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# EURASIAN MATHEMATICAL JOURNAL

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## KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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# TWO–DIMENSIONAL BILINEAR INEQUALITY FOR RECTANGULAR HARDY OPERATOR AND NON–FACTORIZABLE WEIGHTS

**R. Sengupta, E.P. Ushakova**

Communicated by V.D. Stepanov

**Key words:** rectangular Hardy operator, bilinear inequality, weighted Lebesgue norm.

**AMS Mathematics Subject Classification:** 26D10, 47G10.

**Abstract.** Necessary conditions and sufficient conditions are given for the validity of two–dimensional bilinear norm inequalities with rectangular Hardy operators in weighted Lebesgue spaces. The results are applicable for non–factorizable weights.

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## 1 Introduction

Let $\mathfrak{M}$ be the set of all Lebesgue measurable functions on $\mathbb{R}^2_+ := (0,\infty)^2$, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ be the subset of all non-negative functions.

For fixed parameters $1 < p_1, p_2, q < \infty$ and weight functions $u, v_1, v_2 \in \mathfrak{M}^+$, we consider the problem of characterizing of the bilinear Hardy inequality

$$\left(\int_0^\infty\int_0^\infty (I_2 f)^q\,(I_2 g)^q\,u\right)^{\frac{1}{q}} \le C\left(\int_0^\infty\int_0^\infty f^{p_1} v_1\right)^{\frac{1}{p_1}}\left(\int_0^\infty\int_0^\infty g^{p_2} v_2\right)^{\frac{1}{p_2}} \tag{1.1}$$

for all $f, g \in \mathfrak{M}^+$, where

$$I_2 f(x_1,x_2) := \int_0^{x_1}\int_0^{x_2} f(t_1,t_2)\,dt_1\,dt_2$$

is the two–dimensional Hardy operator. Here, $C > 0$ is supposed to be the best (least possible) constant that does not depend on $f$ and $g$.

Integral transforms, which map a product of function spaces into another function space (multi–linear integral operators), have applications, in particular, to smoothness properties and approximation of function classes (see e.g. [13] and references therein). In the one–dimensional case a multi–linear analogue of (1.1) was considered in [3, 4] as an illustration of results about multi–linear inequalities. Other types of one–dimensional linear and bilinear integral operators in Lebesgue spaces and subclasses were studied in [1, 2, 5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 21]. For product type weight functions (or factorizable weights) inequality (1.1) was completely studied in [16].

The goal of our work is to solve the same problem without such restrictions on weight functions.

The paper is organized as follows. In Section 2, we review auxiliary results which pertain to estimates of the best constant $C$ in weighted two-dimensional linear Hardy inequality (2.1). In Section 3, the results are given on characterization of the best constant $C$ in bilinear Hardy inequality (1.1). We consider thirteen cases depending on relations between the norm parameters $p_1, p_2$ and $q$.

The subsections correspond those relations between the numerical parameters for which the proofs of estimates are similar.

The dual operator to $I_2$ is defined as

$$I_2^* f(x_1, x_2) := \int_{x_1}^{\infty}\int_{x_2}^{\infty} f(t_1, t_2) dt_1 dt_2.$$

The results of this paper for the operator $I_2$ can be proved in a similar way for the operator $I_2^*$.

Throughout the paper, products of the form $0 \cdot \infty$ are taken to be equal to 0. By $A \lesssim B$ we mean that there exists $k > 0$, which depends only on some insignificant numerical parameters, such that $A \le kB$. If $A \lesssim B$ and $B \lesssim A$ then we write $A \approx B$. If $p > 1$ then $p' = p/(p-1)$.

## 2 Auxiliary results

Let us first recall Sawyer's Theorem (see [12, Theorem A] or [18, Theorem 1]).

**Theorem 2.1.** *Let $1 < p \le q < \infty$ and $w, v \in \mathfrak{M}^+$ be weights. Then the inequality*

$$\left(\int_0^{\infty}\int_0^{\infty} (I_2 f)^q\, w\right)^{\frac{1}{q}} \le C\left(\int_0^{\infty}\int_0^{\infty} f^p v\right)^{\frac{1}{p}} \tag{2.1}$$

*holds for some $C > 0$ and for all $f \in \mathfrak{M}^+$ if and only if*

$$D_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \big(I_2^* w(t_1, t_2)\big)^{\frac{1}{q}} \big(I_2\sigma(t_1, t_2)\big)^{\frac{1}{p'}} < \infty,$$

$$D_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (I_2\sigma)^q w\right)^{\frac{1}{q}} \big(I_2\sigma(t_1, t_2)\big)^{-\frac{1}{p}} < \infty,$$

$$D_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} (I_2^* w)^{p'} \sigma\right)^{\frac{1}{p'}} \big(I_2^* w(t_1, t_2)\big)^{-\frac{1}{q'}} < \infty,$$

*where $\sigma := v^{1-p'}$. Moreover, if $C$ is the best constant in* (2.1)*, then*

$$C \approx D_1 + D_2 + D_3. \tag{2.2}$$

Now, if $p < q$ there is an alternative estimate ([18], Theorem 2).

