ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2023, Volume 14, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 14, Number 4 (2023), 23 – 46

COMPARISON OF POWERS OF DIFFERENTIAL POLYNOMIALS

H.G. Ghazaryan

Communicated by V.I. Burenkov

Key words: the power of a differential operator (polynomial), comparison of polynomials, generalized-homogeneous polynomial, Newton polyhedron.

AMS Mathematics Subject Classification: 12E10, 12D05, 26D05, 35A23.

Abstract. Necessary and sufficient conditions are obtained for a polynomial P to be more powerful then a polynomial Q. These conditions are formulated in terms of the orders of generalized-homogeneous sub-polynomials, corresponding to these polynomials, and the multiplicity of their zeros. Applying these results, conditions are obtained, under which a monomial ξ^ν for a certain set of multi-indices $\nu \in \Re^*$ can be estimated via terms of a given degenerate polynomial P.

DOI: https://doi.org/10.32523/2077-9879-2023-14-4-23-46

1 Introduction

Let $\mathbb{E}^n$ and $\mathbb{R}^n$ be n-dimensional Euclidean spaces of points (vectors) $x = (x_1, \cdots, x_n)$ and $\xi = (\xi_1, \cdots, \xi_n)$ respectively, $\mathbb{R}^{n,+} := \{\xi \in \mathbb{R}^n, \xi_j \geq 0,\ j = 1, \cdots, n\}$, $\mathbb{R}^{n,0} := \{\xi \in \mathbb{R}^n, \xi_1 \cdots \xi_n \neq 0\}$. Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ be the set of all $n-$dimensional multi-indices, i.e. the set of all points with non-negative integer coordinates $\{\alpha = (\alpha_1, ..., \alpha_n) :\ \alpha_i \in \mathbb{N}_0\ (i = 1, ..., n)\}$.

For $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n :\ \lambda_j > 0\ (j = 1, ..., n)$ and $\nu \in \mathbb{R}^{n,+}$ we denote $|\xi| := \sqrt{\xi_1^2 + \cdots + \xi_n^2}$, $|\xi, \lambda| := \sqrt{|\xi_1|^{2/\lambda_1} + \cdots + |\xi_n|^{2/\lambda_n}}$, $|\nu| := \nu_1 + ... + \nu_n$, $\xi^\nu := \xi_1^{\nu_1} \cdots \xi_n^{\nu_n}$, $|\xi^\nu| := |\xi_1|^{\nu_1} \cdots |\xi_n|^{\nu_n}$.

For $\alpha \in \mathbb{N}_0^n$, we denote $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = \frac{1}{i}\, \partial/\partial x_j$ or $D_j = \partial/\partial \xi_j$ $(j = 1, ..., n)$.

Let $\mathcal{A} = \{\nu^j = (\nu_1^j, \cdots, \nu_n^j)\}_{j=1}^M$ be a finite set of points $\nu^j \in \mathbb{R}^{n,+}$. By the Newton polyhedron (further, when it does not cause misunderstanding, we will briefly write N.P.) of the set $\mathcal{A}$ we mean the least convex hull (which is a polyhedron) $\Re = \Re(\mathcal{A})$ in $\mathbb{R}^n$, containing all points of $\mathcal{A}$ (see [23] or [33]).

A polyhedron $\Re$ with vertices in $\mathbb{R}^{n,+}$ is said to be complete if $\Re$ has a vertex at the origin of coordinates and further vertices on each coordinate axis of $\mathbb{R}^{n,+}$.

The $k-$dimensional faces of a polyhedron $\Re$ are denoted by $\Re_i^k$ $(i = 1, ..., M_k^{'}, k = 0, 1, ..., n-1)$. The faces of the N.P. (by definition) are closed sets.

The unit outward normal to a supporting hyper-plane of a polyhedron $\Re$, containing some face $\Re_i^k$ and not containing any other face of dimension greater than k, will be simply called the outward normal (or $\Re-$normal) of the face $\Re_i^k$. Thus, a given unit vector λ can serve as an outward normal to one (and only one) face of $\Re$. We denote by Λ_i^k the set of all outward normals of the face $\Re_i^k$ $(i = 1, ..., M_k^{'}, k = 0, 1, ..., n-1)$. Note that either the set Λ_i^k consists of one vector (when $k = n-1$), or it is an open set (when $0 \leq k < n-1$).

For any $\lambda \in \Lambda_i^k$ $(1 \leq i \leq M_k^{'},\ k = 0 \leq k \leq n-1)$ there exists a number $d = d_{i,k} = d_{i,k}(\lambda) \geq 0$ such that $(\lambda, \alpha) = d$ for all $\alpha \in \Re_i^k$, and $(\lambda, \alpha) < d$ for any $\alpha \in \Re \setminus \Re_i^k$. Moreover, the $\Re-$normal of the $(n-1)-$dimensional (and only $(n-1)-$dimensional) face $\Re_i^{n-1}$ of the polyhedron $\Re$ and the number $d_{i,n-1}(\lambda)$ $(1 \le i \le M_{n-1})$ are determined uniquely.

Definition 1. A face $\Re_i^k$ of a polyhedron $\Re$ is said to be principal, if one of the following (obviously equivalent) conditions is satisfied: 1) $\Re_i^k$ does not go through the origin, 2) among the $\Re-$normals of this face there is one with at least one positive component. We say that a point $\alpha \in \Re$ is principal if α lies on some (recall, closed) principal face.

Obviously, all sub-faces of a principal face are principal. The number of $k-$dimensional principal faces of the polyhedron $\Re$ is denoted by M_k, obviously $M_k \le M_k'$.

Let $P(D) = P(D_1, ..., D_n) = \sum_\beta \gamma_\beta D^\beta$ be a linear differential operator with constant coefficients and $P(\xi) = \sum_\beta \gamma_\beta \xi^\beta$ be its complete symbol (the characteristic polynomial). Here the sum goes over a finite set of multi-indices $(P) := \{\beta \in \mathbb{N}_0^n; \gamma_\beta \ne 0\}$.

The Newton polyhedron of the set $(P) \cup \{0\}$ is called the Newton polyhedron of the operator $P(D)$ (polynomial $P(\xi)$) and is denoted by $\Re(P)$. Thus, the Newton polyhedron of any operator $P(D)$ (polynomial $P(\xi)$) is actually constructed as the Newton polyhedron of the operator $I+P(D)$ (polynomial $1+P(\xi)$), where I is the identity operator. Note that a polyhedron $\Re(P)$ may have dimensionality less than n. However, in our considerations, we will assume that for both general and generalized-homogeneous polynomials P the polyhedrons $\Re(P)$ are $n-$dimensional (for complete polyhedrons this is obvious).

Let $\Re(P)$ be the N.P. of a polynomial $P(\xi)$ and $\Re_i^k$ $(i = 1, ..., M_k'; k = 0, 1, ..., n-1)$ be its faces. The polynomial $P^{i,k}(\xi) := \sum_{\alpha \in \Re_i^k} \gamma_\alpha \xi^\alpha$ $(1 \le i \le M_k; 0 \le k \le n))$ will be called the sub-polynomial of polynomial $P(\xi)$, corresponding to the face $\Re_i^k$.

Definition 2. Let $\mu \in \mathbb{R}^n$ be a vector with rational components. A polynomial $R(\xi) = R(\xi_1, ..., \xi_n)$ is called $\mu-$homogeneous (generalized-homogeneous) of $\mu-$order $d = d(\mu)$ (which is also a rational number), if $R(t^\mu \xi) := R(t^{\mu_1} \xi_1, ..., t^{\mu_n} \xi_n) = t^d R(\xi)$ for all $t > 0$, $\xi \in \mathbb{R}^n$. When $\lambda_1 = \lambda_2 = ... = \lambda_n (= 1)$, it is an ordinary homogeneous polynomial, wherein $|\xi, \lambda| = |\xi|$.

We will often use the following proposition, proved by V.P. Mikhailov

Lemma 1.1. ([33]) *Let $\Re = \Re(P)$ be the N.P. of a polynomial $P(\xi)$ and λ be any $\Re-$normal to the face $\Re_i^k$ $(\lambda \in \Lambda_i^k, 1 \le i \le M_k'; 0 \le k \le n-1)$ of the polyhedron $\Re$. Then the sub-polynomial $P^{i,k}$ is $\lambda-$homogeneous.*

Remark 1. It is obvious that if λ is a unit vector and $P(\xi) = \sum_{\alpha \in (P)} \gamma_\alpha \xi^\alpha$ is a polynomial, then there exist (uniquely defined) numbers $d_j(\lambda)$ and $\lambda-$homogeneous polynomials $P_j := P_{d_j(\lambda)}$ $(j = 0, 1, ..., M(\lambda))$: $d_0(\lambda) > d_1(\lambda) > ... > d_{M(\lambda)}(\lambda)$, such that the polynomial $P(\xi)$ can be represented in the form

$$P(\xi) = \sum_{j=0}^{M(\lambda)} P_j(\xi) = \sum_{j=0}^{M(\lambda)} P_{d_j(\lambda)}(\xi) = \sum_{j=0}^{M(\lambda)} \sum_{(\lambda,\alpha)=d_j(\lambda)} \gamma_\alpha \xi^\alpha, \tag{1.1}$$

where the set of numbers $\{d_j = d_j(\lambda)\}$ coincides with the finite set of values $\{(\lambda, \alpha)\}$ for all $\alpha \in \Re(P)$.

Note that

1) if $\Re_i^k$ $(i = 1, ..., M_k; k = 0, 1, ..., n-1)$ is some principal face of $\Re(P)$ and $\lambda \in \Lambda(\Re_i^k)$, then $(\lambda, \alpha) = d_0(\lambda)$ is the equation of the $(n-1)-$dimensional supporting hyperplane to $\Re(P)$ with the outward (with respect to $\Re(P)$) normal λ, containing the face $\Re_i^k$, where $P_{d_0(\lambda)}(\xi) \equiv P^{i,k}(\xi)$.

2) it follows from Lemma 1.1 that a sub-polynomial $P^{i,k}$ $(1 \le M_k', 0 \le k \le n-1)$ of the polynomial P is $\lambda-$homogeneous for any $\lambda \in \Lambda_i^k(\Re(P))$, i.e. there exists a number $d_{i,k} = d_{i,k}(\lambda) \ge 0$ such that $P^{i,k}$ can be represented in the form $P^{i,k}(\xi) = \sum_{(\lambda,\beta)=d_{i,k}} \gamma_\beta \xi^\alpha$.

