1,n}$.

This lemma is proven by standard means.

Lemma 5.2. *Let $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, $1 \le \overline{p} \le \infty$, $\gamma_i \in \mathbb{R}$, $i = \overline{1,n}$, then the following inequality is true:*

$$\left\|\Pi_\rho \varphi\ ;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| = C\left(\rho^{\gamma^*}\right) \cdot \left\|\varphi\ ;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|, \tag{5.1}$$

where

$$C\left(\rho^{\gamma^*}\right) = \prod_{i=1}^n C\left(\rho_i^{\gamma_i^*}\right),\ C\left(\rho_i^{\gamma_i^*}\right) = \begin{cases} \rho_i^{\frac{\gamma_i}{p_i}},\ 1 \le p_i < \infty, \\ \rho_i^{\gamma_i},\quad \rho = \infty, i = \overline{1,n}. \end{cases} \tag{5.2}$$

In addition, the dilation operator approximates the unit operator in the space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$:

$$\lim_{\rho\to 1-0} \left\|\Pi_\rho \varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| = 0. \tag{5.3}$$

Proof. Equality (5.1) is proved by obvious changes of variables. Let us prove the statement (5.3). We have

$$\left\|\Pi_\rho \varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| \le \left\|[1 - \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}]\Pi_\rho\varphi + \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}\Pi_\rho\varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|,$$

where $C\left(\rho^{\gamma^*}\right)$ is the function given in (5.2). Hence, on the basis of the generalized Minkowski inequality (see [5], p. 22), we obtain

$$\left\|\Pi_\rho \varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| \le \left\|[1 - \left(C\left(\rho^{\gamma^*}\right)\right)^{-1}]\Pi_\rho\varphi\ \ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| + \left\|\left(C\left(\rho^{\gamma^*}\right)\right)^{-1}\Pi_\rho\varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|.$$

By (5.1) and (1.1), we have

$$\left\|\Pi_\rho \varphi - \varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| \le \left|1 - C\left(\rho^{\gamma^*}\right)\right| \left\|\varphi\ \ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| + \left\|\Pi_\rho g - g\ ;\mathfrak{L}^{\overline{p}}\right\|, \tag{5.4}$$

where $g(x) := x^{-\overline{\gamma}\cdot\overline{p}}\varphi(x)$, $g(x) \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$ at $1 \le \overline{p} < \infty$, $g(x) := x^{-\overline{\gamma}}\varphi(x)$, $g(x) \in C\left(\dot{\mathbb{R}}^n_+\right)$ at $\overline{p} = \overline{\infty}$. Statement (5.3) follows from inequality (5.4). □

The following lemmas relate to convolution-type operators that are invariant with respect to dilation and to their approximation of the unities in the spaces $\mathfrak{L}_{\overline{p},\overline{\gamma}}$. Consider the operators of the form:

$$(A_\rho\varphi)(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(y_1, \dots, y_n)\,\varphi\left(x_1 \cdot \rho_1^{y_1}, \dots, x_n \cdot \rho_n^{y_n}\right) dy_1 \dots dy_n$$

and

$$(B_\omega\varphi)(x) = \int_0^\infty \dots \int_0^\infty B(\xi_1, \dots, \xi_n)\,\varphi\left(x_1 \cdot \xi_1^{\omega_1}, \dots, x_n \cdot \xi_n^{\omega_n}\right) d\xi_1 \dots d\xi_n,$$

where $\rho_i > 0$, $\omega_i > 0$, $i = \overline{1,n}$.

Lemma 5.3. *Let* $1 \le p_i \le \infty,\ \gamma_i \in \mathbb{R}, \rho_i > 0,\ \omega_i > 0,\ i = \overline{1,n}.$

1) *If* $K\left(\rho^{\gamma^*}\right) := \int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} |K(y_1,\ldots,y_n)| \prod\limits_{i=1}^{n} \rho_i^{\gamma_i^*\cdot y_i} dy_1 \ldots dy_n < \infty,$ *where* $\gamma_i^*,\ i = \overline{1,n}$- *are the constants from* (1.2), *then the operator* A_ρ *is bounded in the space* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, *and*

$$\left\|A_\rho\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le K\left(\rho^{\gamma^*}\right) \left\|\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|. \tag{5.5}$$

2) *If* $d(\gamma^*,\omega) := \int\limits_{0}^{\infty} \cdots \int\limits_{0}^{\infty} |B(\xi_1,\ldots,\xi_n)| \prod\limits_{i=1}^{n} \xi_i^{\gamma_i^*\cdot\omega_i} d\xi_1 \ldots d\xi_n < \infty,$ *where* $\gamma_i^*,\ i = \overline{1,n}$- *are the constants from* (1.2), *then the operator* B_ω *is bounded in the space* $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ *and*

$$\left\|B_\omega\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le d(\gamma^*,\omega) \left\|\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

Proof. Representing $A_\rho\varphi$ as

$$(A_\rho\varphi)(x) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} K(y_1,\ldots,y_n) (\Pi_{\rho^y}\varphi)(x)\, dy_1 \ldots dy_n$$

and using the generalized Minkowski inequality, we have

$$\left\|A_\rho\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} |K(y_1,\ldots,y_n)|\ \left\|(\Pi_{\rho^y}\varphi)(x)\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| dy_1 \ldots dy_n.$$

Taking into account equality (5.1) we obtain (5.5). The operator $B_\omega\varphi$ is considered similarly. □

Lemma 5.4. *Let* $K(y) = k_1(y_1)\ldots k_n(y_n), k_i(y_i) \in L_1(\mathbb{R}^1),\ k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}.$ *Then*

$$\left\|A_\rho\varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \le \left\|k_1\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \ldots \left\|k_n\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \cdot \left\|\varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

at $0 < \rho_i \le 1,\ i = \overline{1,n}.$

The proof of Lemma 5.4 follows from Lemma 5.3.

Lemma 5.5. *Let* $K(y) = k_1(y_1)\cdots k_n(y_n), k_i(y_i) \in L_1(\mathbb{R}^1),\ k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}$ *and* $\int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} K(y_1,\ldots,y_n)\, dy_1 \ldots dy_n = 1.$ *Then*

$$\lim_{\rho\to 1-0} \left\|A_\rho\varphi - \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| = 0 \tag{5.6}$$

for all $1 \le p_i \le \infty,\ \gamma_i \ge 0,\ 0 < \rho_i \le 1, i = \overline{1,n}.$

Proof. First, note that $A_\rho\varphi \in\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ for $\varphi \in\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ at $0 < \rho_i < 1,\ i = \overline{1,n}$, according to Lemma 5.4. To prove equality (5.6) note that since $\int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} K(y_1,\ldots,y_n)\, dy_1 \ldots dy_n = 1$, then

$$(A_\rho\varphi)(x) - \varphi(x) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} K(y_1,\ldots,y_n)\, ((\Pi_{\rho^y}\varphi)(x) - \varphi(x)) dy_1 \ldots dy_n.$$

Using the generalized Minkowski inequality, we obtain

$$\left\|A_\rho\varphi - \varphi\, ;\, \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|$$

$$\leq \int_0^\infty \dots \int_0^\infty |k_1(y_1)| \dots |k_n(y_n)| \cdot \left\| (\Pi_{\rho^y}\varphi)(x) - \varphi(x)\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| dy_1 \dots dy_n. \tag{5.7}$$

