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KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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BOUNDEDNESS OF THE GENERALIZED RIEMANN-LIOUVILLE OPERATOR IN LOCAL MORREY-TYPE SPACES

M.A. Senouci

Communicated by M.L. Goldman

Key words: Morrey-type spaces, generalized Riemann-Liouville operator, boundedness.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. The objective of this paper is to establish the boundedness of the generalized multi-dimensional Riemann-Liouville integral operator from one local Morrey-type space to another one under some conditions on numerical parameters p and q.

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1 Introduction

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < p \leq \infty, 0 \leq \lambda \leq \dfrac{n}{p}$. We denote by $LM_p^{\lambda}(\Omega)$, local Morrey-type spaces, the spaces of all functions $f \in L_p^{loc}\Omega$ with finite quasi-norms

$$\|f\|_{LM_p^{\lambda}(\Omega)} = \sup_{r>0} r^{-\lambda} \|f\|_{L_p(\Omega \bigcap B(0,r))} < \infty.$$

For properties of Morrey-type spaces, introduced in [7], see, for example, [1]-[5].

Definition 2. The left multidimensional fractional Riemann-Liouville integral operator $I_{a+}^{\overline{\alpha}} f$ of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $i = 1, ..., n$, $a = (a_1, ..., a_n) \in \mathbb{R}^n$, is defined as follows

$$\left(I_{a+}^{\overline{\alpha}} f\right)(x) = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_{a_n}^{x_n} ... \int_{a_1}^{x_1} \left(\prod_{i=1}^n (x_i - t_i)^{\alpha_i - 1}\right) f(t_1, ..., t_n) dt_1 ... dt_n \tag{1.1}$$

for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $x_i > a_i$, $i = 1, .., n$, where Γ is the Euler Gamma-function.

Let $\tau_i = t_i - a_i$, then

$$\left(I_{a+}^{\overline{\alpha}} f\right)(x)$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^{x_n - a_n} ... \int_0^{x_1 - a_1} \left(\prod_{i=1}^n (x_i - a_i - \tau_i)^{\alpha_i - 1}\right) f(\tau_1 + a_1, ..., \tau_n + a_n) d\tau_1 ... d\tau_n$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^{x_n - a_n} ... \int_0^{x_1 - a_1} \left(\prod_{i=1}^n (x_i - a_i - \tau_i)^{\alpha_i - 1}\right) g(\tau_1, ..., \tau_n) d\tau_1 ... d\tau_n$$

$$= (I_{0+}^{\overline{\alpha}} g)(x_1 - a_1, ..., x_n - a_n), \tag{1.2}$$

where $g(\tau_1, ..., \tau_n) = f(\tau_1 + a_1, ..., \tau_n + a_n)$.

The right multidimensional fractional Riemann-Liouville integral operator of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $b = (b_1, ..., b_n) \in \mathbb{R}^n$ is defined similarly:

$$\left(I_{b_-}^{\overline{\alpha}} f\right)(x) = \frac{1}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \int_{x_n}^{b_n} ... \int_{x_1}^{b_1} \left(\prod_{i=1}^{n} (t_i - x_i)^{\alpha_i - 1} \right) f(t_1 ... t_n) dt_1 ... dt_n$$

for all $x \in \mathbb{R}^n$ such that $x_i < b_i, \ i = 1, ..., n$.

Definition 3. Let $f \in L_p(\Omega)$, where $0 < p \leq \infty$, $\overline{k} = (k_1, ..., k_n)$, $k_i \geq 0$, $i = 1, ..., n$. The generalized Riemann-Liouville fractional integral operator $I^{\overline{\alpha}, \overline{k}} f$ of order $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $0 < \alpha_i < 1$, $i = 1, ..., n$, $n \in \mathbb{N}$, is defined by

$$\left(I_{a_+}^{\overline{\alpha}, \overline{k}} f\right)(x)$$

$$= \prod_{i=1}^{n} \frac{(k_i + 1)^{1 - \alpha_i}}{\Gamma(\alpha_i)} \int_{a_n}^{x_n} ... \int_{a_1}^{x_1} \left(\prod_{i=1}^{n} \left[(x_i^{k_i + 1} - t_i^{k_i + 1})^{\alpha_i - 1} t_i^{k_i} \right] \right) f(t_1, ..., t_n) dt_1 ... dt_n \tag{1.3}$$

Remark 1. If $k_i = 0$, $\forall i = 1, ..., n$, $\overline{k} = 0$ in Definition 3; we get the usual Riemann-Liouville integral operator defined by (1.1).