A face $\Re_i^k$ $(1 \le i \le M_k',\ 0 \le k \le n-1)$ of the polyhedron $\Re(R)$ of a polynomial $R(\xi)$ is said to be non-degenerate ([33]) if $R^{i,k}(\xi) \neq 0$ for $\xi \in \mathbb{R}^{n,0}$. If there exists a point $\eta \in \mathbb{R}^{n,0}$, such that $P^{i,k}(\eta) = 0$, then the face $\Re_i^k$ is said to be degenerate. A polynomial $P(\xi)$ with P° N.P. $\Re(P)$ is said to be non-degenerate, if all its principal faces are non-degenerate.

Definition 3. An operator $P(D)$ (a polynomial $P(\xi)$) is called hypoelliptic ([12], Definition 11.1.2 and Theorem 11.1.1) if the following equivalent conditions are satisfied:

1) all the solutions $u \in D' = D'(\mathbb{E}^n)$ of the equation $P(D)u = f$ are continuously differentiable (belong to C^∞) for any $f \in C^\infty$,

2) $P^{(\alpha)}(\xi)/P(\xi) := D^\alpha P(\xi)/P(\xi) \to 0$ if $|\xi| \to \infty$, and $0 \neq \alpha \in \mathbb{N}_0^n$.

Definition 4. 1) ([36] or [16]) We say that a polynomial P is more powerful than a polynomial Q (a polynomial Q is less powerful than a polynomial P) and write $P > Q$ ($Q < P$), if there exists a constant $c > 0$ such that

$$|Q(\xi)| \leq c[|P(\xi)| + 1] \; \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

2) ([12], Definition 10.3.4) We say that a polynomial P is stronger (by L.Hörmander) than a polynomial Q (Q is weaker than P) and write $P \succ Q$ $(Q \prec P)$, if there exists a constant $c > 0$ such that

$$\tilde{Q}(\xi) \leq c \, \tilde{P}(\xi) \quad \forall \xi \in \mathbb{R}^n, \tag{1.3}$$

where for a polynomial R the function $\tilde{R}$ is defined by the formula

$$\tilde{R}(\xi) = [\sum_{|\alpha| \geq 0} |D^\alpha R(\xi)|^2]^{1/2}, \quad \xi \in \mathbb{R}^n.$$

Denote by $\mathbb{I}_n$ the set of all polynomials in n variables, such that $|P(\xi)| \to \infty$ for $|\xi| \to \infty$.

Many properties of the solutions of a general linear differential equation $P(D)u = 0$ are determined by the behavior at infinity of the symbol $P(\xi)$ of corresponding operator $P(D)$ as the modulus of the argument tends to infinity. For example, the symbol of a hypoelliptic operator tends to infinity (i.e. $P \in \mathbb{I}_n$).

In this case, it is important (and sometimes determining) not only that the symbol of a given operator tends to infinity, but also that this happens at a certain rate. For example, the symbol of an elliptic (and only elliptic) operator tend to infinity at an "optimal" rate, i.e. if $P(D)$ is an elliptic operator of order m, then there exits a number $c > 0$ such that

$$c^{-1}\,[1 + |\xi|^m] \leq 1 + |P(\xi)| \leq c\,[1 + |\xi|^m] \;\; \forall \xi \in \mathbb{R}^n.$$

In accordance with this, all continuous solutions of the elliptic equation $R(D)u = 0$ are real-analytic functions.

Solutions to a hypoelliptic equation (the symbols $P(\xi)$ of which belongs to $\mathbb{I}_n$) are infinitely differentiable functions. But they can also have better smoothness properties, for example, they can belong to certain Gevrey classes ([9], [37] or [38]). As is known, the Gevrey class $G^{(\sigma)}$ $(0 < \sigma < 1)$ is intermediate between the class of all infinitely continuously differentiable functions and the class of all real-analytic functions. Moreover, if for a differential operator $P(D)$ there are positive constants c and k such that

$$1 + |P(\xi)| \geq c\,[1 + |\xi|^k] \forall \xi \in \mathbb{R}^n,$$

then the value of σ directly depends on the value of k ([9], [37], [3], [29], [30]). Therefore, the need naturally arises to describe the set of multi-indices $\mathbb{B} = \mathbb{B}(P) := \{\beta\}$ for which the estimate

$$1 + |P(\xi)| \geq c \sum_{\beta \in \mathbb{B}} |\xi^\beta| \;\; \forall \xi \in \mathbb{R}^n \tag{1.4}$$

is valid with some constant $c > 0$.

V.P. Mikhailov in [33] described the class of all non-degenerate polynomials P with a complete Newton polyhedron, for which the set $\mathbb{B}$ coincides with the set $\Re(P)$, which is (in a certain sense) an "optimal" result. Similar result for an incomplete polyhedron was obtained by S. G. Gindikin in [10]. The classes of polynomials considered by these authors are certainly different from the class of elliptic ones, but they are close in character to an elliptic operator in the sense that they are non-degenerate.

The case in which the polynomial P is degenerate was first considered in the work [17]. The following proposition was proved there.

Theorem 1.1. ([17]) *Let $\Re = \Re(P)$ be the complete N.P. of a polynomial P. Suppose that all principal faces $\Re_i^k$ $(i = 1, 2, ..., M_k < M_k', k = 0, 1, ..., n-1)$ of the polyhedron $\Re$ except one $(n-1)-$dimensional principal face $\Gamma := \Re_{i_0}^{n-1}$ are non-degenerate, and the face Γ with the outward normal μ (which in this case is determined uniquely) is degenerate. Let the polynomial P be represented as the sum of $\mu-$ homogeneous polynomials* (*see representation* (1.1))

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) = \sum_{j=0}^{M} P_{d_j(\mu)}(\xi) = \sum_{j=0}^{M} \sum_{(\mu,\alpha)=d_j(\mu)} \gamma_\alpha \, \xi^\alpha, \tag{1.5}$$

where $P_0(\xi) = P^{i_0, n-1}(\xi)$, $M = M(P) = M(P, \mu)$.

Suppose that $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0) := \{\eta \in \mathbb{R}^{n,0}, |\eta, \lambda| = 1, P_0(\eta) = 0\}$ and denote by $\Re^*$ the N.P. of the set $\{\beta \in \Re, (\mu, \beta) \leq d_1\}$.

Then

1) *in order to have the estimate*

$$|\xi^\nu| \leq c\,[|P(\xi)| + 1] \;\; \forall \xi \in \mathbb{R}^n \tag{1.6}$$

for all points $\nu \in \Re^*$ with some constant $c = c(\nu, P) > 0$, it is necessary and sufficient, that for each point $\eta \in \Sigma(P_0)$ there exists a neighbourhood $U(\eta)$ such that $P_1(\eta) \neq 0$ and $P_0(\xi)\, P_1(\xi) \geq 0$ for all $\xi \in U(\eta)$.

2) *if $\nu \notin \Re^*$, then inequality (1.4) cannot hold for any constant c.*

As for the fact that only one face is degenerate, moreover it is a $(n-1)-$dimensional face, it is obvious that in case of the presence of several $(n-1)-$dimensional degenerate faces, the set $\nu \in \Re^*$ narrows and is obtained as the intersection of the sets. In [19], the case, in which $k-$dimensional faces for $k < n-1$ were present was also studied. Namely, the following proposition was proved in [19] (see also [17], Lemma 1.1), which in terms of the set $\mathbb{I}_n$ can be rephrased as follows (below $\Re^* := \{\nu \in \Re, (\lambda, \nu) \leq d_1(\lambda) \;\; \forall \lambda \in \Lambda(\Gamma)\}$)

Theorem 1.2. *Let $\Re$ be the complete Newton polyhedron of a polynomial $P \in \mathbb{I}_n$. Let all the principal faces $\Re_i^k$ $(i = 1, ..., M_k,\; k = 0, 1, ..., n-1)$ of the polyhedron $\Re$, except for (possibly) one k_0-dimensional face $\Gamma := \Re_{i_0}^{k_0}$ $(1 \leq i \leq M_k : 1 \leq k_0 < n-1)$ are non-degenerate, and the face Γ is degenerate. Let the polynomial P be represented with respect to any vector $\lambda \in \Lambda(\Gamma)$ in form* (1.1)(*for the definition of the set $\Lambda(\Gamma)$ see* [17]).

Then inequality (2.1) *holds for $\nu \in \Re^*$ if and only if $P_{d_1(\lambda)}(\eta) \neq 0$ for all $\eta \in \Sigma(\Gamma)$ and for all $\lambda \in \Lambda(\Gamma)$.*

The main limitation in these theorems is that at the points of the set $\Sigma(P_0)$, on which the polynomial P_0 vanishes, the next (or, which is the same, the first after P_0) polynomial P_1 must be nonzero. The author (and not only him) has not yet been able to overcome this limitation.

Our goal in this work is to overcome this limitation. Namely, we consider the casein which $P_1(\eta) = P_2(\eta) = ... = P_{l-1}(\eta) = 0, \quad P_l(\eta) \neq 0, \, l \geq 2$ for some point $\eta \in \Sigma(P_0)$.

First, let us make the following remarks important for the sequel.

Remark 2. 1) When we compare a monomial ξ^ν and P° polynomial P, (or two polynomials Q and P), we can assume that the coefficients of these polynomials are real. Otherwise, we can compare the polynomials $|Q(\xi)|^2$ and $|P(\xi)|^2$. This is possible thanks to a simple lemma proved in [21], which says that if $\Re = \Re(R)$ is the N.P. of a polynomial R and $\mathfrak{M} = \mathfrak{M}(|R|^2)$ is the N.P. of the polynomial $|R|^2$, then $\Re$ is similar to $\mathfrak{M}$ with a similarity coefficient is equal to 2 and the similarity center at the origin. Moreover, if the similar faces are denoted by the same indices (i, k), then $[|P|^2]^{i,k}(\xi) = |P^{i,k}(\xi)|^2$. In particular, this means that if the face $\Re_i^k$ of the polyhedron $\Re$ is principal (degenerate, non-degenerate), then the face $\mathfrak{M}_i^k$ of the polyhedron $\mathfrak{M}$ is also principal (degenerate, non - degenerate) and vice versa.

2) If a polynomial P satisfies the conditions of Theorem 1.1 and the polyhedron $\Re^*$ is complete, then $P \in \mathbb{I}_n$.

3) If $P \in \mathbb{I}_n$, then outside of some ball the polynomial P does not change its sign. Therefore, if necessary, multiplying by (-1) and adding a positive constant (which does not affect their power), we can assume that the polynomials $P \in \mathbb{I}_n$ are everywhere positive. [21]

4) For polynomials $P \in \mathbb{I}_n$, the following simple proposition holds.