Since $0 < \rho_i \leq 1$, $i = \overline{1,n}$, then in (5.7) the passage to the limit under the sign of the integral is possible on the basis of the majorant Lebesgue theorem. The application of the latter is substantiated by statements (5.1), (5.3) of Lemma 5.2. □

6 On the boundedness of mixed fractional Hadamard and Hadamard type integration in space $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$

Theorem 6.1. *Let $\gamma_i \in \mathbb{R}^1$, $1 \leq p_i \leq \infty$, $\alpha_i > 0$ and $\mu_i \in \mathbb{C}$, $i = \overline{1,n}$. If $Re\mu_i > -\gamma_i^*$, , $i = \overline{1,n}$, where γ_i^*, $i = \overline{1,n}$ are the constants from (1.2), then the operator $J_{+\dots+,\mu}^{\alpha}$ is bounded in the $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, and*

$$\left\| J_{+\dots+,\mu}^{\alpha}\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \prod_{i=1}^{n} (\mu_i + \gamma_i^*)^{-\alpha_i} \left\| \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|. \tag{6.1}$$

Proof. First consider the case $1 \leq \overline{p} < \overline{\infty}$. By the generalized Minkowski inequality, we have

$$\left\| J_{+\dots+,\mu}^{\alpha}\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \int_0^\infty \dots \int_0^\infty \left\| \varphi(x \circ y)\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \prod_{i=1}^{n} \left| k_{\mu_i,\alpha_i}^{+}(y_i) \right| \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n}.$$

After substitution $\tau_i = x_i \cdot y_i$, $i = \overline{1,n}$, we obtain

$$\left\| J_{+\dots+,\mu}^{\alpha}\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} \left| k_{\mu_i,\alpha_i}^{+}(y_i) \right| y_i^{\frac{\gamma_i}{p_i}} \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n} \left\| \varphi; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|.$$

So,

$$\left\| J_{+\dots+,\mu}^{\alpha}\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} y_i^{\mu_i + \frac{\gamma_i}{p_i}} \left(\ln \frac{1}{y_i} \right)^{\alpha_i - 1} \frac{dy_1}{y_1} \dots \frac{dy_n}{y_n} \left\| \varphi; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|$$

$$\leq \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} e^{-(\mu_i + \frac{\gamma_i}{p_i})\xi_i} (\xi_i)^{\alpha_i - 1} d\xi_1 \dots d\xi_n \left\| \varphi; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|$$

$$\leq \prod_{i=1}^{n} \left(\frac{p_i}{\mu_i\, p_i + \gamma_i} \right)^{\alpha_i} \left\| \varphi; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|. \tag{6.2}$$

At $\overline{p} = \overline{\infty}$ in (6.2) substitute p_i, $i = \overline{1,n}$ for 1. Then we get (6.1). □

Theorem 6.2. 1) *Let $\gamma_i \in \mathbb{R}^1$, $1 \leq p_i \leq \infty$, $\alpha_i > 0, i = \overline{1,n}$. If $\gamma_i > 0$, $i = \overline{1,n}$, then operator $J_{+\dots+}^{\alpha}$ is bounded in the $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, and*

$$\left\| J_{+\dots+}^{\alpha}\varphi\ ;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \prod_{i=1}^{n} (\gamma_i^*)^{-\alpha_i} \left\| \varphi\ ; \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\|.$$

2) *Let $1 \leq p_i \leq \infty$, $1 \leq q_i \leq \infty$, $0 < \alpha_i < 1$, $i = \overline{1,n}$. Operators of fractional integration $J_{+\dots+}^{\alpha}\varphi$ and $J_{-\dots-}^{\alpha}\varphi$ are bounded from $\mathfrak{L}^{\overline{p}}\left(\mathbb{R}_+^n, \frac{dx}{x}\right)$ into $\mathfrak{L}^{\overline{q}}\left(\mathbb{R}_+^n, \frac{dx}{x}\right)$ if and only if $1 < p_i < \frac{1}{\alpha_i}$, $q_i = \frac{p_i}{1-\alpha_i p_i}$, $i = \overline{1,n}$.*

Proof. The first statement follows from Lemma 5.3. Then, the operators $J^{\alpha}_{+\cdots+}\varphi$ and $J^{\alpha}_{-\cdots-}\varphi$ are related to the Riemann-Liouville operators $I^{\alpha}_{\pm\cdots\pm}\varphi$ by the equalities

$$J^{\alpha}_{+\cdots+}\varphi = Q^{-1}I^{\alpha}_{+\cdots+}Q\varphi,\, J^{\alpha}_{-\cdots-}\varphi = Q^{-1}I^{\alpha}_{-\cdots-}Q\varphi, \tag{6.3}$$

where $(Q\varphi)(x) = \varphi(e^{x}) = \varphi(e^{x_1}, \ldots, e^{x_n})$. By virtue of (6.3), the second statement of the theorem follows from the well-known Hardy-Littlewood theorem for ordinary fractional integration over $\mathbb{R}^n$ (see [20], p. 494). □

Theorem 6.3. *Operators $J^{\alpha,l}_{+\cdots+,\mu;\tau}$, $J^{\alpha,l}_{+\cdots+,\tau}$ is bounded in the space $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ for all $1 \le p_i \le \infty$, $\gamma_i \ge 0$, $i = \overline{1,n}$,*

$$\left\| J^{\alpha,l}_{\pm\cdots\pm,\tau}\varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| \le \prod_{i=1}^{n} c_i(\tau_i, \mu_i)\left\|\varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|,$$

where $0 < c_i(\tau_i, \mu_i) < 1$ at $Re\mu_i + \gamma_i^* \ge 0,\ 0 < \tau_i \le 1,\ l_i > \alpha_i > 0,\ i = \overline{1,n}$,

$$\left\| J^{\alpha,l}_{\pm\cdots\pm,\tau}\varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\| \le \prod_{i=1}^{n} c_i(\tau_i)\left\|\varphi\ ;\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right\|,$$

where $0 < c_i(\tau_i) < 1$ at $0 < \tau_i \le 1,\ l_i > \alpha_i > 0,\ i = \overline{1,n}$.

The proof of this theorem follows from Lemma 5.3.

7 Integral representation of the truncated mixed Marchaud-Hadamard and Marchaud-Hadamard-type fractional derivatives

Lemma 7.1. *Let $f(x) = \left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, where $1 \le p_i < \infty,\ \ \gamma_i > 0, \mu_i \ge 0, \mu_i > -\frac{\gamma_i}{p_i},\ 0 < \alpha_i < 1, i = \overline{1,n}$, and $0 < \rho_i < 1$, $i = \overline{1,n}$, the truncated mixed fractional derivative $D^{\alpha}_{+\cdots+,\mu;\,\rho}f$ has the following integral representation*

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = \int\limits_{R^n} K^{+}_{\alpha,\,\mu}(t,\,\rho)\,\varphi\left(x\circ\rho^{t}\right)dt, \tag{7.1}$$

where

$$K^{+}_{\alpha,\,\mu}(t,\,\rho) = K^{+}_{\alpha_1,\,\mu_1}(t_1,\,\rho_1)\ldots K^{+}_{\alpha_n,\,\mu_n}(t_n,\,\rho_n)\,, K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) =$$

$$= \frac{\sin\alpha_i\pi}{\pi}\frac{\rho_i^{\mu_i\,t_i}}{t_i}[(\alpha_i\,\Gamma\left(-\alpha_i,\,\mu_i\ln\frac{1}{\rho_i}\right) + \Gamma(1-\alpha_i))\left(\mu_i\ln\frac{1}{\rho_i}\right)^{\alpha_i}(t_i)^{\alpha}_{+} - (t_i-1)^{\alpha_i}_{+}],$$

$\Gamma\left(-\alpha_i,\,\mu_i\ln\frac{1}{\rho_i}\right)$, $i = \overline{1,n}$, the upper incomplete gamma function. In this case, the kernel $K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) \in L_1\left(\mathbb{R}^1_+\right)$ is an averaging one:

$$\int\limits_0^{\infty} K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i)\,dt_i = 1, K^{+}_{\alpha_i,\,\mu_i}(t_i,\,\rho_i) > 0 \tag{7.2}$$

at $0 < t_i < 1$.