**Theorem 2.2.** *Let $1 < p < q < \infty$ and $w, v \in \mathfrak{M}^+$ be weights. Then inequality* (2.1) *holds for some $C > 0$ and for all $f \in \mathfrak{M}^+$ if and only if $D_1 < \infty$. Moreover, if $C$ is the best constant in* (2.1)*, then*

$$C \approx D_1. \tag{2.3}$$

For the case $q < p$ the following results are known [20], Theorem 3).

**Theorem 2.3.** *Let $1 < q < p < \infty$, $1/r := 1/q - 1/p$ and let $w, v \in \mathfrak{M}^+$ be weights. Then inequality* (2.1) *holds for some $C > 0$ and for all $f \in \mathfrak{M}^+$ if*

$$B_v := \left(\int_0^{\infty}\int_0^{\infty} \sigma(u, z)\left(\int_u^{\infty}\int_z^{\infty} (I_2\sigma)^{q-1} w\right)^{\frac{r}{q}} du\, dz\right)^{\frac{1}{r}} < \infty.$$

*Reversely, if inequality* (2.1) *true then* $B < \infty$, *where*

$$B := \left( \int_0^\infty \int_0^\infty d_y \big( I_2\sigma(x,y) \big)^{\frac{r}{p'}} d_x \Big( - \big( I_2^* w(x,y) \big)^{\frac{r}{q}} \Big) \right)^{\frac{1}{r}}$$

$$= \left( \int_0^\infty \int_0^\infty \big( I_2\sigma(x,y) \big)^{\frac{r}{p'}} d_x\, d_y \big( I_2^* w(x,y) \big)^{\frac{r}{q}} \right)^{\frac{1}{r}}$$

$$= \left( \int_0^\infty \int_0^\infty \big( I_2^* w(x,y) \big)^{\frac{r}{q}} d_x\, d_y \big( I_2\sigma(x,y) \big)^{\frac{r}{p'}} \right)^{\frac{1}{r}}.$$

*Moreover, if* $C$ *is the best constant in* (2.1), *then*

$$B \lesssim C \lesssim B_v.$$

**Theorem 2.4.** *Let* $1 < q < p < \infty$ *and* $w, v \in \mathfrak{M}^+$ *be weights. Assume that the following is satisfied:*

(1) *there exists* $\gamma \in [\frac{q}{p}, 1)$ *such that* $\dfrac{\partial^2 \Big( \big[ I_2\sigma(x,y) \big]^{\gamma} \Big)}{\partial x \partial y} \geq 0$ *for almost all* $(x,y) \in \mathbb{R}_+^2$;

(2) *there exists* $\gamma^* \in [\frac{p'}{q'}, 1)$ *such that* $\dfrac{\partial^2 \Big( \big[ I_2 w(x,y) \big]^{\gamma^*} \Big)}{\partial x \partial y} \geq 0$ *for almost all* $(x,y) \in \mathbb{R}_+^2$.

*Then inequality* (2.1) *holds for some* $C > 0$ *and for all* $f \in \mathfrak{M}^+$ *if and only if* $B < \infty$. *Moreover, if* $C$ *is the best constant in* (2.1), *then*

$$C \approx B. \tag{2.4}$$

# 3 Main results

We denote $\sigma_i := v_i^{1-p_i'}$ and $V_i := I_2\sigma_i$, where $i = 1, 2$.

There are thirteen ways to arrange three numbers $p_1, p_2, q$ considering that some of them may be equal. We will break the thirteen cases into subcases based upon similarity of the proof.

## 3.1 Case $\max(p_1, p_2) \leq q$

The following cases arise when $\max(p_1, p_2) \leq q$:

1) $p_1 < p_2 = q$,

2) $p_2 < p_1 = q$,

3) $\max\{p_1, p_2\} < q$,

4) $p_1 = p_2 = q$.

**Theorem 3.1.** *Let* $p_1 \neq p_2$ *and* $\max(p_1, p_2) = q$ *or* $\max\{p_1, p_2\} < q$. *Assume* $w, v \in \mathfrak{M}^+$ *are weights. Then the best constant* $C$ *in inequality* (1.1) *can be estimated as:*

1) $C \approx \mathcal{A}_1$, *if* $p_1 < p_2 = q$, *where*

$$\mathcal{A}_1 := \sup_{(x,y)\in\mathbb{R}_+^2} \big( V_1(x,y) \big)^{\frac{1}{p_1'}} \Big( \widetilde{D}_1(x,y) + \widetilde{D}_2(x,y) + \widetilde{D}_3(x,y) \Big)$$

*and* $\widetilde{D}_i(x,y)$, $i = 1, 2, 3$ *are defined by equations* (3.3), (3.4), (3.5), *respectively;*

2) $C \approx \mathcal{A}_2$, *if* $p_2 < p_1 = q$, *where*

$$\mathcal{A}_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y) + \widehat{D}_2(x,y) + \widehat{D}_3(x,y)\Big)$$

*and* $\widehat{D}_i(x,y)$, $i = 1,2,3$ *are defined by equations* (3.6), (3.7), (3.8), *respectively;*