Lemma 1.2. *Let $\Re = \Re(P)$ be the N.P. of a polynomial $P \in I_n$ and $\Re_i^k$ $(i = 1, 2, ..., M_k, \, k = 0, 1, ..., n-1)$ be the principal faces of $\Re$. Then*

a) *the polyhedron $\Re = \Re(P)$ is complete,*

b) $P^{i,k}(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}^n$ $(i = 1, 2, ..., M_k, \, k = 0, 1, ..., n-1)$,

c) *let a pair of indices (i, k) $(1 \leq i \leq M_k, \, 0 \leq k \leq n-1)$, a vector $\lambda \in \Lambda(\Re_i^k)$ and a point $\eta \in \Sigma(P^{i,k})$ be fixed; moreover (see representation (1.5)) $P_j(\eta) = 0 \quad (j = 0, 1, ..., l-1), \quad P_l(\eta) \neq 0$ $(1 \leq l \leq M)$, then $P_l(\eta) > 0$.*

Proof. Property a) is obvious. In both cases b) and c), assuming the converse, that $P^{i,k}(\eta) < 0$ (respectively, $P_l(\eta) < 0$) for some point $\eta \in \Sigma(P^{i,k})$, we get that on the sequence $\{\xi^s := s^\lambda \, \eta\}_{s=1}^{\infty}$ $P(\xi^s) \to -\infty$ for $s \to \infty$, which contradicts our assumption $P(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. □

With all this in mind, Theorem 1.1 can be rephrased as follows.

Theorem 1.1′ *Let $\Re = \Re(P)$ be the Newton polyhedron of a polynomial $P \in \mathbb{I}_n$. Let all the principal faces $\Re_i^k$ $(i = 1, ..., M_k, \, k = 0, 1, ..., n-1)$ of the polyhedron $\Re$, except (possibly) one $(n-1)-$dimensional face $\Gamma := \Re_{i_0}^{n-1}$ $(1 \leq i \leq M_k : 1 \leq k_0 \leq n-1)$ (with the outward normal μ), are non-degenerate. Then, if Γ is also non-degenerate, for any $\nu \in \Re$ estimate (1.6) holds. If Γ is degenerate, then with respect to the vector μ, we represent the polynomial P by formula (1.5). Suppose $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0)$ and let $\Re^*$ denote the Newton polyhedron of the set $\{\beta \in \Re, (\mu, \beta) \leq d_1\}$.*

Then estimate (1.6) holds if and only if $\nu \in \Re^*$.

Corollary 1.1. *Obviously, under the assumptions of Theorem 1.1′ $P_0 < P$ and $P_1 < P$.*

Remark 3. It goes without saying that in Theorems 1.1 and 1.2, in essence, only the cases in which the polyhedron $\Re^*$ is complete are interesting. Moreover in this case, obviously, Theorems 1.1 and 1.1′ are equivalent (see also Lemma 1.2).

We are now in a position to move on to our main task. Namely, let a degenerate polynomial $P \in \mathbb{I}_n$ be represented in form (1.5), with $P_l(\eta) \neq 0$ $(1 < l \leq M)$ for all $\eta \in \Sigma(P_0)$, and each of the polynomials P_j $(1 \leq j \leq l-1)$ vanishes at least at one point $\eta \in \Sigma(P_0)$. Let $\Re^{**}$ denote the Newton polyhedron of the set $\{\beta \in \Re, (\mu, \beta) \leq d_l\}$. Under what conditions on the polynomials P_j $(1 \leq j \leq l-1)$ inequality (1.6) is valid for all $\nu \in \Re^{**}$?

Let us paraphrase the problem in terms convenient for us. Let a polynomial P be represented in form (1.5) and satisfy the above conditions. Denote $\mathcal{P}(\xi) := P_0(\xi) + P_l(\xi) + P_{l+1}(\xi) + ... + P_M(\xi)$,

$\mathcal{P}_1(\xi) := P_1(\xi)+...+P_{l-1}(\xi)$. Then $P(\xi) = \mathcal{P}(\xi) + \mathcal{P}_1(\xi)$. If $l = 1$, then $\mathcal{P}(\xi) \equiv P(\xi)$, $\Re^{**} = \Re^*$ and from Theorem 1.1$'$ it follows that $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^{**}$. Let $l \geq 2$, what should be the polynomials P_j $(j = 1, ..., l-1)$ in order for the polynomial P to satisfy the conditions $\xi^\nu < P = \mathcal{P} + \mathcal{P}_1$ for all $\nu \in \Re^{**}$?

Since the polynomial $\mathcal{P}$ satisfies the assumptions of Theorem 1.1$'$ (with P_1 replaced by P_l) then $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^{**}$, it is clear that the polynomials P_j $(j = 1, ..., l-1)$ must be such, that the relation $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$ holds.

The question posed is a special case of the following more general question (which, in addition to having numerous applications in the general theory of linear differential equations, is of independent interest): what polynomials $\{ r(\xi) \}$ can be added to a polynomial $R(\xi)$, so that

a) $\Re(R + r) = \Re(R)$,

b) $r < R$,

c) the polynomials R and $R + r$ have the same power, i.e. $R < R + r < R$

We will call such polynomials r the lower-order terms of R.

Except that (as we will see below) the method of adding lower-order terms to a given differential operator (polynomial) that preserve (do not change) the power of the original operator (polynomial) will be directly applied to solving the problem posed by us, we present a number of other uses to illustrate the importance of this capability.

1) ([12], Theorem 11.1.9) If the operators $P(D)$ and $Q(D)$ have the same strength (by L. Hörmander) and $P(D)$ is hypoelliptic, then $Q(D)$ is also hypoelliptic.

2) ([31], Theorem 2) Let P and Q be polynomials with real coefficients with degrees m_P and m_Q respectively $(m_P > m_Q)$. If for any real number a the polynomial $P + a\,Q$ is hypoelliptic, then the polynomial Q is also hypoelliptic.

3) ([20],Theorem 1) Let a hypoelliptic polynomial P be represented by a vector $\lambda \in \mathbb{E}^n$ in form (1.5), where $M = 1$. Let R be a $\lambda-$homogeneous polynomial of $\lambda-$degree $d(R) : d_1 < d(R) < d_0$ and $R < P_0$. Then $P + R$ is also hypoelliptic

4) ([20], Theorem 2) If a polynomial P (with generally speaking complex coefficients) is hypoelliptic and $Q < P$, then there exists a number $\varepsilon > 0$, such that for any complex number $a : |a| < \varepsilon$ the polynomial $P + a\,Q$ is hypoelliptic.

5) ([12, Section 12.4], [11], [3]) Let P_m be a homogeneous polynomial, hyperbolic with respect to the vector $N \in \mathbb{R}^n$ and Q be a polynomial such that $ord\,Q < m$. Then the polynomial $P_m + Q$ is hyperbolic (with respect to the N) if and only if $Q \prec P_m$ (see Definition 4)

These and other examples show the importance of finding the widest possible classes of lower-order terms for a given (in particular, generalized-homogeneous) polynomial.

Thus, our problem is reduced to finding conditions under which the polynomials $P_1, \cdots, P_{l-1}$ are the lower- order terms of the polynomial $\mathcal{P}$, i.e. for which a) $P_j < \mathcal{P}\,(j = 1, 2, ..., l-1)$, b) $P < \mathcal{P} < P$.

We will deal with this issue in the next section.

2 Comparison of powers of polynomials

Note that everywhere below, when comparing polynomials (or monomials and polynomials), we will consider only the case of the presence of an $(n-1)-$dimensional degenerate face. The case of the presence of a degenerate face of dimension $k < n-1$ is staded by comparing the method of proving Theorem 2.2 of this paper and the method of proving Theorem 1.2 formulated in present paper and proved in [19].

Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{E}^n$ be a vector with positive rational coordinates and $R(\xi) = \sum_{(\lambda,\alpha)=d_R} \gamma_\alpha^R\, \xi^\alpha$ be a $\lambda-$homogeneous polynomial. As usual, we denote by (R) the set of multi

- indices $\{\alpha\}$ for which $\gamma_\alpha^R \neq 0$ and by $\Re(R)$ we denote the Newton polyhedron of the set $(R) \cup 0$. Further, we will assume that the polyhedron $\Re(R)$ has a dimension n. Also put $\Sigma(R) := \{\xi; \xi \in \mathbb{R}^{n,0}, |\xi, \lambda| = 1, R(\xi) = 0\}$ and for the points $\eta \in \Sigma(R)$ denote

$$\mathfrak{A}(\eta, R) := \{\nu; \nu \in \mathbb{N}_0^n, D^\nu R(\eta) \neq 0\},\ \Delta(\eta, R) := \min_{\nu \in \mathfrak{A}(\eta, R)} (\lambda, \nu). \tag{2.1}$$

It is natural to start the comparison with the simplest case, namely with the comparison of generalized- homogeneous polynomials.

2.1 Comparison of powers of generalized-homogeneous polynomials

First, let us make the following remark

Remark 4. It is geometrically obvious that a sub - polynomial corresponding to the face $\Re_i^k$ of the polyhedron $\Re(R) = \Re(R \cup 0)$ has the form $R^{i,k}(\xi)$ or $R^{i,k}(\xi) + 1$. Therefore, only the faces $\Re_i^k$ that are formed without taking into account the point zero (that is the faces $\Re_i^k$ to which the polynomials $R^{i,k}(\xi)$ correspond without the participation of unity), can be degenerate, because the remaining faces correspond polynomials of the form $R^{i,k}(\xi) + 1$, where $R^{i,k}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ (see Remark 2).

The next proposition was proved in [16]

Theorem 2.1. *Let R be a $\lambda-$homogeneous polynomial of $\lambda-$order d_R. Let all the principal faces $\Re(R)$ of the polynomial R be non - degenerate, except possibly the $(n-1)-$dimensional face Γ containing the set (R). Then*

I) If the face Γ is non - degenerate, then $r < P$ for any $\lambda-$homogeneous polynomial r of $\lambda-$order $d_r \leq d_R$ such that $\Re(r) \subset \Re(R)$;

II) If the face Γ is degenerate, then $r < R$ if and only if the following conditions are simultaneously satisfied

1) $d_r \leq d_R$,
2) $\Sigma(r) \supset \Sigma(R)$,
3) $\Re(r) \subset \Re(R)$,
4)(*see notation* (2.1))

$$\frac{d_r}{d_R} \leq \frac{\Delta(\eta, r)}{\Delta(\eta, R)} \quad \forall \eta \in \Sigma(R), \tag{2.2}$$

5) *for each point $\eta \in \Sigma(R)$ there exists a neighborhood $U(\eta)$ and a constant $c = c(\eta) > 0$ such that*

$$|r(\xi)|^{1/\Delta(\eta,r)} \leq c\,|R(\xi)|^{1/\Delta(\eta,R)} \quad \forall \xi \in U(\eta). \tag{2.3}$$

In general, for generalized polynomials Q and P the relation $Q < P$ does not guarantee, that the polynomial Q is a lower-order term of the polynomial P, that is, $P < P + Q < P$ However, it turns out that for generalized - homogeneous polynomials r and R from $r < R$ it follows that $R < R + r < R$.