Proof. The proof is easily reduced to known facts for the one-dimensional case ([25]). Namely, we have

$$J^{\alpha}_{+\cdots+,\mu}\varphi = J^{\alpha_1}_{+,\mu_1} \otimes \cdots \otimes J^{\alpha_n}_{+,\mu_n}\varphi,$$
$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = D^{\alpha_1}_{+,\mu_1;\,\rho_1} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\rho_n}f \ .$$

Since $f(x) = \left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)(x)$, then

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = D^{\alpha_1}_{+,\mu_1;\,\rho_1}J^{\alpha_1}_{+,\mu_1} \otimes D^{\alpha_2}_{+,\mu_2;\,\rho_2}J^{\alpha_2}_{+,\mu_2} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\rho_n}J^{\alpha_n}_{+,\mu_n}\varphi \ .$$

It is known, that (see [15], [25])

$$D^{\alpha_i}_{+,\mu_i;\,\rho_i}J^{\alpha_i}_{+,\mu_i}g = \mathrm{K}^{+}_{\alpha_i,\,\mu_i}(\tau,\,\rho_i)\,g,\, i=\overline{1,n},\, g=g(t)\in\mathfrak{L}^p\left(\mathbb{R}_+,\, t^{-\gamma}\frac{dx}{x}\right)$$

for the function of one variable and

$$D^{\alpha_i}_{+,\mu_i;\,\rho_i}J^{\alpha_i}_{+,\mu_i}g = \int\limits_0^\infty K^{+}_{\alpha_i,\,\mu_i}(\tau,\,\rho_i)\,g\left(t\cdot\rho_i^{\tau}\right)d\tau.$$

Then

$$D^{\alpha}_{+\cdots+,\mu;\,\rho}f = K^{+,\alpha_1}_{\mu_1;\,\rho_1} \otimes K^{+,\alpha_2}_{\mu_2;\,\rho_2} \otimes \cdots \otimes K^{+,\alpha_n}_{\mu_n;\,\rho_n}\varphi,$$

for $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i=\overline{1,n}$, taking into account the density of functions of form (3.1). This implies representation (7.1). Operator (7.1) on the right-hand side is also bounded by Lemma 5.4. Therefore, by virtue of Lemma 5.1, identity (7.1) applies with $C_0^\infty\left(\mathbb{R}^n_+\right)$ to all functions $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i=\overline{1,n}$. □

Lemma 7.2. *Let* $f(x) = \left(J^{\alpha}_{+\cdots+}\varphi\right)(x)$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ *, where* $\alpha_i > 0$, $1 \le p_i < \infty$, $\gamma_i > 0$, $i=\overline{1,n}$, *or* $0 < \alpha_i < 1$, $1 < p_i < \frac{1}{\alpha}$, $\gamma_i = 0$, $i=\overline{1,n}$ *and* $0 < \rho_i < 1$, $i=\overline{1,n}$. *Then the truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\,\rho}f$ *has the following integral representation*

$$\left(D^{\alpha}_{+\cdots+,\rho}f\right)\ (x) = \int\limits_0^\infty \ldots \int\limits_0^\infty \prod_{i=1}^{n} K^{+}_{l_i,\alpha_i}(y_i)\varphi(x\circ\rho^y)dy_1\ldots dy_n, \tag{7.3}$$

where the kernel

$$K^{+}_{l_i,\,\alpha_i}(y_i) = \frac{\sum\limits_{k=0}^{l_i}(-1)^k\begin{pmatrix} l_i \\ k\end{pmatrix}(y_i-k)^{\alpha_i}_{+}}{\vartheta\left(\alpha_i,l_i\right)\Gamma(1+\alpha_i)y_i} \in L_1\left(\mathbb{R}^1_+\right) \tag{7.4}$$

at $l > \alpha > 0$,

$$\int\limits_0^\infty K^{+}_{l_i,\,\alpha_i}(y_i)dy_i = 1,\ l_i > \alpha_i > 0. \tag{7.5}$$

The proof of Lemma 7.2 is similar to the proof of Lemma 7.1

Lemma 7.3. *Let* $f \in \mathfrak{L}^{\overline{r}}_{\overline{\lambda}}$, $1 \le r_i \le \infty, \lambda_i \ge 0$, $i=\overline{1,n}$ *be such that its difference* $(\tilde{\Delta}^l_t f)(x)$ *of order* l *is represented by a modified mixed Hadamard fractional integral* (2.7) *of a function from* $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$*:*

$$(\tilde{\Delta}^l_\tau f)(x) = (J^{\alpha,l}_{+\cdots+,\,\tau}\varphi)(x) = \int\limits_0^\infty \ldots \int\limits_0^\infty \left(\tilde{\Delta}^l_{\tau^{-1}}k^{+}_{\alpha}\right)(y)\,\varphi\left(x\circ y\right)\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}, \tag{7.6}$$

where $l_i > \alpha_i > 0,\ 0 < \tau_i < 1,\ \varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}},\ 1 \le p_i \le \infty,\quad \gamma_i \ge 0,\ i = \overline{1,n}$ *and* $0 <\ h_i < 1,\ i = \overline{1,n}$. *Then the truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\rho} f$ *allows integral representation* (7.3) *for all* $1 \le p_i < \infty,\ \gamma_i \ge 0,\ i = \overline{1,n}$ *and integral representation*

$$\left(D^{\alpha}_{+\cdots+,\rho} f\right)\ (x) = K_1\left(\Pi^1_{\rho_1^{t_1}} - \Pi^1_0\right) \otimes \cdots \otimes K_n\left(\Pi^n_{\rho_n^{t_n}} - \Pi^n_0\right) \varphi\left(x\right) \tag{7.7}$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = \overline{1,n}$, *where the operator* Π^i_0 *is:*

$$\left(\Pi^i_0 \varphi\right)(x) = \varphi\left(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\right)$$

In particular, for $n = 2$

$$\left(D^{\alpha}_{++,\rho} f\right)(x) = K_1\left(\Pi^1_{\rho_1^{t_1}} - \Pi^1_0\right) \otimes K_2\left(\Pi^2_{\rho_2^{t_2}} - \Pi^2_0\right) \varphi\left(x\right)$$

$$= \int_0^{\infty}\int_0^{\infty} K^+_{l_1,\alpha_1}\left(t_1\right) K^+_{l_2,\alpha_2}\left(t_2\right) \varphi\left(x_1 \cdot \rho_1^{t_1}, x_2 \cdot \rho_2^{t_2}\right) dt_1 dt_2$$

$$- \int_0^{\infty} K^+_{l_1,\alpha_1}\left(t_1\right) \varphi\left(x_1 \cdot \rho_1^{t_1}, 0\right) dt_1 - \int_0^{\infty} K^+_{l_2,\alpha_2}\left(t_2\right) \varphi\left(0, x_2 \cdot \rho_2^{t_2}\right) dt_2 + \varphi\left(0,0\right),$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = 1,2$, *where* $K^+_{l_i,\alpha_i}\left(t_i\right)$ *is kernel* (7.4).