3) $C \approx \min\big\{\mathcal{B}_1, \mathcal{B}_2\big\}$, *if* $p_1 < p_2 < q$ *or* $p_2 < p_1 < q$ *or* $p_1 = p_2 < q$, *where*

$$\mathcal{B}_1 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y)\Big),$$

$$\mathcal{B}_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y)\Big).$$

*Proof.* For a given weight $v \in \mathfrak{M}^+$ and a fixed parameter $p > 1$ denote by

$$\|h\|_{p,v} := \left(\int_0^\infty\!\!\int_0^\infty |h|^p v\right)^{\frac{1}{p}}$$

the weighted Lebesgue norm of $h$. Then we have the following equality for the best constant $C$ in (1.1):

$$C = \sup_{g\neq 0}\sup_{f\neq 0} \frac{\left(\int_0^\infty\!\!\int_0^\infty (I_2 f)^q\,(I_2 g)^q\,u\right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}}. \tag{3.1}$$

1) Consider the case $p_1 < p_2 = q$. By virtue of (2.3) and (2.2),

$$C \overset{(2.3)}{\approx} \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2} \sup_{(x,y)\in\mathbb{R}^2_+} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \Big(I_2 v_1^{1-p_1'}(x,y)\Big)^{\frac{1}{p_1'}} \tag{3.2}$$

$$= \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left(\int_0^\infty\!\!\int_0^\infty (I_2 g)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2}$$

$$\overset{(2.2)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y) + \widetilde{D}_2(x,y) + \widetilde{D}_3(x,y)\Big),$$

where

$$\widetilde{D}_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}}, \tag{3.3}$$

$$\widetilde{D}_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\!\!\int_0^{t_2} (V_2)^q \chi_{(x,\infty)\times(y,\infty)} u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}}, \tag{3.4}$$

$$\widetilde{D}_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^\infty\!\!\int_{t_2}^\infty \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)\Big)^{p_2'} \sigma_2\right)^{\frac{1}{p_2'}}}{\Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}. \tag{3.5}$$

2) The proof for the case $p_2 < p_1 = q$ is analogous to case 1). In this case

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y) + \widehat{D}_2(x,y) + \widehat{D}_3(x,y)\Big),$$

where

$$\widehat{D}_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}}, \tag{3.6}$$

$$\widehat{D}_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int\limits_0^{t_1}\int\limits_0^{t_2} (V_1)^q \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}}, \tag{3.7}$$

$$\widehat{D}_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int\limits_{t_1}^{\infty}\int\limits_{t_2}^{\infty} \Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)\Big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}}}{\Big(I_2^*\big(\chi_{(x,\infty)\times(y,\infty)}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}. \tag{3.8}$$

3) Let $p_1 < p_2 < q$ or $p_2 < p_1 < q$ or $p_1 = p_2 < q$. Analogously to (3.2),

$$\begin{aligned} C &\overset{(2.3)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0}\left(\int\limits_0^{\infty}\int\limits_0^{\infty} (I_2 g)^q\, \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2} \\ &\overset{(2.3)}{\approx} \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(\widetilde{D}_1(x,y)\Big) := \mathcal{B}_1. \end{aligned}$$

Similarly, we can obtain an alternative estimate:

$$C \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_2(x,y)\big)^{\frac{1}{p_2'}} \Big(\widehat{D}_1(x,y)\Big) := \mathcal{B}_2.$$

Therefore, $C \approx \min\big\{\mathcal{B}_1, \mathcal{B}_2\big\}$. $\square$

It remains to consider the case 4) $p_1 = p_2 = q$. By (2.2), we have

$$C \approx C_1 + C_2 + C_3,$$

where

$$C_1 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \sup_{g\neq 0}\left(\int\limits_0^{\infty}\int\limits_0^{\infty} (I_2 g)^q\, \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2},$$

$$C_2 := \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{-\frac{1}{p_1}} \sup_{g\neq 0}\left(\int\limits_0^{\infty}\int\limits_0^{\infty} (V_1)^q\,(I_2 g)^q\, \chi_{(0,x)\times(0,y)}u\right)^{\frac{1}{q}} \|g\|^{-1}_{p_2,v_2},$$

$$C_3 := \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2} \sup_{(x,y)\in\mathbb{R}^2_+} \frac{\left(\int\limits_x^{\infty}\int\limits_y^{\infty}\left(\int\limits_0^{\infty}\int\limits_0^{\infty} (I_2 g)^q\, \chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int\limits_0^{\infty}\int\limits_0^{\infty} (I_2 g)^q\, \chi_{(x,\infty)\times(y,\infty)}u\right)^{\frac{1}{q'}}}. \tag{3.9}$$

From (2.2) it follows that

$$C_1 \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{\frac{1}{p_1'}} \Big(S_1(x,y)+S_2(x,y)+S_3(x,y)\Big) =: Q_1,$$

where

$$S_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^\infty\!\!\int_0^\infty u\chi_{(\max(t_1,x),\infty)\times(\max(t_2,y),\infty)}\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$

$$S_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_x^{t_1}\!\!\int_y^{t_2} (V_2)^q u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$S_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^\infty\!\!\int_{t_2}^\infty \left(\int_0^\infty\!\!\int_0^\infty u\chi_{(\max(x,\rho),\infty)\times(\max(y,\tau),\infty)}\right)^{p_2'} \sigma_2(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_2'}}}{\left(\int_0^\infty\!\!\int_0^\infty u\chi_{(\max(t_1,x),\infty)\times(\max(t_2,y),\infty)}\right)^{\frac{1}{q'}}},$$

and

$$C_2 \approx \sup_{(x,y)\in\mathbb{R}^2_+} \big(V_1(x,y)\big)^{-\frac{1}{p_1}} \Big(T_1(x,y)+T_2(x,y)+T_3(x,y)\Big) =: Q_2,$$

where

$$T_1(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{x}\!\!\int_{t_2}^{y} (V_1)^q\,u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$

$$T_2(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^\infty\!\!\int_0^\infty (V_1V_2)^q u\chi_{(0,\min(t_1,x)\times(0,\min(t_2,y))}\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$T_3(x,y) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^\infty\!\!\int_{t_2}^\infty \left(\int_\rho^x\!\!\int_\tau^y (V_1)^q u\right)^{p_2'} \sigma_2(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_2'}} \left(\int_{t_1}^x\!\!\int_{t_2}^y (V_1)^q u\right)^{-\frac{1}{q'}}.$$