Let us prove the last statement. Firstly note, that since the numbers $\lambda_1, ..., \lambda_n$ are positive and rational, and any $\lambda-$homogeneous polynomial R is also $(k\,\lambda)-$homogeneous, then choosing a natural number k in an appropriate way (which does not affect the d_Q/d_P ratios), we can assume, that the numbers d_Q and d_P are natural, hence the functions P^{d_Q} and Q^{d_P} are also polynomials. Therefore, we can compare their power. Moreover, the following proposition holds

Lemma 2.1. *Let P and Q be $\lambda-$homogeneous polynomials of $\lambda-$orders d_P and d_Q respectively, where $d_P \geq d_Q$. Then*

a) $Q < P$ *if and only if* $Q^{d_P} < P^{d_Q}$, *(what is the same* $Q < P^{d_Q/d_P}$*) i.e. there is a number* $c > 0$ *such that*

$$|Q(\xi)|^{d_P} \leq c\,[1 + |P(\xi)|^{d_Q}] \ \ \forall \xi \in \mathbb{R}^n, \tag{2.4}$$

b) *if* $P > Q$ *and* $d_Q < d_P$, *then*

b.1) $|Q(\xi)|/|P(\xi)| \to 0$ *and* $|Q(\xi)|/|P(\xi) + Q(\xi)| \to 0$ *for* $|Q(\xi)| \to \infty$ *(hence* $|P(\xi)| \to \infty$*)*,

b.2) $P < P + Q < P$.

Proof. Let us prove item a). Since $d_P \geq d_Q$, it is obviously followed from $Q^{d_P} < P^{d_Q}$ that $Q < P$. Consequently, the sufficiency of estimation (2.4) for the ratio $Q < P$ is obvious. We need to prove that estimation (2.4) follows from $Q < P$. On the other hand, to prove the estimate (2.4), it suffices to prove it for sequences $\{\xi^s\}$ such that $|Q(\xi^s)| \to \infty$ for $|\xi^s| \to \infty$.

So, let $\{\xi^s\}$ be such a sequence. From the condition $P > Q$ it also follows that $|P(\xi^s)| \to \infty$ for $s \to \infty$.

Denote $t_s := |P(\xi^s)|$ and $\tau_i^s := t_s^{-\lambda_i/d_P}\,\xi_i^s$ $(i = 1, ..., n)$ i.e. $\xi^s = t_s^{\lambda/d_P}\,\tau^s$, $P(\tau^s) = 1$ $(i = 1, 2, ..., n;\ s = 1, 2, ...\cdot)$ Consider the individual parts of the inequality (2.4) on this sequence.

Due to the $\lambda-$homogeneity of the polynomials Q and P and bearing in mind that $P(\tau^s) = 1$ $(s = 1, 2, ...)$, we will have

$$|Q(\xi^s)| = t_s^{d_Q/d_P}\,|Q(\tau^s)|, \quad |P(\xi^s)|^{d_Q/d_P} = t_s^{d_Q}\,|P(\tau^s)|^{d_Q/d_P} = t_s^{d_Q}.$$

Since $Q < P$, from these representations and from $P(\tau^s) = 1$ $(s = 1, 2, ...)$ we have

$$|Q(\xi)^s|/[1 + |P(\xi^s)|^{d_Q/d_P}] = t_s^{d_Q/d_P}\,|Q(\tau^s)|/[1 + t_s^{d_Q}]$$

$$\leq c\,t_s^{d_Q/d_P}\,[1 + P(\tau^s)]/[1 + t_s^{d_Q}] = 2\,c\,t_s^{d_Q/d_P - d_Q} = 2\,c\,t_s^{d_Q(1/d_P - 1)}.$$

Since $d_P \geq 1$ and $t_s \to \infty$ for $|\xi^s| \to \infty$ we obtain item a) of the lemma.

Item b.1) directly follows from item a), if both sides of the (already proved) inequality (2.4) are divided by $|P(\xi)|^{d_P}$ and $|P(\xi)|$ tends to infinity. Similarly, it turns out that b.1) implies b.2). $\square$

Corollary 2.1. *From part* b.2) *of Lemma* 2.1 *it follows that if* P *and* Q *are generalized - homogeneous polynomials satisfying the conditions* $P > Q$ *and* $d_Q < d_P$, *then polynomial* Q *is the lower - order term of the polynomial* P, *that is* $P < P + Q < P$. *As mentioned above, below we will make sure that, generally speaking, this is not the case for general polynomials (see examples* 2.3 *and* 3.1 *below).*

2.2 Comparison of powers of general polynomials

In this section, we set ourselves the task of comparing the powers of two general polynomials. Exactly: let P be a given polynomial with the complete Newton polyhedron $\Re(P)$ and Q be some polynomial. Find the conditions under which $Q < P$. If polynomial P is non - degenerate and $\Re(Q) \subset \Re(P)$, then by Theorem 2.1 $Q < P$. Therefore, we only need to consider the case when polynomial P is degenerate.

Before proceeding to the comparison of general polynomials, we prove one simple proposition, which, comparing a general polynomial with a generalized homogeneous polynomial reduces to comparing two generalized homogeneous polynomials and which, in our opinion, is also independent interest.

Lemma 2.2. *Let* R *be a* $\lambda-$*homogeneous polynomial of* $\lambda-$*order* d_R *and* Q *be a general polynomial represented in form* (1.1) *of as the sum of* $\lambda-$*homogeneous polynomials, i.e.*

$$Q(\xi) = \sum_{j=1}^{N(Q)} Q_j(\xi) = \sum_{j=1}^{N(Q)} \sum_{(\lambda,\alpha)=\delta_j} \gamma_\alpha^Q \, \xi^\alpha, \;\; \delta_1 > \delta_2 > ... > \delta_{N(Q)} \geq 0.$$

Then relation $R > Q$ *holds if and only if* $Q_j < R \quad (j = 1, ..., N = N(Q))$

Proof. The proof of sufficiency is obvious. Let us prove the **necessity**. Let $R > Q$. We must proof, that $Q_k < R \;\; (k = 1, .., N)$.

Since $R > Q$, for any $t > 0 \quad Q(t^\lambda \, \xi) < R(t^\lambda \, \xi) = t^{d_R} \, R(\xi)$. Consequently $Q(t^\lambda \, \xi) < R(\xi)$ for any $t > 0$.

Choose (and fix) N positive numbers $t_1, ..., t_N$ sPs that the matrix $(t_j^{\delta_k})$ is non - degenerate. We obtain from representation of Q that $Q(t_j^\lambda \, \xi) = \sum_{k=1}^N t_j^{\delta_k} \, Q_k(\xi)$. Therefore each of polynomials $Q_k(\xi) \;\; (k = 1, .., N)$ is a linear combination of polynomials $Q(t_j^\lambda \, \xi)$. This means that there exist numbers $\{a_i^j = a_i^j(t_1, ..., t_N)\}_{i,j=1}^N$ and $\{b_i^j = b_i^j(t_1, ..., t_N)\}_{i,j=1}^N$ such that for all j = 1,...,N

$$Q_j(\xi) = a_1^j \, Q(t_1^\lambda \, \xi) + ... + a_N^j \, Q(t_N^\lambda \, \xi) \;\; \leq b_1^j \, [1 + |R(t_1^\lambda \, \xi)| \,]$$
$$+ ... + b_N^j(t)[1 + |R(t_N^\lambda \, \xi)| \,] = b_1^j \, [1 + t_1^{d_R} \, |R(\xi)| \,] + ... + b_1^j \, [1 + t_N^{d_R} \, |R(\xi)|.$$

Since the vector $t = (t_1, ..., t_N)$ is fixed, denoting $B_j := max \, \{b_i^j; i = 1, ..., N\} \;\; (j = 1, ..., N)$ and $T := max\{t_i^{d_R}, i = 1, ..., N\}$, we obtain for some constant $c_j = c_j(B_j, T, R, Q) > 0$

$$|Q_j(\xi)| \leq c_j[1 + |R(\xi)|] \;\; \forall \xi \in \mathbb{R}^n, \;\; j = 1, ..., N.$$

□

To describe the set of all polynomials which are estimated via a given non-homogeneous, degenerate polynomial P, as above, we first consider the simplest case in which $P \in \mathbb{I}_n$, only one principal $(n-1)-$dimensional face of the polyhedron $\Re(P)$ of the polynomial P is degenerate, and $P_1(\eta) \neq 0$ for all $\eta \in \Sigma(P_0)$.

So, let us compare a degenerate polynomial P, represented as the sum of $\mu-$homogeneous polynomials in form (1.5) and a polynomial Q represented in the form (below $\delta_j = \delta_j(\mu) = \delta_j(Q, \mu) \; (j = 0, 1, ..., M(Q); \; \delta_0 > \delta_1 > ... > \delta_{M(Q)})$

$$Q(\xi) = \sum_{j=0}^{M(Q)} Q_j(\xi) = \sum_{j=0}^{M(Q)} Q_{\delta_j(\mu)}(\xi) = \sum_{j=0}^{M(Q)} \sum_{(\mu,\alpha)=\delta_j} \gamma_\alpha^Q \, \xi^\alpha. \tag{2.5}$$

We want to find under which conditions $Q < P$.

First, note the following

1)if $\Re(Q) \subset \Re(P)$, and $\delta_{j_0} \leq d_1$, for some number $j_0 : 1 \leq j_0 \leq M(Q)$, then by Theorem 1.1′ $Q_j < P$ for all $j = j_0, j_0 + 1, ..., M(Q)$. Therefore, it remains to consider the polynomials Q_j for $j = 0, 1, ..., j_0 - 1$.

2) If $Q_j < P_0$ for all $j = 0, 1, ..., j_0-1$, then by Corollary 1.1 $Q_j < P_0 < P$ for all $j = 0, 1, ..., j_0-1$. As a result, we get that $Q < P$.

So, it suffices to consider the case $Q_{j_1} \not< P_0$ for some number $j_1 : 0 \leq j_1 \leq j_0 - 1$, wherein $d_1 < \delta_{j_1} \leq d_0$.

Let us prove two numerical inequalities which will be used in the proof of Theorem 2.2.

Lemma 2.3. *In order the inequality*

$$x^a \, y^b \leq 1 + x^c \, y^d$$

hold for all $x \geq 1, y \in [0, 1]$, *it is necessary and sufficient that the positive numbers* a, b, c, d *satisfy the inequalities:*

1) $a \leq c$

2) $d/b \le c/a$.