Proof. Lemma 7.3 is proven in the same way as Lemma 6.3 from [25]. Since at $\gamma_i > 0,\ i = \overline{1,n}$ the procedure is substantiated in the proof of Lemma 7.1, it suffices to consider the case at $\gamma_i = 0,\ i = \overline{1,n}$ for any $1 \le p_i \le \infty,\ \alpha_i > 0,\ i = \overline{1,n}$. Similarly, as in the one-dimensional case, it is necessary to substantiate the following equality

$$\int_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int_{\ln\frac{1}{\rho_n}}^{\infty} \prod_{i=1}^{n} \frac{d\xi_i}{\xi_i^2} \int_0^{\infty} \cdots \int_0^{\infty} (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \varphi(x \circ e^{-\tau}) d\tau$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} \varphi(x \circ e^{-\tau}) d\tau \int_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int_{\ln\frac{1}{\rho_n}}^{\infty} (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \prod_{i=1}^{n} \frac{d\xi_i}{\xi_i^2},$$

where $(\Delta_1^l k_\alpha^+)(y) = \Delta_1^{l_1}[\Delta_1^{l_2} \dots (\Delta_1^{l_n} k_\alpha^+)](y),\ \ (k_\alpha^+)(y) = \prod\limits_{i=1}^{n} \frac{(y_i)_+^{\alpha_i - 1}}{\Gamma(\alpha_i)}$. Here the change of order of integration is substantiated by Fubini's theorem at $1 \le p_i < \infty,\ \gamma_i = 0,\ i = \overline{1,n}$. Next, we prove that at $1 \le p_i < \infty,\ i = \overline{1,n}$, the iterated integral converges (for almost all x, $x \in \mathbb{R}^n$)

$$\int_0^{\infty} \cdots \int_0^{\infty} \left|\varphi(x \circ e^{-\tau})\right| d\tau_1 \dots d\tau_n \int_{\ln\frac{1}{\rho_1}}^{\infty} \cdots \int_{\ln\frac{1}{\rho_n}}^{\infty} \left|(\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi})\right| \frac{d\xi_1}{\xi_1^2} \dots \frac{d\xi_n}{\xi_n^2},$$

for all $\varphi \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$. Changing of the variables $\frac{\tau_i}{\xi_i} = s_i$ and $\tau_i = t_i \ln\frac{1}{h_i},\ i = \overline{1,n}$, leads to the necessity to prove the convergence of the integral

$$A := \int_0^{\infty} \cdots \int_0^{\infty} \left|\varphi(x \circ h^t)\right| K^*(t) dt_1 \dots dt_n, \tag{7.8}$$

where $K^*(t) = \frac{1}{t_1 \ldots t_n} \int_0^{t_1} \ldots \int_0^{t_n} \left|(\Delta_1^l k_\alpha^+)(s)\right| ds$. Since $(\Delta_1^l k_\alpha^+)(s) \in L_1(\mathbb{R}^n)$ (see Theorem 6.3), then $K^*(t) \leq \frac{c}{t_1 \ldots t_n}$ at $t \to \infty$. Then it is evident that $K^*(t) \leq ct^{\alpha-1}$ at $t \to 0$ and, $K^*(t)$ is continuous at $t \in \mathbb{R}_+^n$. We have

$$\overline{K^*(t)} = \sum_{1 \leq |j| < n} k_j(t), \quad k_j(t) = t^{a_j(t)} = t_1^{a_{j_1}(t_1)} \ldots t_n^{a_{j_n}(t_n)},$$

where $a_{j_i}(t_i) = \begin{cases} \alpha_i - 1, & 0 < t_i < 1, \\ -1, & t_i \geq 1. \end{cases}$ Then from (7.8) we obtain

$$A \leq \int_0^\infty \ldots \int_0^\infty \left|\varphi(x \circ \rho^t)\right| \overline{K^*(t)} dt_1 \ldots dt_n$$

and it remains to refer to Young's theorem for spaces with mixed norm ([5], p. 25).

Substantiate the case $p_i = \infty$, $i = \overline{1,n}$, for $\varphi \in C\left(\mathbb{R}_+^n\right)$. Consider the "two-sided" mixed truncated Marchaud-Hadamard fractional derivative, i.e.

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x) = \frac{1}{\aleph(\alpha,l)} \int_{\delta_1}^{\rho_1} \ldots \int_{\delta_n}^{\rho_n} \left(\ln \frac{1}{t}\right)^{-1-\alpha} \left(\tilde{\Delta}_t^l f\right)(x) \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}, \tag{7.9}$$

at $l_i > \alpha_i > 0$, where $0 < \delta_i < \rho_i < 1$, $i = \overline{1,n}$, then refer to the limit $\delta \to 0$. From (7.6) we have

$$\left(\tilde{\Delta}_t^l f\right)(x) = \left(\ln \frac{1}{t}\right)^\alpha \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(y) \varphi\left(x \circ t^y\right) dy_1 \ldots dy_n, \tag{7.10}$$

where $0 < t_i < 1$, $i = \overline{1,n}$. Substituting (7.10) into (7.9), we obtain

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x)$$

$$= \frac{1}{\aleph(\alpha,l)} \int_{\delta_1}^{\rho_1} \ldots \int_{\delta_n}^{\rho_n} \prod_{i=1}^n \left(\ln \frac{1}{t_i}\right)^{-1} \frac{dt_i}{t_i} \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(y) \varphi\left(x \circ t^y\right) dy_1 \ldots dy_n.$$

The changes of variables $\ln \frac{1}{t_i} = \xi_i$ and $y_i \xi_i = \tau_i$, $i = \overline{1,n}$, give:

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x)$$

$$= \frac{1}{\aleph(\alpha,l)} \int_{\ln \frac{1}{\rho_1}}^{\ln \frac{1}{\delta_1}} \ldots \int_{\ln \frac{1}{\rho_n}}^{\ln \frac{1}{\delta_n}} \frac{d\xi_1}{\xi_1^2} \ldots \frac{d\xi_n}{\xi_n^2}{}_i \int_0^\infty \ldots \int_0^\infty (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \varphi\left(x \circ e^{-\tau}\right) d\tau_1 \ldots d\tau_n$$

and the change of the order of integration leads to the equality

$$\left(D_{+\cdots+,\rho,\delta}^\alpha f\right)(x) =$$

$$= \frac{1}{\aleph(\alpha,l)} \int_0^\infty \ldots \int_0^\infty \varphi\left(x \circ e^{-\tau}\right) d\tau \int_{\ln \frac{1}{\rho_1}}^{\ln \frac{1}{\delta_1}} \ldots \int_{\ln \frac{1}{\rho_n}}^{\ln \frac{1}{\delta_n}} (\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi}) \prod_{i=1}^n \xi_i^{-2} d\xi_i. \tag{7.11}$$