Since $\chi_{(\rho,\infty)\times(\tau,\infty)} \le \chi_{(x,\infty)\times(y,\infty)}$ in (3.9) and $p'=q'$ then

$$\begin{aligned}
C_3 &\lesssim \sup_{g\neq 0} \frac{\sup\limits_{(x,y)\in\mathbb{R}^2_+} \left(\int_x^\infty\!\!\int_y^\infty \left(\int_0^\infty\!\!\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'-1} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\|g\|_{p_2,v_2}} \qquad (3.10)\\
&= \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left(\left(\int_0^\infty\!\!\int_0^\infty \left(\int_0^\infty\!\!\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'-1} \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'-1}}\right)^{\frac{1}{p_1}}.
\end{aligned}$$

Let $p_1'-1\geq 1$. Applying Minkowskii's integral inequality with $p_1'-1$, we obtain

$$\begin{aligned} C_3 &\lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty \big(I_2g(t,z)\big)^q u(t,z)\left(\int_0^t\int_0^z \sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'-1}} dt\,dz\right)^{\frac{1}{p_1}} \\ &= \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty (I_2g)^q u(V_1)^{p_1-1}\right)^{\frac{1}{p_1}} \overset{(2.2)}{\lesssim} R_1+R_2+R_3. \end{aligned}$$

Here,

$$\begin{aligned} R_1 &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_{t_1}^\infty\int_{t_2}^\infty u(V_1)^{p_1-1}\right)^{\frac1q}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_1'}}, \\ R_2 &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} (V_2)^q u(V_1)^{p_1-1}\right)^{\frac1q}\big(V_2(t_1,t_2)\big)^{-\frac{1}{p_1}}, \\ R_3 &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_{t_1}^\infty\int_{t_2}^\infty\left(\int_t^\infty\int_z^\infty u(V_1)^{p_1-1}\right)^{p_1'}\sigma_2(t,z)\,dt\,dz\right)^{\frac{1}{p'}}\left(\int_{t_1}^\infty\int_{t_2}^\infty u(V_1)^{p_1-1}\right)^{-\frac{1}{q'}}. \end{aligned}$$

If $p_1'-1<1$ then from (3.10) it follows that

$$\begin{aligned} C_3 &\lesssim \sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty\left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{p_1'-1}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}} \\ &\leq \left(\int_0^\infty\int_0^\infty\left(\sup_{g\neq 0}\|g\|^{-1}_{p_2,v_2}\left(\int_0^\infty\int_0^\infty (I_2g)^q\,\chi_{(\rho,\infty)\times(\tau,\infty)}u\right)^{\frac{1}{p_1}}\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}} \\ &\overset{(2.2)}{\lesssim} \left(\int_0^\infty\int_0^\infty\Big(J_1(\rho,\tau)+J_2(\rho,\tau)+J_3(\rho,\tau)\Big)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}} =: Q_3, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} J_1(\rho,\tau) &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,\rho),\infty)\times(\max(t_2,\tau),\infty)}\right)^{\frac1q}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}}, \\ J_2(\rho,\tau) &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left(\int_\rho^{t_1}\int_\tau^{t_2}(V_2)^q u\right)^{\frac1q}\big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}}, \\ J_3(\rho,\tau) &:= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\frac{\left(\int_{t_1}^\infty\int_{t_2}^\infty\left(\int_0^\infty\int_0^\infty u\chi_{(\max(\rho,z_1),\infty)\times(\max(\tau,z_2),\infty)}\right)^{p_2'}\sigma_2(z_1,z_2)\,dz_1\,dz_2\right)^{\frac{1}{p_2'}}}{\left(\int_0^\infty\int_0^\infty u\chi_{(\max(t_1,\rho),\infty)\times(\max(t_2,\tau),\infty)}\right)^{\frac{1}{q'}}}. \end{aligned} \tag{3.12}$$

Note that estimates (3.11)–(3.12) hold for the case $p'-1\ge 1$ as well.

For a lower bound for $C_3$ we obtain from (3.9) by setting $g=\sigma_2\chi_{(0,t_1)\times(0,t_2)}$:

$$C_3 \ge \sup_{\substack{(x,y)\in\mathbb{R}^2_+\\ (t_1,t_2)\in\mathbb{R}^2_+}} \frac{\left(\int_x^\infty\int_y^\infty\left(\int_\rho^\infty\int_\tau^\infty \big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int_x^\infty\int_y^\infty \big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{\frac{1}{q'}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2}}}.$$

Summarizing the above, we can state the following theorem.