For the operator I_{0+}^{α}, the following theorem was proved in [8].

Let a, $b \in \mathbb{R}^n$, $0 < a_i < b_i < \infty$, $i = 1, ..., n$, and

$$Q(a, b) = \{x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, ..., n\}.$$

Theorem 1.1. *Let $1 < p \leq \infty$, $0 < q \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$, $0 \leq \mu \leq \frac{n}{q}$, $\frac{1}{p} < \alpha_i < 1$, $i = 1, ..., n$. Then there exists $C_1 > 0$ such that*

$$\| I_{a_+}^{\alpha} f \|_{M_q^{\mu}(Q(a,b))} \leq C_1 \mid b - a \mid^{\nu} \| f \|_{M_p^{\lambda}(Q(a,b))}, \tag{1.4}$$

where

$$\nu = \lambda + \alpha_1 + ... + \alpha_n - \frac{n}{p} + \frac{n}{q} - \mu, \tag{1.5}$$

for all finite parallelepipeds $Q(a, b)$ and for all $f \in M_p^{\lambda}(Q(a, b))$.

The exponent ν cannot be replaced by any other one.

2 Main results

Lemma 2.1. ([8]) *Let $0 < p \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$. Then*

$$\| f \|_{L_p(Q(0,y))} \leq |y|^{\lambda} \| f \|_{LM_p^{\lambda}(Q(0,b))} \tag{2.1}$$

for any parallelepipeds $Q(0, b)$ and for any $y \in Q(0, b)$.

Theorem 2.1. *Let $1 < p < \infty$, $0 < q \leq \infty$, $0 \leq \lambda \leq \frac{n}{p}$, $0 \leq \mu \leq \frac{n}{q}$, $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$, $\frac{1}{p} < \alpha_i < 1$, $\overline{k} = (k_1, ..., k_n)$, $k_i \geq 0$, $i = 1, ..., n$.*

Then there exists $C_2 > 0$ such that

$$\| I_{0_+}^{\overline{\alpha}\overline{k}} f \|_{LM_q^{\mu}(Q(a,b))} \leq C_2 \mid b \mid^{\nu} \| f \|_{LM_p^{\lambda}(Q(a,b))}, \tag{2.2}$$

where

$$\nu = \lambda + \frac{n}{q} - \frac{n}{p} + \sum_{i=1}^{n} (k_i + 1)\alpha_i - \mu, \tag{2.3}$$

for all finite parallelepipeds $Q(a, b)$ and for all $f \in LM_p^{\lambda}(Q(a, b))$.

The exponent ν cannot be replaced by any other one.

Proof. Part 1. By (1.1) with $a=0$ and Hölder's inequality it follows that

$$\left|\left(I_{0+}^{\overline{\alpha},\overline{k}}f\right)(x)\right|$$

$$\leq C_3\left\|\prod_{i=1}^{n}(x_i^{k_i+1}-t_i^{k_i+1})^{(\alpha_i-1)}t_i^{k_i}\right\|_{L_{p'}(Q(0,x))}\|f\|_{L_p(Q(0,x))}(t_i=x_i\tau_i)$$

$$\leq C_3\prod_{i=1}^{n}x_i^{\alpha_i(k_i+1)-\frac{1}{p}}\prod_{i=1}^{n}\left(\int_0^1(1-\tau_i^{k_i+1})^{(\alpha_i-1)p'}\tau_i^{k_ip'}d\tau_i\right)^{\frac{1}{p'}}\|f\|_{L_p(Q(0,x))},$$