Here the change of order of integration is easily substantiated by introducing $\delta=(\delta_1,\ldots,\delta_n)$, $0<\delta_i<1$, $i=\overline{1,n}$ (considering that $|\varphi|\le c$ and $\int\limits_{\ln\frac{1}{\rho_1}}^{\ln\frac{1}{\delta_1}}\ldots\int\limits_{\ln\frac{1}{\rho_n}}^{\ln\frac{1}{\delta_n}}\frac{d\xi_1}{\xi_1^2}\ldots\frac{d\xi_n}{\xi_n^2}\int\limits_0^\infty\ldots\int\limits_0^\infty\left|(\Delta_1^l k_\alpha^+)(\frac{\tau}{\xi})\right|d\tau_1\ldots d\tau_n<\infty$). Equality (7.11) means that

$$\left(D^\alpha_{+\cdots+,\rho,\delta}f\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^\infty\ldots\int\limits_0^\infty\varphi\left(x\circ e^{-\tau}\right)\times$$

$$\times\prod_{i=1}^n[\frac{1}{\ln\frac{1}{\rho_i}}K^+_{l_i,\alpha_i}\left(\frac{\tau_i}{\ln\frac{1}{\rho_i}}\right)-\frac{1}{\ln\frac{1}{\delta_i}}K^+_{l_i,\alpha_i}\left(\frac{\tau_i}{\ln\frac{1}{\delta_i}}\right)]d\tau_1\cdots d\tau_n,$$

where $K^+_{l_i,\alpha_i}(t_i)$, $i=\overline{1,n}$, is kernel (7.4). Here, the integral representation can be written in terms of tensor products, i.e.

$$\left(D^\alpha_{+\cdots+,1-\rho,\delta}f\right)(x)=K_1\left(\Pi^1_{\rho_1^{t_1}}-\Pi^1_{\delta_1^{t_1}}\right)\otimes\cdots\otimes K_n\left(\Pi^n_{\rho_n^{t_n}}-\Pi^n_{\delta_n^{t_n}}\right)\varphi(x),\tag{7.12}$$

where

$$K_i\left(\Pi^i_{\rho_i^{t_i}}-\Pi^i_{\delta_i^{t_i}}\right)g(x_i)=\int\limits_0^\infty K^+_{l_i,\alpha_i}(t_i)\left[g\left(x_i\rho_i^{t_i}\right)-g\left(x_i\delta_i^{t_i}\right)\right]dt_i,$$

$\left(\Pi^i_{\rho_i^{t_i}}\varphi\right)(x)=\varphi\left(x_1,\ldots,x_{i-1},x_i\rho_i^{t_i},x_{i+1},\ldots,x_n\right)$ is the dilation operator. Since $\varphi\in C\left(\dot{\mathbb{R}}^n_+\right)$ and $K^+_{l_i,\alpha_i}(t_i)\in L_1(\mathbb{R}^1)$, $i=\overline{1,n}$, a passage to the limit is possible at $\delta\to0$ under the sign of the integral. By (7.5) from (7.12) we obtain (7.7). □

8 Inversion of mixed fractional integrals of functions belonging to $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$

Theorem 8.1. . *Let $f=J^\alpha_{+\cdots+,\mu}\varphi$, $\varphi\in\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, where $\gamma_i>0$, $0<\alpha_i<1$, $1\le p_i\le\infty$, $\mu_i\ge0$, $\mu_i>-\gamma_i^*$, $i=1,\ldots,n$. Then*

$$\left(D^\alpha_{+\cdots+,\mu}f\right)(x)=\lim_{\substack{\rho\to1\\(\mathfrak{L}^{\overline{p}}_{\overline{\gamma}})}}\left(D^\alpha_{+\cdots+,\mu;\rho}f\right)(x)=\varphi(x).$$

Proof. Convergence in norm follows from Lemmas 7.1 and 5.5. □

Theorem 8.2. *Let $f=J^\alpha_{+\cdots+}\varphi$, $\varphi\in\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, where either $\gamma_i>0$, $\alpha_i>0, 1\le p_i\le\infty$, $i=1,\ldots,n$, or $\gamma_i=0$, $0<\alpha_i<1, 1\le p_i<\frac{1}{\alpha_i}, i=1,\ldots,n$. Then*

$$\left(D^\alpha_{+\cdots+}f\right)(x)=\lim_{\substack{\rho\to1\\(\mathfrak{L}^{\overline{p}}_{\overline{\gamma}})}}\left(D^\alpha_{+\cdots+,\rho}f\right)(x)=\varphi(x),$$

where the limit is taken both in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, and almost everywhere.

Proof. Convergence in norm follows from Lemmas 7.2 and 5.5. The proof of convergence almost everywhere is obtained by using Theorem 2 ([24], p. 77-78), applying it for each variable. In this case, equality (7.4) and the property of the kernel $\left|K^+_{l_i,\alpha_i}(y_i)\right|\le\frac{c}{(1+y_i)^{l_i+1-\alpha_i}}$ at $l_i>\alpha_i$, $y_i>1$, $i=1,\ldots,n$ (see [20], p. 379) are taken into account, so, the kernel $K^+_{l_i,\alpha_i}(y_i)$ has a monotone summable majorant. □

Theorem 8.3. *Let* $\left(\tilde{\Delta}_\tau^l f\right)(x) = J^{\alpha,l}_{+\cdots+,\tau}\varphi$, $\varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, *where* $\gamma_i \geq 0$, $l > \alpha_i > 0$, $1 \leq p_i \leq \infty$, $0 < \tau_i < 1$, $i = 1, \ldots, n$. *Then*

$$\left(D^\alpha_{+\cdots+}f\right)(x) = \lim_{\substack{\rho \to 1 \\ \left(\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\right)}} \left(D^\alpha_{+\cdots+,\,\rho}f\right)(x) = \varphi(x),$$

where the limit is taken both in $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$, *and almost everywhere.*

The proof of the convergence in norm follows from Lemmas 7.3 and 5.5. The convergence almost everywhere is proven as in Theorem 8.2.

Remark 1. One can admit the case in which $\alpha_i = 0$ for some i. In particular, if $f = J^\alpha_{+\cdots+}\varphi$, $\varphi \in \mathfrak{L}_{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right)$, where $\alpha_i > 0$ at $i = 1, \ldots, k-1, k+1, \ldots, n$, $\alpha_k = 0$, $1 \leq p_i < \frac{1}{\alpha_i}$ at $i = 1, \ldots, k-1, k+1, \ldots, n$ and $1 \leq p_k \leq \infty$. Then

$$\left(D^\alpha_{+\cdots+}f\right)(x) = \lim_{\rho\to 1}(D^{(\alpha_1,\ldots,\alpha_{k-1},0,\alpha_{k+1},\ldots,\alpha_n)}_{+\cdots+;\rho}f)(x) = \varphi(x),$$

where the limit is taken both in $\mathfrak{L}_{\overline{p}}$, and almost everywhere.