Proof. The necessity of condition 1) is obvious. Let us prove **the necessity** of condition 2).

Let condition 2) be violated, i.e. $d/b > c/a$. Let us prove that the required inequality cannot hold. Put $y = x^{-c/d}$. Then

$$x^a y^b = x^{a-b\,(c/d)} = x^{b[(a/b)-(c/d)]};\ \ x^c y^d = 1,$$

Since, according to our assumption $(a/b)-(c/d) > 0$, the obtained relations show that for sufficiently large values of x the required inequality does not hold.

Sufficiency. If $b \ge d$, then the required inequality is obvious. Let $b < d$. Denoting $x^a =: u$, $y^b =: v$, we arrive at the equivalent inequality

$$u\,v \le 1 + u^{\frac{c}{a}}\,v^{\frac{d}{b}} \quad \forall u \ge 1, v \in [0,1].$$

When $u\,v \le 1$ this inequality is obvious. If $u\,v > 1$, then by the conditions of the lemma and the assumption $b < d$ we have

$$u\,v \le (u\,v)^{\frac{d}{b}} = u^{\frac{d}{b}}\,v^{\frac{d}{b}} \le u^{\frac{c}{a}}\,v^{\frac{d}{b}},$$

which proves the required inequality. □

Lemma 2.4. *In order the inequality*

$$x^a\,y^b \le 1 + C[\sigma_1 x^c\,y^d + \sigma_2\,x^{c-d}]$$

to hold for all $x \ge 1, y \in [0,1]$ *and a pair of positive numbers* σ_1, σ_2, *with some constant* $C = C(\sigma_1,\sigma_2) > 0$, *it is necessary and sufficient that the positive numbers* a,b,c,d *satisfy the inequalities:*

1) $a \le c$,

2) $a - b \le c - d$.

Proof. The necessity of condition 1) is obvious. We prove the necessity of condition 2). Let condition 2) be violated, i.e. $a-b > c-d$. and let $y = x^{-1}$, then for $x \to \infty$ we have

$$x^a\,y^b/\{1 + [\sigma_1 x^c\,y^d + \sigma_2\,x^{c-d}]\} = x^{a-b}/[1 + C\,(\sigma_1+\sigma_2)\,x^{c-d}] \to \infty,$$

which proves the necessity of condition 2).

Sufficiency. If $b \ge d$ or $d/b \le c/a$, then the required inequality is a corollary of inequalities in Lemma 2.3. If, however, $d/b > c/a \ge 1$, then the substitution $y = t/x$ yields the equivalent inequality

$$x^{a-b}\,t^b \le C\,[1 + \sigma_1 x^{c-d}\,t^d + \sigma_2 x^{c-d}],$$

which can be easily proved (with any constant $C \ge max\{1, |\sigma_1|, |\sigma_2|\,\}$) if we consider separately the cases $t \ge 1$ and $t < 1$. □

Now, let us turn to the comparison of generalized polynomials P and Q represented forms (1.5) and (2.5), respectively. Moreover, it is obvious that to prove the relation $Q < P$, it is suffices to prove the relations $Q_j < P$ for each $j = 0, 1, ..., M(Q)$. It means, that it is suffices for us to compare the generalized-homogeneous polynomial Q with the generalized polynomial P. In a certain sense the following theorem allows to solve the problem in this case.

Theorem 2.2. I) *Let* $P \in \mathbb{I}_n$ *be a degenerate polynomial with a complete Newton polyhedron* $\Re$, *all principal faces of which are non - degenerate, except one* $(n-1)-$*dimensional face* $\Gamma = \Re^{n-1}_{i_0}$, *(with the outward normal* μ*), which is degenerate. Assume that the polynomial* P *is represented by formula* (1.5) *and that* $P_1(\eta) \ne 0$ *for all* $\eta \in \Sigma(P_0)$. *Let* Q *be a* $\mu-$ *homogeneous polynomial of* $\mu-$*order* $\delta_Q : d_1 < \delta_Q < d_0$ *and* $\Re(Q) \subset \Re(P)$. *Then* $Q < P$ *if and only if*

1) $\Sigma(P_0) \subset \Sigma(Q)$,

2) $(d_0 - d_1)/(\delta_Q - d_1) \geq \Delta(\eta, P_0)/\Delta(\eta, Q), \ \forall \eta \in \Sigma(P_0)$,

3) *if* $n > 2$, *then for every point* $\eta \in \Sigma(P_0)$ *there exists a constant* $c = c(\eta) > 0$ *and a neighborhood* $U(\eta)$ *such that*

$$|Q(\xi)| \leq c|P_0(\xi)|^{(\delta_Q - d_1)/(d_0 - d_1)} \ \forall \xi \in U(\eta).$$

II) *Moreover, if* $Q < P$, *the points of the set* (Q) *are interior points of the polyhedron* $\Re(P)$ *and for each point* $\eta \in \Sigma(P_0)$ *there exists a neighborhood* $U(\eta)$ *such that* $Q(\xi) \geq 0$ *for all* $\xi \in U(\eta)$, *then* $P < P + Q < P$.

Proof. The necessity of condition I.1) is obvious.

The necessity of condition I.2). Assume the converse, i.e. the condition $Q < P$ is satisfied, but there exists a point $\eta \in \Sigma(P_0)$ such that

$$(d_0 - d_1)/(\delta_Q - d_1) < (\Delta(\eta, P_0)/(\Delta(\eta, Q)). \tag{2.6}$$

For $t > 0, \theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n, \ \kappa > 0$ set $\xi_i = \xi_i(t) = \xi_i(t, \theta, \kappa) = t^{\mu_i}(\eta_i + \theta_i t^{-\kappa \mu_i}), \quad i = 1, ..., n.$

Since $D^\alpha P(\eta) = 0$ for all $\alpha \in \mathbb{N}_0^n$ and the condition $(\mu, \alpha) < \Delta(\eta, P_0)$, is satisfied, then according to Taylor's formula, for sufficiently large values of t we have

$$Q(\xi(t)) = t^{\delta_Q} Q(\eta + \theta t^{-\kappa \mu}) = t^{\delta_Q} \sum_\alpha t^{-\kappa(\mu, \alpha)} [D^\alpha Q(\eta)/(\alpha!)] \theta^\alpha$$

$$= t^{\delta_Q - \kappa \Delta(\eta, Q)} \sum_{(\mu, \alpha) = \Delta(\eta, Q)} [D^\alpha Q(\eta)/(\alpha!)] \theta^\alpha + o(t^{\delta_Q - \kappa \Delta(\eta, Q)}).$$

Choose θ in such a way that

$$c = c(\theta) := \sum_{(\mu, \alpha) = \Delta(\eta, Q)} [D^\alpha Q(\eta)/(\alpha!)] \theta^\alpha \neq 0.$$

The existence of such a vector θ obviously follows from the definition of the number $\Delta(\eta, Q)$. In fact, otherwise, it turns out that all the coefficients of the polynomial $c(\theta)$ are equal to zero, which contradicts the definition of $\Delta(\eta, Q)$. Then (for a fixed such θ), we have

$$|Q(\xi(t))| \geq c\, t^{\delta_Q - \kappa \Delta(\eta, Q)}. \tag{2.7}$$

For the polynomials P_0 and P_1 we obviously have for a constant $c_1 > 0$ such that for sufficiently large t

$$|P_0(\xi(t))| \leq c_1 t^{d_0 - \kappa \Delta(\eta, P_0)}, \ \ |P_1(\xi(t))| = t^{d_1} P_1(\eta) (1 + o(1)). \tag{2.8}$$

Obvious geometric arguments show that as $t \to +\infty$

$$r((\xi(t)) := P((\xi(t)) - [P_0((\xi(t)) + P_1((\xi(t))] = o(t^{d_1}). \tag{2.9}$$

We put $\kappa = (d_0 - d_1)/\Delta(\eta, P_0)$, then $d_0 - \kappa \Delta(\eta, P_0) = d_1$, and from (2.8) - (2.9), for a constant $c_2 > 0$ we have

$$|P(\xi(t))| \leq c_2 t^{d_1}. \tag{2.10}$$

It is easy to calculate, that from assumption (2.6) it follows that $d_1 < \delta_Q - \kappa \Delta(\eta, Q)$. From estimates (2.7), (2.10) it follows, that $|Q(\xi(t))|/[1 + P(\xi(t))] \to \infty$ for $t \to \infty$, which contradicts the condition $Q < P$ and proves the necessity of condition I.2).

The necessity of condition I.3). Assume that for some point $\eta \in \Sigma(P_0)$ there ib a sequence $\{\eta^s\}$ such that $P_0(\eta^s) \neq 0$ $(s = 1, 2, ...)$, $\eta^s \to \eta$ for $s \to \infty$ and

$$R(\eta^s) := |Q(\eta^s)|/[|P_0(\eta^s)|^{(\delta_Q - d_1)/(d_0 - d_1)}] \to \infty. \tag{2.11}$$

Set $t_s = |P_0(\eta^s)|^{-1/(d_0 - d_1)}$, $\xi^s = t_s^{\mu} \eta^s$, $s = 1, 2, ...\cdot$ Since $\eta^s \to \eta \in \Sigma(P_0)$ we have $t_s \to \infty$ as $s \to \infty$. Then, as a corollary of the $\mu-$homogeneity of $P_0(\xi)$, $P_1(\xi)$ and $Q(\xi)$, for sufficiently large s we have

$$|P_1(\xi^s)| = t_s^{d_1} |P_1(\eta^s)| = t_s^{d_1} |P_1(\eta)| (1 + o(1)), \tag{2.12}$$

$$|P_0(\xi^s)| = t_s^{d_0} |P_0(\eta^s)| = t_s^{d_1}, \ \ r(\xi) = o(t_s^{d_1}) \tag{2.13}$$

Representations (2.12), (2.13) show that a constant $c_3 > 0$ exists such that for sufficiently large s

$$|P(\xi^s)| + 1 \leq c_3 t_s^{d_1}. \tag{2.14}$$

For $Q(\xi)$ we obtain analogously (see also (2.11))

$$|Q(\xi^s)| = t_s^{\delta_Q} |Q(\eta^s)| = t_s^{\delta_Q} R(\eta^s) |P_0(\eta^s)|^{(\delta_Q - d_1)/(d_0 - d_1)} = R(\eta^s) t_s^{d_1}. \tag{2.15}$$

Estimates (2.14) and (2.15), together with assumption (2.11), show that as $s \to \infty$ we have $|Q(\xi^s)|/[|P(\xi^s)| + 1] \geq [1/c_3] R(\eta^s) \to \infty$. This proves the necessity of condition I.3) for $Q < P$.
Sufficiency. When proving sufficiency, we will use the method, proposed by Mikhailov in the study of non - degenerate polynomials (see [33]) and the method, modified by us, which was used in the study of degenerate polynomials (see, for example, [16] or [19]).