**Theorem 3.2.** *Let $p_1=p_2=q$ and $w,v\in\mathfrak{M}^+$ be weights. Then the best constant $C$ in inequality* (1.1) *can be estimated from above as follows.*

1. *If $p_1'-1<1$ then*
$$C\lesssim Q_1+Q_2+Q_3.$$
2. *If $p_1'-1\ge 1$ then*
$$C\lesssim Q_1+Q_2+\min\big\{R_1+R_2+R_3,Q_3\big\}.$$

*A lower bound for $C$, independently of relations between $p_1'-1$ and 1, is*

$$C\gtrsim Q_1+Q_2+ \\ +\sup_{\substack{(x,y)\in\mathbb{R}^2_+\\ (t_1,t_2)\in\mathbb{R}^2_+}} \frac{\left(\int_x^\infty\int_y^\infty\left(\int_\rho^\infty\int_\tau^\infty \big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{p_1'}\sigma_1(\rho,\tau)\,d\rho\,d\tau\right)^{\frac{1}{p_1'}}}{\left(\int_x^\infty\int_y^\infty \big(V_2(\min(t_1,t),\min(t_2,z))\big)^q u(t,z)\,dt\,dz\right)^{\frac{1}{q'}}\big(V_2(t_1,t_2)\big)^{\frac{1}{p_2}}}.$$

## 3.2 Case $q<\max\{p_1,p_2\}$

There following cases arise when $q<\max\{p_1,p_2\}$:

1) $p_1<q<p_2$ or $p_1<q<p_2$,

2) $q<\min\{p_1,p_2\}$,

3) $q=p_1<p_2$ or $q=p_2<p_1$.

**Theorem 3.3.** *Let $\min\{p_1,p_2\}<q<\max\{p_1,p_2\}$ and $w,v\in\mathfrak{M}^+$ be weights. Then the best constant $C$ in inequality* (1.1) *can be estimated as*

$$\sup_{(x,y)\in\mathbb{R}^2_+}\big(V_1(x,y)\big)^{\frac{1}{p_1'}}\widetilde{B}_v(x,y)\lesssim C\lesssim \sup_{(x,y)\in\mathbb{R}^2_+}\big(V_1(x,y)\big)^{\frac{1}{p_1'}}\widetilde{B}(x,y)$$

*in the case $p_1<q<p_2$ with $\widetilde{B}_v(x,y)$ and $\widetilde{B}(x,y)$ given by equalities* (3.14) *and* (3.15). *If $p_2<q<p_1$ then we have* (3.16).

*Proof.* Let $p_1 < q < p_2$. Then

$$C \overset{(2.3)}{\approx} \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \sup_{(x,y)\in\mathbb{R}_+^2} \left( \int_0^\infty\int_0^\infty (I_2 g)^q \, \chi_{(x,\infty)\times(y,\infty)} u \right)^{\frac{1}{q}} \left( I_2 v_1^{1-p_1'}(x,y) \right)^{\frac{1}{p_1'}}$$

$$= \sup_{(x,y)\in\mathbb{R}_+^2} \left( V_1(x,y) \right)^{\frac{1}{p_1'}} \sup_{g\neq 0} \left( \int_0^\infty\int_0^\infty (I_2 g)^q \, \chi_{(x,\infty)\times(y,\infty)} u \right)^{\frac{1}{q}} \|g\|_{p_2,v_2}^{-1}. \tag{3.13}$$

Therefore, from Theorem 2.3 it follows that

$$\sup_{(x,y)\in\mathbb{R}_+^2} \left( V_1(x,y) \right)^{\frac{1}{p_1'}} \widetilde{B}(x,y) \lesssim C \lesssim \sup_{(x,y)\in\mathbb{R}_+^2} \left( V_1(x,y) \right)^{\frac{1}{p_1'}} \widetilde{B}_v(x,y),$$

where with $1/r_2 := 1/q - 1/p_2$

$$\widetilde{B}_v(x,y) := \left( \int_0^\infty\int_0^\infty \sigma_2(t,z) \left( \int_t^\infty\int_z^\infty (V_2)^{q-1} \chi_{(x,\infty)\times(y,\infty)} u \right)^{\frac{r_2}{q}} dt\, dz \right)^{\frac{1}{r_2}}, \tag{3.14}$$

$$\widetilde{B}(x,y) := \left( \int_0^\infty\int_0^\infty d_z \left( V_2(t,z) \right)^{\frac{r_2}{p_2'}} d_t \left( - \left( I_2^*(\chi_{(x,\infty)\times(y,\infty)} u)(t,z) \right)^{\frac{r_2}{q}} \right) \right)^{\frac{1}{r_2}}. \tag{3.15}$$

The proof of the case $p_2 < q < p_1$ is analogous. Here,

$$\sup_{(x,y)\in\mathbb{R}_+^2} \left( V_2(x,y) \right)^{\frac{1}{p_2'}} \widehat{B}(x,y) \lesssim C \lesssim \sup_{(x,y)\in\mathbb{R}_+^2} \left( V_2(x,y) \right)^{\frac{1}{p_2'}} \widehat{B}_v(x,y) \tag{3.16}$$

and with $1/r_1 := 1/q - 1/p_1$

$$\widehat{B}_v(x,y) := \left( \int_0^\infty\int_0^\infty \sigma_1(t,z) \left( \int_t^\infty\int_z^\infty (V_1)^{q-1} \chi_{(x,\infty)\times(y,\infty)} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}},$$

$$\widehat{B}(x,y) := \left( \int_0^\infty\int_0^\infty d_z \left( V_1(t,z) \right)^{\frac{r_1}{p_1'}} d_t \left( - \left( I_2^*(\chi_{(x,\infty)\times(y,\infty)} u)(t,z) \right)^{\frac{r_1}{q}} \right) \right)^{\frac{1}{r_1}}.$$