9 Characterization of mixed fractional integrals of functions from $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$

Denote by $J^\alpha_{\pm\cdots\pm,\mu}(\mathfrak{L}^{\overline{p}})$ the operator image of mixed fractional integration

$$J^\alpha_{\pm\cdots\pm,\mu}\left(\mathfrak{L}^{\overline{p}}\right) = \{\, f : f = J^\alpha_{\pm\cdots\pm,\mu}\varphi,\ \ \varphi \in \mathfrak{L}^{\overline{p}}\left(\mathbb{R}^n_+, \frac{dx}{x}\right) \}$$

defined for $0 < \alpha_i < 1$, $1 \leq p_i < \frac{1}{\alpha_i}, i = 1, \ldots, n$. Actually, at $1 < p_i < \frac{1}{\alpha_i}, i = 1, \ldots, n$, they coincide, that is, do not depend on the sign choice, so, denote them by

$$J^\alpha := J^\alpha_{+\cdots+,\mu}\left(\mathfrak{L}^{\overline{p}}\right) = J^\alpha_{++--\cdots+,\mu}\left(\mathfrak{L}^{\overline{p}}\right) = \cdots = J^\alpha_{-\cdots-,\mu}\left(\mathfrak{L}^{\overline{p}}\right)$$

Denote similar modified operator of mixed fractional integration by $J^{\alpha,l}_{\pm\cdots\pm,\mu}(\mathfrak{L}^{\overline{p}})$:

$$J^{\alpha,l}_{\pm\cdots\pm,\mu}\left(\mathfrak{L}^{\overline{p}}\right) = \left\{g : g = J^{\alpha,l}_{\pm\cdots\pm,\mu;\tau}\varphi,\ \ \varphi \in \mathfrak{L}^{\overline{p}}\ \right\}.$$

This space is defined for $l_i > \alpha_i > 0$, $1 \leq p_i \leq \infty$, $\mu_i > 0$, $0 < \tau_i < 1$, $i = \overline{1, n}$.

Introduce into consideration the space

$$\mathfrak{L}^{\overline{p},\overline{r},\alpha}_{\overline{\gamma},\overline{\lambda}}\left(\mathbb{R}^n_+\right) = \left\{f : f \in \mathfrak{L}^{\overline{r}}_{\overline{\lambda}},\ \lim_{\delta\to 0} D^{\alpha,l}_{+\cdots+,\mu;\delta}f = \varphi,\ \ \varphi \in \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\ \right\},$$

where $\gamma_i \geq 0$, $\lambda_i \geq 0, 1 \leq p_i, r_i \leq \infty$, $\alpha_i > 0$, $i = 1, ..., n$.

Lemma 9.1. *The operator*

$$\left(B^\alpha_h\varphi\right)(x) = \int\limits_0^1 \cdots \int\limits_0^1 \left(\Delta^\alpha_{\ln\frac{1}{h}} k^+_\alpha\right)\left(\ln\frac{1}{t}\right)\varphi(x \circ t)\,\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}, \tag{9.1}$$

where $\left(\Delta^\alpha_\xi k^+_\alpha\right)(y) = \Delta^{\alpha_1}_{\xi_1}[\Delta^{\alpha_2}_{\xi_2}\ldots(\Delta^{\alpha_n}_{\xi_n}k^+_\alpha)](y), k^+_\alpha(y) = \prod\limits_{i=1}^n \frac{(y_i)_+^{\alpha_i-1}}{\Gamma(\alpha_i)}$, *is bounded in the space* $\mathfrak{L}^{\overline{p}}_{\overline{\gamma}}$ *for every* $1 \leq p_i \leq \infty$, $\alpha_i > 0, \gamma_i \geq 0$, $0 < h_i < 1$, $i = 1, \ldots, n$, *and*

$$\left\|B^\alpha_h\varphi;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\ \right\| \leq C\prod_{i=1}^n\left(\ln\frac{1}{h_i}\right)^{\alpha_i}\left\|\varphi;\ \mathfrak{L}^{\overline{p}}_{\overline{\gamma}}\ \right\|,$$

where C *does not depend on* h_i, $i = 1, \ldots, n$.

Proof. From (9.1) by the generalized Minkowski inequality, we have

$$\left\| B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \int_0^1 \ldots \int_0^1 \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{1}{t}\right) \left\|\varphi\left(x\circ t\right);\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\| \frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

By (5.1) we obtain

$$\left\| B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \int_0^1 \ldots \int_0^1 \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{1}{t}\right) \prod_{i=1}^{n} t_i^{\gamma_i^*} \frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n} \left\|\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|,$$

where $\gamma_i^*,\ \ i=1,\ldots,n$, are constants from (1.2). The substitution $\ln\frac{1}{t_i} = \xi_i \ln\frac{1}{h_i}, i=1,\ldots,n$ gives

$$\left\| B_h^\alpha \varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}} \right\| \leq \prod_{i=1}^{n}\left(\ln\frac{1}{h_i}\right)^{\alpha_i} \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{n} \left|P_{\alpha_i}\left(z_i\right)\right| dz_1\ldots dz_n \left\|\varphi;\ \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}\right\|, \tag{9.2}$$

where $P_{\alpha_i}\left(z_i\right) = \frac{1}{\Gamma(\alpha_i)}\sum\limits_{j_i=1}^{\infty}(-1)^{j_i}\binom{\alpha_i}{j_i}\left(z_i-j_i\right)_+^{\alpha_i-1} \in L_1\left(\mathbb{R}_+^1\right)$ (see 20, p. 282). So, inequality (9.1) follows from (9.2). □

Theorem 9.1. *Let* $f \in \mathfrak{L}_{\overline{\gamma},\overline{\lambda}}^{\overline{p},\overline{r},\alpha}\left(\mathbb{R}_+^n\right)$, $\gamma_i \geq 0,\ \ \lambda_i \geq 0,\ 1 \leq p_i, r_i < \infty,\ \ \alpha_i > 0,\ \ i=1,\ldots,n$. *Then for the mixed fractional difference* $\left(\tilde{\Delta}_h^\alpha f\right)(x)$, *at fixed* $h=(h_1,\ldots,h_n)$, $0<h_i<1,\ i=1,\ldots,n$, *the following integral representation is true*

$$\left(\tilde{\Delta}_h^\alpha f\right)(x) = \int_0^\infty \ldots \int_0^\infty \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\left(D_{+\cdots+}^\alpha f\right)(t)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}, \tag{9.3}$$

where $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y) = \Delta_{\xi_1}^{\alpha_1}[\Delta_{\xi_2}^{\alpha_2}\ldots(\Delta_{\xi_n}^{\alpha_n} k_\alpha^+)](y)$, $k_\alpha^+(y) = \prod\limits_{i=1}^{n}\frac{(y_i)_+^{\alpha_i-1}}{\Gamma(\alpha_i)}$.