Assume that $Q \not< P$ under the hypotheses of Theorem 2.2, i.e. there exists a sequence $\{\xi^s\}$ such that $\xi^s \to \infty$ as $s \to \infty$ and

$$|Q(\xi^s)|/[|P(\xi^s)| + 1] \to \infty. \tag{2.16}$$

Without loss of generality, it can assumed, that all coordinates of the vectors ξ^s are positive. Let

$$\rho_s := exp\sqrt{\sum_{k=1}^{n} (\ln \xi_k^s)^2}, \ \ \lambda_i^s := \frac{\ln \xi_i^s}{\ln \rho_s} \ \ (i = 1, ..., n, \ s = 1, 2, ...). \tag{2.17}$$

Then $\lambda^s = (\lambda_1^s, ..., \lambda_n^s)$ is a unit vector and

$$\xi^s = \rho_s^{\lambda^s} \ \ (\xi_i^s = \rho_s^{\lambda_i^s}, \ i = 1, ..., n), \tag{2.17$'$}$$

It is clear, that $\rho_s \to \infty$ if $|\xi_i^s| \to \infty$ or $|\xi_i^s| \to +0$ for some $i = 1, 2, ..., n$.

Since the vectors λ^s are placed on the unit sphere, the sequence $\{\lambda^s\}$ has a limit point λ^∞. It can be assumed, that $\lambda^s \to \lambda^\infty$, $|\lambda^\infty| = 1$. From the convexity of the polyhedron $\Re(P)$ it follows, that λ^∞ is an outward normal to one and only one face of $\Re(P)$.

Denote λ^∞ by $e^{1,1}$, and choose $n-$ dimensional vectors $(e^{1,1}, e^{1,2}, ..., e^{1,n})$ so that this system forms an orthonormal basis in $\mathbb{R}^n$. Then $\lambda^s = \sum_{i=1}^n \lambda_{1,i}^s e^{1,i}$ $(s = 1, 2, ...)$. Since $\lambda^s \to \lambda^\infty = e^{1,1}$ for $s \to \infty$, then $\lambda_{1,1}^s \to 1$, $\lambda_{1,i}^s = o(\lambda_{1,1}^s)$ for $i = 2, 3, ..., n$.

If it is possible to choose a sub-sequence in a such way, that $\sum_{j=2}^n \lambda_{1,i}^s e^{1,j} = 0$ for all sufficiently large s, , then the basis $(e^{1,2}, ..., e^{1,n})$ we shall denote by $e^1, ..., e^n$. Otherwise, by appropriate choice of a sub-sequence we may assume that $\sum_{j=2}^n \lambda_{1,i}^s e^{1,j} \neq 0$ for all $s = 1, 2, ...$ and for $s \to \infty$

$$[\sum_{i=2}^{n} \lambda_{1,i}^s e^{1,i}]/|\sum_{i=2}^{n} \lambda_{1,i}^s e^{1,i}| \to e^{2,2}.$$

In the subspace spanned by $(e^{1,2}, e^{1,3}, ..., e^{1,n})$ we pass to a new orthonormal basis $(e^{2,2}, e^{2,3}, ..., e^{2,n})$ with the vector $e^{2,2}$ defined above. Then, if $n \geq 3$

$$\lambda^s = \lambda^s_{1,1}\, e^{1,1} + \lambda^s_{2,2}\, e^{2,2} + \sum_{i=3}^{n} \lambda^s_{2,i}\, e^{2,i}, \ (s = 1, 2, ...),$$

hence $\lambda^s_{1,1} \to 1$, $\lambda^s_{2,2} = o(\lambda^s_{1,1})$, $\lambda^s_{2,i} = o(\lambda^s_{2,2})$, $i = 3, ..., n$ for $s \to \infty$.

Reasoning analogously, as in the subspace with the basis $(e^{2,3}, ..., e^{2,n})$ etc., we finally obtain (after modifying the notation) that $\lambda^s = \sum_{i=1}^n \lambda^s_i\, e^i$, where $(e^1, ..., e^n)$ is an orthonormal basis, and $\lambda^s_1 \to 1, \lambda^s_{i+1} = o(\lambda^s_i)$, $i = 1, ..., n-1$ for $s \to \infty$.

Moreover, there exist numbers s_0 and $m : 1 \leq m \leq n$ such that for all $s \geq s_0$ we have $\lambda^s_i > 0$ for $(i = 1, ..., m)$ and $\lambda^s_i = 0$ $(i = m+1, ..., n)$. By choosing a sub-sequence, we may assume, that $s_0 = 1$, $\lambda^s_i > 0$ for all $(i = 1, ..., m)$ and $s \in \mathbb{N}$.

Now we associate the constructed basis with the polyhedron $\Re$. We select the faces $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ as follows: denote by $\Re^{k_1}_{i_1}$ the faces of $\Re(P)$ which lie in the supporting hyperplane of $\Re(P)$ with the outward normal e^1, and each face $\Re^{k_j}_{i_j}$ $(j = 2, ..., n)$ either coincides with the previous one, or is its sub-face, which lies in the supporting hyperplane with the normal e^j. If there are several sub-faces $\Re^{k_j}_{i_j}$ with the normal e^{j+1}, then as $\Re^{k_{j+11}}_{i_{j+1}}$ we agree to take the one for which points α the expression (e^{j+1}, α) is maximal.

From the construction of the faces $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ it is obvious, that their dimensions are subject to the relation: $k_1 \geq k_2 \geq ..., \geq k_m$ and (see (2.17) - (2.17′))

$$\xi^s = \rho_s^{\sum_{i=1}^{n} \lambda^s_i e^i} \ (s = 1, 2, ...),$$

wherein, it can be assumed that $\rho_s \to \infty$ for $s \to \infty$ and some r $(1 \leq r \leq m)$

$$\rho_s^{\lambda^s_j} \to \infty \ (j = 1, ..., r), \ \ \rho_s^{\lambda^s_{r+1}} \to b \geq 1, (s = 1, 2, ..).$$

When $r = m = n$, then we shall assume, that $\lambda^s_{n+1} = 0$ $(s = 1, 2, ..)$, and e^{n+1} is an arbitrary unit vector.

Let, as above, $P^{i_j,k_j}(\xi)$ be the sub-polynomial of $P(\xi)$, corresponding to the face $\Re^{k_j}_{i_j}$, i.e $P^{i_j,k_j}(\xi) := \sum_{\beta \in \Re^k_i} \gamma_\beta\, \xi^\beta$, and α be an arbitrary multi-index belonging to all $\Re^{k_j}_{i_j}$ $(j = 1, ..., m)$, i.e $\alpha \in \Re^{k_m}_{i_m}$. We will study the behaviour of polynomials $P(\xi)$ and $Q(\xi)$ for $\rho_s \to \infty$ and $\xi^s = \rho_s^{\lambda^s_1 e^1 + \lambda^s_2 e^2 + ... + \lambda^s_n e^n}$.

Further, for brevity, when this does not cause misunderstanding, we omit the index s in the notation.

Then, from e^j-homogeneity of polynomials $\{P^{i_j,k_j}(\xi)\}$ and convexity of $\Re(P)$ and its faces, for certain positive numbers $\sigma_1, ..., \sigma_r$ and multi-index $\alpha \in \Re^{k_r}_{i_r}(P)$ we get

$$\begin{aligned} P(\xi) &= \rho^{(\alpha, \lambda_1 e^1)}[P^{i_1,k_1}(\rho^{\sum_{j=2}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_1 \lambda_1})] \\ &= \rho^{(\alpha, \lambda_1 e^1 + \lambda_2 e^2)}[P^{i_2,k_2}(\rho^{\sum_{j=3}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_2 \lambda_2})] = ... \\ &= \rho^{(\alpha, \sum_{j=1}^{r} \lambda_j e^j)}[P^{i_r,k_r}(\rho^{\sum_{j=r+1}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma_r \lambda_r})]. \end{aligned} \tag{2.18}$$

Similarly, for the polynomial Q, for a number σ'_r and a multi-index $\beta \in \Re^{k_r}_{i_r}(Q)$ we have

$$Q(\xi) = \rho^{(\beta, \sum_{j=1}^{r} \lambda_j e^j)}[Q^{i_r,k_r}(\rho^{\sum_{j=r+1}^{n+1} \lambda_j e^j}) + o(\rho^{-\sigma'_r \lambda_r})]. \tag{2.18′}$$

Since $\rho_s^{\lambda_{r+1}^s} \to b \geq 1$, it follows, that $\rho_s^{(\alpha, \sum_{r+1}^{n+1} \lambda_j^s e^j)} \to b^{e^{r+1}} := \eta$ for $s \to \infty$. It is clear, that $0 < \eta_i < \infty$ for all $i = 1, ..., n$ (in accordance with the definition of η_i).

Let us consider two cases: a) $(e^1, \alpha) > 0$ and b) $(e^1, \alpha) = 0$. The case $(e^1, \alpha) < 0$ is impossible because of the fact, that equation for supporting hyperplane with the outward normal λ of $\Re$ can be written in the form $(\lambda, \alpha) = d$, where $d \geq 0$ is the distance from the origin to the given hyperplane and α is a point of the hyperplane (see, for example, [1]).

Case a.1) Firstly suppose, that $P^{i_r, k_r}(\eta) \neq 0$. Since $(e^1, \alpha) > 0$, $\lambda_1^s \to 1$ and $\lambda_i = o(\lambda_1)$ for $i = 2, ..., n$,, for sufficiently large s eventually we have, that $(\alpha, \sum_1^r \lambda_j e^j) > 0$. Therefore, (2.18) implies that

$$P(\xi) = \rho^{(\alpha, \sum\limits_{j=1}^{r} \lambda_j e^j)} [P^{i_r, k_r}(\eta) + o(1)]. \tag{2.19}$$

Similarly, for the polynomial $Q(\xi)$

$$Q(\xi) = \rho^{(\beta, \sum\limits_{j=1}^{r} \lambda_j e^j)} [Q^{i_r, k_r}(\eta) + o(1)]. \tag{2.20}$$

We show, that

$$(\beta, \sum_{j=1}^{r} \lambda_j e^j) \leq (\alpha, \sum_{j=1}^{r} \lambda_j e^j). \tag{2.21}$$

Since $\beta \in \Re_{i_r}^{k_r}(\Re(Q))$, $\alpha \in \Re_{i_r}^{k_r}(\Re(P))$, $\Re(Q) \subset \Re(P)$ and e^1 is the normal of the face $\Re_{i_1}^{k_1}(\Re(P))$, hence $(\beta, e^1) \leq (\alpha, e^1)$. If $(\beta, e^1) < (\alpha, e^1)$, then inequality (2.21) follows from the fact that $\lambda_1 \to 1$ and $\lambda_{j+1} = o(\lambda_j)$ for $j = 1, 2, ..., r-1$. If $(\beta, e^1) = (\alpha, e^1)$, then this means that the points β and α belong to the same face $\Re_{i_1}^{k_1}$. Since $\beta \in \Re_{i_r}^{k_r}(\Re(Q))$ and $\Re(Q) \subset \Re(P)$, hence $(\beta, e^2) \leq (\alpha, e^2)$. If $(\beta, e^2) < (\alpha, e^2)$, then inequality (2.21) follows from the same fact, regarding the numbers λ_j. If $(\beta, e^2) = (\alpha, e^2)$, this means that the points β and α belong to the same face $\Re_{i_2}^{k_2}$, $(\beta, e^3) \leq (\alpha, e^3)$ and so on.