□

**Remark 1.** *If the weights $u$ and $\sigma_2$ in* (3.13) *satisfy properties* (1) *and* (2), *respectively, from Theorem* 2.4, *then by virtue of* (2.4) *we obtain*

$$C \approx \sup_{(x,y)\in\mathbb{R}_+^2} \left( V_1(x,y) \right)^{\frac{1}{p_1'}} \left( \int_0^\infty\int_0^\infty \left( I_2^*[u\chi_{(x,\infty)\times(y,\infty)}](\rho,\tau) \right)^{\frac{r_2}{q}} d_\rho\, d_\tau \left( V_2(\rho,\tau) \right)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}}.$$

*Analogically, in the case $p_2 < q < p_1$, if the weights $u$ and $\sigma_1$ satisfy properties* (1) *and* (2) *from Theorem* 2.4, *then we obtain*

$$C \approx \sup_{(x,y)\in\mathbb{R}_+^2} \left( V_2(x,y) \right)^{\frac{1}{p_2'}} \left( \int_0^\infty\int_0^\infty \left( I_2^*(u\chi_{(x,\infty)\times(y,\infty)})(\rho,\tau) \right)^{\frac{r_1}{q}} d_\rho\, d_\tau \left( V_1(\rho,\tau) \right)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}.$$

**Theorem 3.4.** *Let $q < p_1 = p_2$ or $q < p_1 < p_2$ or $q < p_2 < p_1$ and $w, v \in \mathfrak{M}^+$ be weights. Then the best constant $C$ in inequality (1.1) can be estimated from above as*

$$C \lesssim \min\left\{ \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z)\big(E(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}}, F \right\},$$

*where $E$ is defined by equation (3.17) and $F$ is defined by equation (3.19).*

*If, in addition, the weights $w_1 := u(V_1)^{q-1}$ and $w_2 := u(V_1)^{\frac{q}{p_1'}}$ satisfy condition (1) of Theorem 2.4 and the weight $\sigma_2$ is of type (2), then*

$$C \lesssim \min\{\tilde{J}, \hat{J}\},$$

*where $\tilde{J}$ and $\hat{J}$ are defined by equations (3.18) and (3.20), respectively.*

*A lower bound for the best constant $C$ in (1.1) is given in (3.22).*

*Proof.* From (3.1) and Theorem 2.3 we have

$$\begin{aligned} C &\lesssim \sup_{g \neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \int_t^\infty\!\!\int_z^\infty (I_2 g)^q (V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}} \\ &= \sup_{g \neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \int_t^\infty\!\!\int_z^\infty (I_2 g)^q w_1 \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \sup_{g \neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty\!\!\int_0^\infty (I_2 g)^q w_1 \chi_{(t,\infty)\times(z,\infty)} \right)^{\frac{1}{q}} \right)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z)\big(E(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}}. \end{aligned}$$

Here,

$$E(t,z) = \left( \int_0^\infty\!\!\int_0^\infty \sigma_2(\rho,\tau) \left( \int_0^\infty\!\!\int_0^\infty (V_1 V_2)^{q-1} u \chi_{(\max(t,\rho),\infty)\times(\max(z,\tau),\infty)} \right)^{\frac{r_2}{q}} d\rho\, d\tau \right)^{\frac{1}{r_2}}. \tag{3.17}$$

If the weights $w_1$ and $\sigma_2$ satisfy properties (1), (2) of Theorem 2.4, then

$$\begin{aligned} C &\lesssim \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \sup_{g \neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty\!\!\int_0^\infty (I_2 g)^q w_1 \chi_{(t,\infty)\times(z,\infty)} \right)^{\frac{1}{q}} \right)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} \\ &\lesssim \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z)\big(J(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} =: \tilde{J}, \end{aligned} \tag{3.18}$$

where

$$J(t,z) = \left( \int_0^\infty\!\!\int_0^\infty \Big( \big(I_2^* w_1 \chi_{(t,\infty)\times(z,\infty)}\big)(x,y) \Big)^{\frac{r_2}{q}} d_x\, d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}}.$$

Alternatively, we can write

$$C \lesssim \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \int_t^\infty\!\!\int_z^\infty (I_2 g)^q\, (V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{1}{r_1}}$$

$$= \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left( \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \left( \int_t^\infty\!\!\int_z^\infty (I_2 g)^q\, (V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{q}{r_1}} \right)^{\frac{1}{q}}.$$