Proof. Consider the operator

$$\left(B_h^\alpha \varphi\right)(x) = \int_0^\infty \ldots \int_0^\infty \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\varphi(t)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

Since $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y) \in L_1\left(\mathbb{R}^n\right)$, the operator $B_h^\alpha\varphi$ is bounded in the space $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ in virtue of Lemma 9.1. Denote $\varphi_\delta = D_{+\cdots+,\mu;\delta}^\alpha f$ and

$$\left(B_h^\alpha \varphi_\delta\right)(x) = \int_0^\infty \ldots \int_0^\infty \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\left(D_{+\cdots+,\delta}^\alpha f\right)(t)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

Note, that B_h^α is a convolution with the summable kernel $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y) \in L_1\left(\mathbb{R}^n\right)$ and so the composition $B_h^\alpha D_{+\cdots+,\delta}^\alpha f$ is (at fixed $\delta=(\delta_1,\ldots,\delta_n)$, $\delta_i>0$, $i=\overline{1,n}$) a bounded operator in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ at all $\lambda_i \geq 0,\ 1 \leq r_i < \infty,\ \ i=1,\ldots,n$. Prove presentation (9.3) first for $f \in C_0^\infty\left(\mathbb{R}_+^n\right)$. We have

$$\left(B_h^\alpha \varphi_\delta\right)(x) = \frac{1}{\aleph\left(\alpha,l\right)}\int_0^\infty \ldots \int_0^\infty \left(\Delta_{\ln\frac{1}{h}}^{\alpha} k_\alpha^+\right)\left(\ln\frac{x}{t}\right)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}\times$$

$$\times\int\limits_0^{1-\delta_1}\ldots\int\limits_0^{1-\delta_n}\prod_{i=1}^{n}\left(\ln\frac{1}{t_i}\right)^{-\alpha_i-1}\left(\tilde{\Delta}_y^l f\right)(t)\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}. \tag{9.4}$$

Since $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y)\in L_1(\mathbb{R}^n)$, then in (9.4) the change of the order of integration is justified by the Fubini's theorem. So

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^{1-\delta_1}\ldots\int\limits_0^{1-\delta_n}\prod_{i=1}^{n}\left(\ln\frac{1}{t_i}\right)^{-\alpha_i-1}\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}\times$$

$$\times\sum_{0\le|k|\le l}(-1)^{|k|}\left(\begin{array}{c}l\\k\end{array}\right)\int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\left(\Delta_{\ln\frac{1}{h}}^\alpha k_\alpha^+\right)\left(\ln\frac{x}{t}\right)f\left(y^k\circ t\right)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}.$$

The substitution $t_i=x_i\cdot\xi_i\cdot y_i^{-k_i}\cdot h_i^{j_i}$, $i=1,\ldots,n$ gives

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^{1-\delta_1}\ldots\int\limits_0^{1-\delta_n}\prod_{i=1}^{n}\left(\ln\frac{1}{t_i}\right)^{-\alpha_i-1}\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}\sum_{0\le|k|\le l}(-1)^{|k|}\left(\begin{array}{c}l\\k\end{array}\right)\times$$

$$\times\int\limits_0^{y_1^{k_1}}\ldots\int\limits_0^{y_n^{k_n}}\left(\ln\frac{y^k}{\xi}\right)^{\alpha-1}\sum_{0\le|j|\le l}(-1)^{|j|}\left(\begin{array}{c}\alpha\\j\end{array}\right)f\left(x\circ h^j\circ t\right)\frac{d\xi_1}{\xi_1}\ldots\frac{d\xi_n}{\xi_n}.$$

Hence,

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^{1}\ldots\int\limits_0^{1}\left(\tilde{\Delta}_h^\alpha f\right)f(x\circ\xi)\frac{d\xi_1}{\xi_1}\ldots\frac{d\xi_n}{\xi_n}\times$$

$$\times\int\limits_0^{1-\delta_1}\ldots\int\limits_0^{1-\delta_n}\prod_{i=1}^{n}\left(\ln\frac{1}{t_i}\right)^{-\alpha_i-1}\left(\Delta_{\ln\frac{1}{y}}^l k_\alpha^+\right)\left(\ln\frac{1}{\xi}\right)\frac{dy_1}{y_1}\ldots\frac{dy_n}{y_n}.$$

Here, the change of the order of integration is possible on the basis of Fubini's theorem, since $\left(\Delta_\xi^\alpha k_\alpha^+\right)(y)\in L_1\left(\mathbb{R}^n,\frac{dx}{x}\right)$. Substituting $\ln\frac{1}{y_i}=\frac{1}{s_i}\cdot\ln\frac{1}{\xi_i}$, $\ln\frac{1}{\xi_i}=u_i\cdot\ln\frac{1}{1-\delta_i}$, $i=1,\ldots,n$, we obtain

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\left(\tilde{\Delta}_h^\alpha f\right)f\left(x\circ(1-\delta)^u\right)\frac{du_1}{u_1}\ldots\frac{du_n}{u_n}\times$$

$$\times\int\limits_0^{u_1}\ldots\int\limits_0^{u_n}\left(\Delta_1^l k_\alpha^+\right)(s)\,ds_1\cdots ds_n. \tag{9.5}$$

The equality $\left(\Delta_1^l k_\alpha^+\right)(s)=\left(\Delta_1^{l_1}k_{\alpha_1}^+\right)(s_1)\cdots\left(\Delta_1^{l_n}k_{\alpha_n}^+\right)(s_n)$, from (9.5), we have

$$\left(B_h^\alpha\varphi_\delta\right)(x)=\int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\left(K_{l_1,\alpha_1}^{+1}\right)(u_1)\ldots\left(K_{l_n,\alpha_n}^{+}\right)(u_1)\left(\tilde{\Delta}_h^\alpha f\right)\left(x\circ(1-\delta)^u\right)du_1\cdots du_n, \tag{9.6}$$

where $K_{l_i,\alpha_i}^{+,\mu_i}$ is kernel (7.4). In (7.4) the right-hand side in (9.6) is an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ by Lemma 5.4. Since $B_h^\alpha D_{+\cdots+,\delta}^\alpha f$ is also an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, then (9.6) follows for functions

f belonging to $C_0^\infty\left(\mathbb{R}_+^n\right)$. Therefore, from (9.6), by passing to the limit at $\delta \to 0$, identity (9.3) is obtained.

Since $\varphi = \lim\limits_{\delta\to 0} \varphi_\delta$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, the left-hand side of (9.6) converges in norm $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ due to the boundedness of operator B_h^α in the $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$. On the other hand, the right-hand side in (9.6) converges at $\delta \to 0$ to $\left(\tilde{\Delta}_h^\alpha f\right)(x)$ in norm $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ by virtue of Lemma 5.5.

Due to the identical coincidence of the left-hand and right-hand sides in (9.6), their limits at $\delta \to 0$, although in different norms $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, must coincide almost everywhere. This leads to (9.3). □

Theorem 9.2. *In order $f(x)$ to be representable by a mixed fractional Hadamard integral $f(x) = \left(J_{+\cdots+}^\alpha \varphi\right)(x)$, $\varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, where either*

1) $\gamma_i > 0$, $\alpha_i > 0$, $1 \le p_i < \infty$, $i = 1, \ldots, n$,

or

2) $\gamma_i = 0$, $0 < \alpha_i < 1$, $1 < p_i < \frac{1}{\alpha_i}$, $i = 1, \ldots, n$,

it is necessary and sufficient that $f \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, where $\lambda_i > 0$, $1 \le r_i < \infty$, $i = 1, \ldots, n$, in case 1) or $\lambda_i = 0$, $r_i = \frac{p_i}{1-\alpha_i p_i}$ in case 2) and the limit exists $\varphi = \lim\limits_{\delta\to 0} D_{+\cdots+,\delta}^\alpha f$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$.