where

$$C_3=\prod_{j=1}^{n}\frac{(k_j+1)^{1-\alpha_j}}{\Gamma(\alpha_j)}.$$

By changing variable $\tau_i^{k_i+1}=z_i$ and taking into account that

$$(\alpha_i-1)p'=\frac{\alpha_ip-1}{p-1}-1$$

and $\frac{(p'-1)k_i}{k_i+1}=\frac{p'k_i+1}{k_i+1}-1$, we obtain

$$\int_0^1(1-\tau_i^{k_i+1})^{(\alpha_1-1)p'}\tau_i^{k_ip'}d\tau_i=\frac{1}{k_i+1}B\left(\frac{\alpha_ip-1}{p-1},\frac{p'k_i+1}{k_i+1}\right),$$

where $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta-function.

So

$$\left|\left(I_{0+}^{\overline{\alpha},\overline{k}}f\right)(x)\right|\leq C_4\left(\prod_{i=1}^{n}x_i^{(k_i+1)\alpha_i-\frac{1}{p}}\right)\|f\|_{L_p(Q(0,x))},$$

where

$$C_4=C_3\prod_{i=1}^{n}\frac{1}{(k_i+1)^{1-\frac{1}{p}}}\left(B\left(\frac{\alpha_ip-1}{p-1},\frac{p'k_i+1}{k_i+1}\right)\right)^{1-\frac{1}{p}}.$$

By Lemma 2.1, since $(k_i+1)\alpha_i-\frac{1}{p}>0$, we get

$$\left|\left(I_{0+}^{\overline{\alpha},\overline{k}}f\right)(x)\right|\leq C_4\mid x\mid^{\lambda-\frac{n}{p}+\sum_{i=1}^{n}(k_i+1)\alpha_i}\| f\|_{LM_p^{\lambda}(Q(0,b))}$$

and

$$\left\||I_{0+}^{\overline{\alpha},\overline{k}}f\right\|_{L_q(Q(0,b)\bigcap B(0,r))}$$

$$\leq C_4\mid b\mid^{\lambda-\frac{n}{p}+\sum_{i=1}^{n}(k_i+1)\alpha_i}\| 1\|_{L_q(Q(0,b))\bigcap B(0,r))}\| f\|_{LM_p^{\lambda}(Q(0,b))}.$$

We consider two cases:

1) If $r<|b|$, since $0<\mu\leq\frac{n}{q}$, then

$$r^{-\mu}\left\|I_{0+}^{\overline{\alpha},\overline{k}}f\right\|_{L_q(Q(0,b)\bigcap B(0,r))}$$

$$\leq C_5\mid b\mid^{\lambda-\frac{n}{p}+\frac{n}{q}-\mu+\sum_{i=1}^{n}(k_i+1)\alpha_i}\| f\|_{LM_p^{\lambda}(Q(0,b))},$$

where

$$C_5 = C_4 \nu_n^{\frac{n}{q}}.$$

2) If $r \geq |b|$, then

$$r^{-\mu} \left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{L_q(Q(0,b) \bigcap B(0,r))}$$
$$\leq C_4 \mid b \mid^{\lambda - \frac{n}{p} + \frac{n}{q} - \mu + \sum_{i=1}^{n}(k_i+1)\alpha_i} \| f \|_{LM_p^{\lambda}(Q(0,b))} .$$

Hence (2.2) follows.