Application of Minkowski's integral inequality with the exponent $r_1/q$ yields

$$C \lesssim \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left( \int_0^\infty\!\!\int_0^\infty \big((I_2 g)(x,y)\big)^q \big(V_1(x,y)\big)^{\frac{q}{p_1'}} u(x,y)\, dx\, dy \right)^{\frac{1}{q}}$$

$$\lesssim \left( \int_0^\infty\!\!\int_0^\infty \sigma_2(\tilde t,\tilde z) \left( \int_{\tilde t}^\infty\!\!\int_{\tilde z}^\infty (V_1)^{\frac{q}{p_1'}} (V_2)^{q-1} u \right)^{\frac{r_2}{q}} d\tilde t\, d\tilde z \right)^{\frac{1}{r_2}} := F. \tag{3.19}$$

Therefore,

$$C \lesssim \min \left\{ \left( \int_0^\infty\!\!\int_0^\infty \sigma_1(t,z) \big(E(t,z)\big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}}, F \right\}.$$

If the weights $w_2$ and $\sigma_2$ are of types (1) and (2) from Theorem 2.4, then

$$C \lesssim \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left( \int_0^\infty\!\!\int_0^\infty (I_2 g)^q\, (x,y) w_2(x,y)\, dx\, dy \right)^{\frac{1}{q}}$$

$$\lesssim \left( \int_0^\infty\!\!\int_0^\infty \Big( \big(I_2^* w_2\big)(x,y) \Big)^{\frac{r_2}{q}} d_x\, d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} =: \hat J. \tag{3.20}$$

For the lower bound for $C$ we use Theorem 2.3, first, to obtain

$$C \gtrsim \sup_{g\neq 0} \|g\|^{-1}_{p_2,v_2} \left( \int_0^\infty\!\!\int_0^\infty \Big( I_2^*\big((I_2 g)^q w\big)(x,y) \Big)^{\frac{r_1}{q}} d_x\, d_y \big(I_2 \sigma_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}$$

$$= \sup_{g\neq 0} \left( \int_0^\infty\!\!\int_0^\infty \left( \|g\|^{-1}_{p_2,v_2} \left( \int_0^\infty\!\!\int_0^\infty \chi_{(x,\infty)\times(y,\infty)} (I_2 g)^q w \right)^{\frac{1}{q}} \right)^{r_1} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}. \tag{3.21}$$

After this, substituting the test function

$$g_0(s,\tau,x,y) := \sigma_2(s,\tau) \left( \int_s^\infty \big(V_2(\rho,\tau)\big)^{\frac{r_1}{q'}} \Big( \big(I_2^* w_0(x,y)\big)(\rho,\tau) \Big)^{\frac{r_1}{p_1}} \left( \int_\tau^\infty w_0(\rho,z)\, dz \right) d\rho \right)^{\frac{1}{p_1}}$$

with $w_0 := \chi_{(x,\infty)\times(y,\infty)} u$ into (3.21) (see [18, pages 627–631] for details) implies

$$C \gtrsim \left( \int_0^\infty\!\!\int_0^\infty \left( \int_0^\infty\!\!\int_0^\infty \Big( \big(I_2^* w_0(x,y)\big)(s,t) \Big)^{\frac{r_2}{q}} d_s\, d_t \big(V_2(s,t)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{r_1}{r_2}} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}. \tag{3.22}$$

□

**Theorem 3.5.** *Let $w, v \in \mathfrak{M}^+$ be weights.*

1. *If $q = p_2 < p_1 < \infty$ then the best constant $C$ in inequality* (1.1) *can be estimated as*

$$C \lesssim \min\{G, K\}.$$

*where functionals $G$, $K$ are defined in* (3.23) *and* (3.24) *respectively. A lower bound for $C$ is as given in* (3.27).

2. *If $q = p_1 < p_2 < \infty$ then the best constant $C$ in inequality* (1.1) *can be estimated as*

$$C \lesssim \min\{\tilde{G}, \tilde{K}\},$$

*where functionals $\tilde{G}$ and $\tilde{K}$ are defined in* (3.25) *and* (3.26) *respectively. A lower estimate for $C$ is as given in* (3.28).

*Proof.* 1. Let $q = p_2 < p_1 < \infty$. By Theorem 2.3 and (2.2), we have

$$\begin{aligned} C &\lesssim \left( \int_0^\infty \int_0^\infty \sigma_1(t,z) \left( \sup_{g \neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty \int_0^\infty (I_2 g)^q \, (V_1)^{q-1} u \chi_{(t,\infty)\times(z,\infty)} \right)^{\frac{1}{q}} \right)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} \\ &\approx \left( \int_0^\infty \int_0^\infty \sigma_1(t,z) \Big( G_1(t,z) + G_2(t,z) + G_3(t,z) \Big)^{r_1} dt\, dz \right)^{\frac{1}{r_1}} := G, \end{aligned} \tag{3.23}$$

where

$$G_1(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big( I_2^* \big( \chi_{(t,\infty)\times(z,\infty)} (V_1)^{q-1} u \big)(t_1,t_2) \Big)^{\frac{1}{q}} \big( V_2(t_1,t_2) \big)^{\frac{1}{p_2'}},$$

$$G_2(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left( \int_0^{t_1} \int_0^{t_2} (V_2)^q (V_1)^{q-1} \chi_{(t,\infty)\times(z,\infty)} u \right)^{\frac{1}{q}} \big( V_2(t_1,t_2) \big)^{-\frac{1}{p_2}},$$

$$G_3(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{ \left( \int_{t_1}^\infty \int_{t_2}^\infty \Big( I_2^* \big( \chi_{(t,\infty)\times(z,\infty)} (V_1)^{q-1} u \big) \Big)^{p_2'} \sigma_2 \right)^{\frac{1}{p_2'}} }{ \Big( I_2^* \big( \chi_{(t,\infty)\times(z,\infty)} (V_1)^{q-1} u \big)(t_1,t_2) \Big)^{\frac{1}{q'}} }.$$