Continuing this process, after a finite number of steps, we either arrive at the equality $(\beta, e^j) = (\alpha, e^j)$ $j = 1, 2, ..., q-1$, $(\beta, e^j) < (\alpha, e^j)$ for some $q < r$, or for the relation $(\beta, e^j) = (\alpha, e^j)$ for all $j = 1, ..., r$. In both cases, the inequality (2.21) is obvious. Thus, inequality (2.21) is proved.

So, relations (2.19) - (2.21) together contradict our assumption (2.16) and complete the consideration of sub - case a.1) of case a).

Consider the case a.2): $P^{i_r, k_r}(\eta) = 0$. In this case, the face $\Re_{i_r}^{k_r}$ coincides with the $(n-1)-$ dimensional degenerate face $\Gamma := \Re_{i_0}^{n-1}$ (with the outward normal μ) and $r = m = 1$, $k_r = k_1 = n-1$, $e^1 = \mu, \eta \in \Sigma(\Gamma)$.

With respect to the vector $e^1 = \mu$, we represent the polynomial $P(\xi)$ in form (1.5)

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) := \sum_{j=0}^{M} \sum_{(e^1, \alpha) = d_j} \gamma_\alpha \xi^\alpha \tag{2.22}$$

and denote $q(\xi) := P(\xi) - [P_0(\xi) + P_1(\xi)]$. Then,

$$P(\xi) = P_0(\xi) + P_1(\xi) + q(\xi). \qquad (2.23).$$

Substituting

$$\xi (= \xi^s) = \rho^{\sum\limits_{j=1}^{n+1} \lambda_j e^j} = \rho_s^{\sum\limits_{j=1}^{n+1} \lambda_j^s e^j}$$

in (2.23) and using e^1-homogeneity of the polynomials $P_0(\xi)$, $P_1(\xi)$ and $Q(\xi)$ we get (below $h^s := \sum_{j=2}^{n+1} \lambda_j^s e^j$)

$$P(\xi^s) = \rho_s^{\lambda_1^s(\mu,\alpha)} P_0(\rho_s^{h^s}) + \rho_s^{\lambda_1^s(\mu,\beta)} P_1(\rho^{h^s}) + q(\xi), \tag{2.24}$$

ord

$$Q(\xi^s) = \rho_s^{\lambda_1^s(\mu,\gamma)} Q(\rho^{h^s}) \tag{2.25}$$

Let $s \to \infty$, i.e. $\rho_s \to \infty$, then $\rho_s^{h^s} \to \eta$, $\lambda_1^s(\mu,\alpha) \to d_0$, $\lambda_1^s(\mu,\beta) \to d_1$, $\lambda_1^s(\mu,\gamma) \to \delta_Q$ and by Lemma 1.2 $P_0(\eta) \geq 0$, $P_1(\eta) > 0$.

On the other hand, since $ord\, q \leq d_2 < d_1$, so with some constant $c_4 > 0$ we have $|q(\xi^s)| \leq c_4\, \rho_s^{d_2}$, for all $s = 1, 2, ...,$ i.e. $|q(\xi^s)| = o(\rho_s^{d_1})$ for $s \to \infty$.

Thus, from (2.24) - (2.25) (for sufficiently large value of s) we have

$$P(\xi^s) = \rho_s^{\lambda_1^s(\mu,\alpha)}\, P_0(\rho_s^{h^s}) + \rho_s^{\lambda_1^s(\mu,\beta)}\, P_1(\rho_s^{h^s}) + o(\rho_s^{\lambda_1^s(\mu,\beta)}), \tag{2.24$'$}$$

and representation (2.25) for polynomial $Q(\xi)$, where $\rho_s^{h^s} \to \eta$, $\lambda_1^s(\mu,\alpha) \to d_0$, $\lambda_1^s(\mu,\beta) \to d_1$, $\lambda_1^s(\mu,\gamma) \to \delta_Q$, as $s \to \infty$.

Since $\rho_s^{h^s} \to \eta \in \Sigma(P_0)$, then for sufficiently large s (i.e. for sufficiently large ρ_s) condition I.3) of our theorem is satisfied. Then, from (2.25) and condition I.3) for sufficiently large s and for a constant $c_5 > 0$ we obtain

$$|Q(\xi^s)| = \rho_s^{\lambda_1^s(\mu,\gamma)}\, |Q(\rho_s^{h^s})| \leq c_5\, \rho_s^{\lambda_1^s(\mu,\gamma)} |P_0(\rho_s^{h^s})|^{(\delta_Q - d_1)/(d_0 - d_1)}. \tag{2.26}$$

According to the conditions of our theorem $P \in \mathbb{I}_n$, therefore, for indicated s $P_0(\rho_s^{h^s}) \geq 0$ and $P_0(\eta) = 0$, $P_1(\eta) > 0$. So, we can assume, that $0 \leq P_0(\rho_s^{h^s}) \leq 1$, $P_1(\rho_s^{h^s}) \geq \frac{1}{2} P_1(\eta) > 0$ for sufficiently large s and $|q(\xi^s)|/|P(\xi^s)| \to 0$ for $s \to \infty$. This and (2.24$'$) in turn show that

$$|P(\xi^s)| \geq \sigma_1 \rho_s^{d_0}\, P_0(\rho_s^{h^s}) + \sigma_2 \rho_s^{d_1}. \tag{2.27}$$

for sufficiently large s and for positive constants σ_1 and σ_2.

From estimates (2.26) - (2.27) it follows that, in order to obtain a contradiction with (2.16), it suffices to prove the existence of a constant $C = C(\sigma_1, \sigma_2) > 0$ such that for sufficiently large s

$$\rho_s^{\delta_Q} P_0(\rho_s^{h^s})^{(\delta_Q - d_1)/(d_0 - d_1)} \leq C[1 + \sigma_1\, \rho_s^{d_0}\, P_0(\rho_s^{h^s}) + \sigma_2 \rho_s^{d_1}]. \tag{2.28}$$

To prove the estimates (2.28), let us apply Lemma 2.4 with the following notations

$$a := \delta_Q, b := \delta_Q - d_1, c = d_0, d := d_0 - d_1, x := \rho_s, y := [P_0(\rho_s^{h^s})]^{[1/(d_0 - d_1)]}.$$

After introducing these notations, the inequality (2.28) takes the following form

$$x^a\, y^b \leq 1 + C\, |\sigma_1\, x^c\, y^d + \sigma_2\, x^{c-d}|. \tag{2.28$'$}$$

Since $x \geq 1$, $y \in [0,1]$, $a \leq c$, $a - b = c - d = d_1$, then all conditions of Lemma 2.4 are satisfied. According to this lemma, inequality (2.28$'$) holds, therefore, inequality (2.28) holds. Resulting inequality (2.28) contradicts our assumption (2.16) and completes the consideration of sub-case a.2) and, therefore, completes the consideration of case a).

Let us move to case b) $(e^1, \alpha) = 0$.

Firstly, note, that if the Newton polyhedron $\Re$ of the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ is complete, then the Newton polyhedron of polynomial $P(\xi)|_{\xi_j=0}$ for $j \in [1,n]$ is also complete in the appropriate $(n-1)-$dimensional subspace. Secondly, in the case b) under consideration, the face, whose outward normal is e^1, clearly passes through the origin and hence is not a principal face of $\Re$; consequently, $e_i^1 \leq 0$ $(i = 1, ..., n)$. In this connection, if non principal face with outward normal e^1 has dimension

$l \le n-1$, then l if the numbers e_i^1 $(1 \le i \le n)$ are equal to zero with the remaining numbers being negative. Without loss of generality it can clearly be assumed, that $e_1^1 = ... = e_l^1 = 0$, $e_{l+1}^1 < 0, ..., e_n^1 < 0$.

Since

$$e_j^1 = \lim_{\xi\to\infty}[ln\xi_j/(\sum_{k=1}^{n}(ln\xi_k)^2)^{1/2}] \ < 0 \ (j = l+1, ..., n),$$

beginning with some number s_0 (we assume that $s_0 = 1$) we have, that $\xi_j^s < 1$ $(j = l+1, ..., n)$ $(s = 1, 2, ...)$. On the other hand, since $|\xi^s| \to \infty$ for $s \to \infty$, we have $\xi_i^s \to \infty$ for certain $i \in [1, l]$. But since $e_i^1 = 0$ for such i, hence (at least for some subsequence of the sequence ξ^s) $\xi_j^s \to 0$ for $s \to \infty$ and at least one $j \in (l, n]$.

Suppose, that (after a possible renumbering) $\xi_l^s \to \infty, ..., \xi_{l_0}^s \to \infty$ $(l_0 \ge l)$ for $s \to \infty$ and $\xi_{l_0+1}^s \to 0, ..., \xi_{l_0+l_1}^s \to 0$ $(l_0 + l_1 \le n)$.