Part 2. Suppose that $I_{0_+}^{\overline{\alpha},\overline{k}} f$ is bounded from $LM_p^{\lambda}(Q(0,b))$ to $LM_q^{\mu}(Q(0,b))$, that is: for some $C_5(b) > 0$ depending on b, p, q, λ, μ and k.

$$\left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^{\mu}(Q(0,b))} \leq C_6(b) \| f \|_{LM_p^{\lambda}(Q(0,b))} . \tag{2.4}$$

Assume that $b_1 = ... = b_n = \beta$, then $\beta = \frac{|b|}{\sqrt{n}}$

Let

$$f(x) = \begin{cases} 1 & x \in Q(\frac{b}{2}, b) \\ 0 & x \in Q(0,b) \backslash Q(\frac{b}{2}, b). \end{cases}$$

$$\|f\|_{L_p(Q(0,b) \bigcap B(0,r))} = \| f \|_{L_p(Q(0,b) \backslash Q(0,\frac{b}{2}) \bigcap B(0,r))}$$

$$= \| 1 \|_{L_p(Q(\frac{b}{2},b) \bigcap B(0,r))} = \left| Q\left(\frac{b}{2}, b\right) \bigcap B(0,r) \right|^{\frac{1}{p}} \leq \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{p}} = \left(\frac{|b|}{2} \right)^{\frac{n}{p}} \tag{2.5}$$

and

$$\|f\|_{LM_p^{\lambda}(Q(\frac{b}{2},b) \bigcap B(0,r))} \leq \sup_{r \geq \frac{|b|}{2}} r^{-\lambda} \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{p}} = C_7 |b|^{\frac{n}{p} - \lambda},$$

where

$$C_7 = 2^{\lambda - \frac{n}{p}}.$$

Let estimate $\left\| I_{a_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^{\mu}(Q(\frac{b}{2},b))}$, so

$$\left\| I_{a_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^{\mu}(Q(\frac{b}{2},b))}$$
$$\geq C_3 \left\| \int_0^{x_n} ... \int_0^{x_1} \prod_{i=1}^{n} (x_i^{k_i+1} - t_i^{k_i+1})^{\alpha_i - 1} t_i^{k_i} dt_1 ... dt_n \right\|_{LM_q^{\mu}(Q(\frac{b}{2},b))} \quad (t_i = x_i \tau_i)$$
$$\geq C_8 r^{-\mu} \left\| \prod_{i=1}^{n} x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b) \bigcap B(0,r))} \Bigg|_{r=|b|} = C_8 |b|^{-\mu} \left\| \prod_{i=1}^{n} x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b))},$$

where

$$C_8 = C_3 \beta^{\frac{1}{q}} (k_i + 1, \alpha_i).$$

$$\left\| \prod_{i=1}^{n} x_i^{\alpha_i(k_i+1)} \right\|_{L_q(Q(\frac{b}{2},b))}$$
$$\geq \left(\frac{|b|}{2} \right)^{\sum_{i=1}^{n}(k_i+1)\alpha_i} \left| Q\left(\frac{b}{2}, b\right) \right|^{\frac{1}{q}} = 2^{-(\frac{n}{q} + \sum_{i=1}^{n}(k_i+1)\alpha_i)} |b|^{\frac{n}{q} + \sum_{i=1}^{n}(k_i+1)\alpha_i}$$
$$\geq C_9 |b|^{\frac{n}{q} + \sum_{i=1}^{n}(k_i+1)\alpha_i},$$

where

$$C_9 = 2^{-(\frac{n}{q}+\sum_{i=1}^{n}(k_i+1)\alpha_i)}.$$

Consequently

$$\left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^\mu(Q(0,b))} \geq C_{10}|b|^{\frac{n}{q}-\mu+\sum_{i=1}^{n}(k_i+1)\alpha_i}, \tag{2.6}$$

where $C_{10} = C_8 C_9$.

Finally, by (2.4), (2.5) and (2.6), we get

$$C_{10}|b|^{\frac{n}{q}-\mu+\sum_{i=1}^{n}(k_i+1)\alpha_i} \leq \left\| I_{0_+}^{\overline{\alpha},\overline{k}} f \right\|_{LM_q^\mu(Q(0,b))} \leq C_6 C_7 |b|^{\frac{n}{p}-\lambda},$$

where $C_{10} = C_8 C_9$.

Hence

$$C_6(b) \geq \frac{C_1 0}{C_7}|b|^{\nu},$$

where ν is defined by (2.3). □

Acknowledgments

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