Alternatively, analogously to the proof of Theorem 3.5 we obtain

$$\begin{aligned} C &\lesssim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left( \left( \int_0^\infty \int_0^\infty \sigma_1(t,z) \left( \int_t^\infty \int_z^\infty (I_2 g)^q (V_1)^{q-1} u \right)^{\frac{r_1}{q}} dt\, dz \right)^{\frac{q}{r_1}} \right)^{\frac{1}{q}} \\ &\qquad \Big[ \text{by Minkowski's integral inequality with the exponent } r_1/q \Big] \\ &\lesssim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty \int_0^\infty \big( (I_2 g)(x,y) \big)^q (V_1)^{\frac{q}{p_1'}}(x,y) u(x,y)\, dx\, dy \right)^{\frac{1}{q}} \\ &\lesssim K_1 + K_2 + K_3 := K. \end{aligned} \tag{3.24}$$

Here,

$$K_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{\frac{1}{p_2'}},$$

$$K_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\!\!\int_0^{t_2} (V_2)^q (V_1)^{\frac{q}{p_1'}}u\right)^{\frac{1}{q}} \big(V_2(t_1,t_2)\big)^{-\frac{1}{p_2}},$$

$$K_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\!\!\int_{t_2}^{\infty} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)\Big)^{p_2'}\sigma_2\right)^{\frac{1}{p_2'}} \Big(I_2^*\big((V_1)^{\frac{q}{p_1'}}u\big)(t_1,t_2)\Big)^{-\frac{1}{q'}}.$$

Therefore,

$$C \lesssim \min\big\{G, K\big\}.$$

2. If $q = p_1 < p_2 < \infty$ then, analogously to Case 1,

$$C \lesssim \left(\int_0^{\infty}\!\!\int_0^{\infty} \sigma_2(t,z)\left(\widetilde{G}_1(t,z)+\widetilde{G}_2(t,z)+\widetilde{G}_3(t,z)\right)^{r_1} dt\,dz\right)^{\frac{1}{r_1}} := \tilde{G}, \tag{3.25}$$

where

$$\widetilde{G}_1(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}},$$

$$\widetilde{G}_2(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\!\!\int_0^{t_2} (V_1)^q (V_2)^{q-1}\chi_{(t,\infty)\times(z,\infty)}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}},$$

$$\widetilde{G}_3(t,z) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \frac{\left(\int_{t_1}^{\infty}\!\!\int_{t_2}^{\infty} \Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}}}{\Big(I_2^*\big(\chi_{(t,\infty)\times(z,\infty)}(V_2)^{q-1}u\big)(t_1,t_2)\Big)^{\frac{1}{q'}}}.$$

Alternatively,

$$C \lesssim \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 := \tilde{K}. \tag{3.26}$$

Here,

$$\tilde{K}_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)(t_1,t_2)\Big)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{\frac{1}{p_1'}},$$

$$\tilde{K}_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\!\!\int_0^{t_2} (V_1)^q (V_2)^{\frac{q}{p_2'}}u\right)^{\frac{1}{q}} \big(V_1(t_1,t_2)\big)^{-\frac{1}{p_1}},$$

$$\tilde{K}_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\!\!\int_{t_2}^{\infty} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)\Big)^{p_1'}\sigma_1\right)^{\frac{1}{p_1'}} \Big(I_2^*\big((V_2)^{\frac{q}{p_2'}}u\big)(t_1,t_2)\Big)^{-\frac{1}{q'}}.$$

Therefore,

$$C \lesssim \min\{\tilde{G}, \tilde{K}\}.$$

To derive the lower estimate for $C$ in Case 1, we start from (3.21)

$$C \gtrsim \sup_{g\neq 0} \|g\|_{p_2,v_2}^{-1} \left( \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty \chi_{(x,\infty)\times(y,\infty)} (I_2 g)^q w \right)^{\frac{r_1}{q}} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}$$

and obtain, by setting $g = \sigma_2 \chi_{(0,s)\times(0,t)}$, that

$$C \gtrsim \sup_{(s,t)\in\mathbb{R}_+^2} \frac{\left( \int_0^\infty \int_0^\infty \left( \int_x^\infty \int_y^\infty (I_2 \sigma_2 \chi_{(0,s)\times(0,t)})^q w \right)^{\frac{r_1}{q}} d_x\, d_y \big(V_1(x,y)\big)^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}}{\big(V_2(s,t)\big)^{\frac{1}{p_2}}}. \tag{3.27}$$

Analogously, in Case 2:

$$C \gtrsim \sup_{(s,t)\in\mathbb{R}_+^2} \frac{\left( \int_0^\infty \int_0^\infty \left( \int_x^\infty \int_y^\infty (I_2 \sigma_1 \chi_{(0,s)\times(0,t)})^q w \right)^{\frac{r_2}{q}} d_x\, d_y \big(V_2(x,y)\big)^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}}}{\big(V_1(s,t)\big)^{\frac{1}{p_1}}}. \tag{3.28}$$

$\square$

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