Proof. The necessity of this theorem follows from Theorem 6.2 and Theorem 8.2.The sufficiency is obtained by the scheme of the proof of Theorem 9.1. □

Theorem 9.3. *In order $\left(\tilde{\Delta}_\tau^l f\right)(x)$ to be representable by a mixed fractional Hadamard integral $\left(\tilde{\Delta}_\tau^l f\right)(x) = J_{+\cdots+,\tau}^{\alpha,l}\varphi$, $\varphi \in \mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, where $\gamma_i \ge 0$, $l > \alpha_i > 0$, $1 \le p_i \le \infty$, $0 < \tau_i < 1$, $i = 1, \ldots, n$ it is necessary and sufficient that, $\left(\tilde{\Delta}_\tau^l f\right)(x) \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$, where $\lambda_i \ge 0$, $1 \le r_i < \infty$, $0 < \tau_i < 1$, $i = 1, \ldots, n$ and the limit exists*

$$\varphi = \lim_{\delta\to 0} D_{+\cdots+,\delta}^\alpha f, \tag{9.7}$$

where the limit is in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$.

Proof. The necessity of this theorem follows from Theorem 6.3 and Theorem 8.3.

Sufficiency. Let $\left(\tilde{\Delta}_\tau^l f\right)(x) \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ and condition (9.7) be satisfied. It is required to prove that

$$\left(\tilde{\Delta}_\tau^l f\right)(x) = J_{+\cdots+,\tau}^{\alpha,l}\varphi. \tag{9.8}$$

From (9.8) we have

$$\tilde{\Delta}_h^\alpha \left(\tilde{\Delta}_\tau^l f\right)(x) = \tilde{\Delta}_h^\alpha J_{+\cdots+,\tau}^{\alpha,l}\varphi. \tag{9.9}$$

At $h = (h_1, \ldots, h_n)$, $0 < h_i < 1$, $i = 1, \ldots, n$. Introduce the notation

$$\left(B_h^{\alpha,l}\varphi\right)(x) = \left(\ln\frac{1}{h}\right)^\alpha \int_0^\infty \ldots \int_0^\infty P_\alpha(z) \left(\tilde{\Delta}_\tau^l \varphi\right)(x \circ h^z)\, dz_1 \ldots dz_n,$$

where $P_\alpha(z) = \frac{1}{\Gamma(\alpha)} \sum\limits_{0\le|j|<\infty} (-1)^{|j|} \begin{pmatrix} \alpha \\ j \end{pmatrix} (z-j)^{\alpha-1} \in L_1(\mathbb{R}^n)$.

Consider the expression $B_h^{\alpha,l}\varphi_\delta$, $\varphi_\delta = D_{+\cdots+,\delta}^\alpha f$. For functions $f(x)$, belonging to $C_0^\infty\left(\mathbb{R}_+^n\right)$, we have

$$B_h^{\alpha,l}\varphi_\delta = B_h^{\alpha,l} D_{+\cdots+,\delta}^\alpha f = \tilde{\Delta}_h^\alpha \tilde{\Delta}_\tau^l J_{+\cdots+,\delta}^\alpha D_{+\cdots+,\delta}^\alpha f. \tag{9.10}$$

With well-known integral representation of finite differences (see [17], p. 101-102), we obtain

$$(\tilde{\Delta}_{\tau}^{l} J_{+\cdots+,\delta}^{\alpha} D_{+\cdots+,\delta}^{\alpha} f)(x) =$$

$$= \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(K_{l_1,\alpha_1}^{+}\right)(y_1) \ldots \left(K_{l_n,\alpha_n}^{+}\right)(y_n) \left(\tilde{\Delta}_{\tau}^{l} f\right)(x \circ (1-\delta)^{y})\, dy_1 \ldots dy_n,$$

where $\left(K_{l_i,\alpha_i}^{+}\right)(y_i)$ is kernel (7.4). Then from (9.10) we have

$$B_h^{\alpha,l} \varphi_\delta = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(K_{l_1,\alpha_1}^{+1}\right)(y_1) \ldots \left(K_{l_n,\alpha_n}^{+}\right)(y_n) \left(\tilde{\Delta}_{h}^{\alpha} \tilde{\Delta}_{\tau}^{l} f\right)(x \circ (1-\delta)^{y})\, dy_1 \ldots dy_n. \tag{9.11}$$

With (7.4) in mind, the right-hand side in (9.11) is an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$. According to Lemma 5.4, since the composition $B_h^{\alpha,l} D_{+\cdots+,\delta}^{\alpha} f$ is (at fixed $\delta = (\delta_1, \ldots, \delta_n)$, $0 < \delta_i < 1$, $i = 1, \ldots, n$) an operator bounded in $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ for $\lambda_i \geq 0$, $1 \leq r_i < \infty$, $i = 1, \ldots, n$, then (9.11) follows for functions f belonging to $C_0^{\infty}\left(\mathbb{R}_{+}^{n}\right)$. By (7.5) the right-hand side in (9.11) converges in the norm of the space $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ to $\left(\tilde{\Delta}_{h}^{\alpha} \tilde{\Delta}_{\tau}^{l} f\right)(x)$.

So, there exists a limit of the left-hand side

$$\lim_{\delta \to 0} B_h^{\alpha,l} D_{+\cdots+,\delta}^{\alpha} f = \left(\tilde{\Delta}_{h}^{\alpha} \tilde{\Delta}_{\tau}^{l} f\right)(x).$$

Since $\varphi = \lim_{\delta \to 0} \varphi_\delta$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, the left-hand side of (9.11) converges in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$ due to the boundedness of the operator $B_h^{\alpha,l}$ in $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$. Then there exists a limit

$$\lim_{\delta \to 0} B_h^{\alpha,l} D_{+\cdots+,\delta}^{\alpha} f = B_h^{\alpha,l} \lim_{\delta \to 0} (D_{+\cdots+,\delta}^{\alpha} f) = B_h^{\alpha,l} \varphi, \tag{9.12}$$

where $\varphi = D_{+\cdots+}^{\alpha} f$. Since $B_h^{\alpha,l} D_{+\cdots+,\delta}^{\alpha} f$ converges both in the norm $\mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$ and norm $\mathfrak{L}_{\overline{\gamma}}^{\overline{p}}$, the limiting functions must coincide almost everywhere. Then from (9.12), we obtain

$$B_h^{\alpha,l} \varphi = \left(\tilde{\Delta}_{h}^{\alpha} \tilde{\Delta}_{\tau}^{l} f\right)(x),$$

which coincides with (9.9). It should be noted that functions $\tilde{\Delta}_{\tau}^{l} f$and $J_{+\cdots+,\tau}^{\alpha,l} D_{+\cdots+}^{\alpha} f$ have identically coinciding mixed finite differences. Therefore, they can differ only by a polynomial (see [19], p. 103)

$$\tilde{\Delta}_{\tau}^{l} f = J_{+\cdots+,\tau}^{\alpha,l} D_{+\cdots+}^{\alpha} f + P(x),$$

where $P(x)$ is a polynomial. Then from (9.9) follows (9.8) taking into account that $\tilde{\Delta}_{\tau}^{l} f$, $J_{+\cdots+,\tau}^{\alpha,l} \varphi \in \mathfrak{L}_{\overline{\lambda}}^{\overline{r}}$. □

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