Let $\psi(\xi) := max_{1\le j\le l_0}\xi_j$, then it is obvious, that as $s \to \infty$

$$ln\psi(\xi^s)/[\sum_{k=1}^{n}(ln\xi_k^s)^2]^{1/2} \to 0. \tag{2.29}$$

On the other hand, there clearly exist positive constants c_6, c_7 such that

$$c_6 \le \sum_{k=1}^{l_0}(ln\xi_k^s)^2\,(ln\psi(\xi^s))^2 \le c_7 \quad (s = 1, 2, ...). \tag{2.30}$$

From (2.29)-(2.30) it follows that

$$\sum_{k=l_0+1}^{n}(ln\xi_k^s)^2\,(ln\psi(\xi^s))^2 \to \infty \quad as \quad s \to \infty. \tag{2.31}$$

From this result, going over a sub-sequence, if necessary, we can get, that for some $j \in [l_0+1, n]$

$$|ln\xi_j^s|/ln\psi(\xi^s) \to \infty \quad as \quad s \to \infty, \tag{2.32}$$

i.e. $|ln\xi_j^s| \to \infty$ "faster" than $ln\psi(\xi^s) \to \infty$. Hence $\xi_j^s = o([\psi(\xi^s)]^{-\sigma})$ for some $\sigma > 0$ or, equivalently,

$$(\xi_j^s)^{\alpha_1}.[\psi(\xi^s)]^{\alpha_2} \to 0 \quad as \quad |\xi| \to \infty \tag{2.33}$$

for $\alpha_1 > 0$ and $\alpha_2 \ge 0$.

Let $\breve{\xi} = (\breve{\xi}_1, ..., \breve{\xi}_n)$, where $\breve{\xi}_j = 0$ if j satisfies the condition (2.32) and $\breve{\xi} = \xi_j$ otherwise.

In view of (2.33) from (2.16) it follows that

$$|Q(\breve{\xi}^s)|/[1 + |P(\breve{\xi}^s)|] \to \infty \quad as \quad s \to \infty \tag{2.34}$$

(under our limit process, i.e. with the possibility of repeatedly going over the sub-sequences of the sequence $\{\xi^s\}$ of (2.16)).

As a result, the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ can be transformed into the polynomial $\breve{P}(\xi) := P(\breve{\xi})$ on less than n variables. Consequently, dimension of the polyhedron $\breve{\Re}(P) := \Re(\breve{P})$ is less than the dimension of the polyhedron $\Re(P)$, while the non - degenerate faces of $\Re$ correspond to the non - degenerate faces of $\breve{\Re}$ and vice versa.

Thus, in the process of proving Theorem, relation (2.16) leads either to a contradiction or to relation (2.34), which is analogous to (2.16) but corresponds to a space of dimension less than or equal to $n-1$.

Repeating the arguments, presented above within the proof of this theorem, now with respect to the polynomial $\breve{P}$, and so on, we clearly arrive after a finite number of steps at either a contradiction or relation (2.34) for polynomials of one variable.

But for polynomials of one variable, the polyhedrons $\Re(P)$ and $\Re(Q)$ have the shape of segment, and a contradiction with (2.16) is due to the fact, that $\Re(Q) \subset \Re(P)$.

Thus, the first part of Theorem 2.2 is proved.

Let us prove the second part of Theorem. Repeating reasoning, carried out in the sufficiency proof of the first part of Theorem, i.e. assuming the converse, that there exists a sequence $\{\xi^s\}$ such that $\xi^s \to \infty$ and

$$|P(\xi^s)|/[1+|P(\xi^s)+Q(\xi^s)] \to \infty \ \ as \ s \to \infty, \tag{2.35}$$

In the case a.1) we obtain representation (2.19) for polynomial P and following representation for polynomial $P+Q$

$$P(\xi^s)+Q(\xi^s) = \rho^{(\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} P^{i_r,k_r}(\eta) + \rho^{(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} Q^{i_r,k_r}(\eta) + o(1). \tag{2.36}$$

Since, based on the condition II) of Theorem, the points of the set (Q) are interior points of the set $\Re(P)$, that is $(\beta, e^1) < (\alpha, e^1)$ and $\lambda_1^s \to 1$, $\lambda_j^s = o(\lambda_1^s)$ for $s \to \infty$ $(j = 2, ..., r)$, then $(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j) < (\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)$ for sufficiently large s. Then

$$\rho^{(\beta, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} / \rho^{(\alpha, \sum\limits_{j=1}^{r} \lambda_j^s e^j)} \to 0 \ \ as \ s \to \infty$$

and representations (2.19), (2.36) together contradict (2.35).

In the case a.2): $P^{i_r,k_r}(\eta) = Q^{i_r,k_r}(\eta) = 0$, the face $\Re_{i_r}^{k_r}$ coincides with $(n-1)-$dimensional degenerate face $\Gamma = \Re_{i_0}^{n-1}$ (with the outward normal μ) and $r = m = 1$, $k_r = k_1 = n-1, e^1 = \mu$, $\eta \in \Sigma(\Gamma)$.

In this case, we obtain the representations (2.24') and (2.25) for the polynomials P and Q, respectively, and following representation for the polynomial $P+Q$

$$P(\xi^s)+Q(\xi^s) = \rho^{\lambda_1^s(\mu,\alpha))} P_0(\rho_s^{h^s}) + \rho^{\lambda_1^s(\mu,\gamma))} Q(\rho_s^{h^s})$$

$$+\rho^{\lambda_1^s(\mu,\alpha))} P_1(\rho_s^{h^s}) + o(1). \tag{2.37}$$

Since, according to the conditions (first and second parts) of Theorem $P_0(\rho_s^{h^s}) \geq 0$, $Q(\rho_s^{h^s}) \geq 0$, $P_1(\rho_s^{h^s}) > 0$ for sufficiently large s, it follows from (2.24'), (2.37) that $|P(\xi^s)+Q(\xi^s)| \geq |P(\xi^s)|$ for sufficiently large s. This contradicts our assumption (2.35). □

Let us give examples, illustrating this theorem.

Example 1. Let us compare the polynomial $Q(\xi) = (\xi_1-\xi_2)^2(\xi_1^6+\xi_2^6)$ with the following two polynomials $P^1(\xi) := P_0^1(\xi)+P_1^1(\xi) = (\xi_1-\xi_2)^4(\xi_1^6+\xi_2^6) + (\xi_1^6+\xi_2^6)$ and $P^2(\xi) := P_0^2(\xi)+P_2^1(\xi) = (\xi_1-\xi_2)^4(\xi_1^6+\xi_2^6) + (\xi_1^4+\xi_2^4)$.

Here $d_0^1 = d_0^2 =: d_0$, $d_1^1 = 6$, $d_1^2 = 4$, $\Delta(\eta, P_0^1) = \Delta(\eta, P_0^2) := 4$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$, Simple calculations show, that the pair (P^1, Q) satisfies all conditions of Theorem 2.2, while the pair (P^2, Q) does not satisfy condition 2) of this theorem. Indeed, $(d_0-d_1^2)/(\delta_Q-d_1^2) = 3/2 < 2 = \Delta(\eta, P_0^2)/\Delta(\eta, Q)$. Therefore, $Q < P^1$, but $Q \not< P^2$.

Remark 5. Note, that conditions of Theorem 2.2 do not guarantee the $Q < P_0$, which can be seen from the following example.

Example 2. Let $n = 2$, $P(\xi) := P_0(\xi)+P_1(\xi) = (\xi_1-\xi_2)^8+(\xi_1^2+\xi_2^2)^2$, $Q(\xi) = (\xi_1-\xi_2)^4\,(\xi_1^2+\xi_2^2)$. Here $d_0 = 8, d_1 = 4$, $\delta_Q = 6$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$, $\Delta(\eta, P_0) = 8$, $\Delta(\eta, Q) = 4$.

It is easy to verify, that all conditions of Theorem 2.2 are satisfied, hence $Q < P$. Moreover, applying the arithmetic inequality $a\,b \leq (1/2)(a^2+b^2))$, we obtain, that $P < P+Q$. However, in this case the (necessary) condition II.4) of Theorem 2.1 is violated, and, therefore, $Q \not< P_0$. This can also be verified directly (without resorting to the help of Theorem 2.1) by taking, for example, $\xi_1^s = s+1, \ \xi_2^s = s \ \ s = 1, 2, ...$.

Remark 6. Note, that as we saw above (see the Corollary 2.1), for a pair of generalized - homogeneous polynomials P and Q the relations $Q < P$ and $P < P + Q < P$ are equivalent, however, in general, this does not apply to generalized polynomials. Here are some examples conforming this.

Example 3. Let $n = 2$. Compare the polynomials $P(\xi) = (\xi_1 - \xi_2)^8 + (\xi_1^2 + \xi_2^2)^2$, and $Q(\xi) = (-2,5)(\xi_1 - \xi_2)^6(\xi_1 + \xi_2)$. Here $P_0(\xi) = (\xi_1 - \xi_2)^8$, $P_1(\xi) = (\xi_1^2 + \xi_2^2)^2$, $\Sigma(P_0) = \{\pm\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})\}$, $d_0 = 8, d_1 = 4$, $\Delta(\eta, P_0) = 8$, $\delta_Q = 7, \Delta(\eta, Q) = 6$, $(\delta_Q - d_1)/(d_0 - d_1) = \Delta(\eta, Q)/\Delta(\eta, P_0) = 3/4$, $\eta = \pm(1/\sqrt{2}, 1/\sqrt{2})$.

Conditions 1) - 2) of Theorem 2.2 are obvious, because $\Sigma(P_0) \subset \Sigma(Q)$ and $(d_0 - d_1)/(\delta_Q - d_1) = \Delta(\eta, P_0)/\Delta(\eta, Q), \ \forall \eta \in \Sigma(P_0)$.

To prove condition 3) of Theorem 2.2 for the couple (P, Q), as a neighborhood of $U(\eta)$ for both η and $-\eta$ one can take, for example, a circle, centered at the point η (or $-\eta$) with unit radius. Then, the condition 3) reduces to the existence of a constant $c > 0$ such that the inequality $|(\xi_1 - \xi_2)^6 (\xi_1 + \xi_2) \le c |\xi_1 - \xi_2|^6$ holds for all $\xi \in U(\eta)$. In this case, this inequality is obvious, since $|\xi - \eta|| \le 1$ for the points $\xi \in U(\eta)$. Thus, by Theorem 2.2 $Q < P$

Let us show, that $P \not< P+Q$, i.e., that Q is not of lower order term for the polynomial P. Indeed, simple calculations show, that on the sequence $\{\xi^s = (s + \sqrt{s}, s)\}$ for $s \to \infty$ $|P(\xi^s)| = O(s^4)$ and $|P(\xi^s) + Q(\xi)| = O(s^{3,5})$, i.e. $|P(\xi^s)|/|P(\xi^s) + Q(\xi)| \to \infty$ for $s \to \infty$. It is also easy to see, that $Q \not< P + Q$.

Thus, in general case, Theorem 2.2 does not answer the question: when (under what conditions on the polynomials $P_j \, (j = 1, 2, ..., l-1)$) $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$? We will do this in the next section. But, before moving to the next section, we note the following

Remark 7. 1)from Theorem 1.1′ it follows, that if the polynomial P, with the complete Newton polyhedron $\Re(P)$, is non - degenerate, then $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1 < \mathcal{P}$

2) when $\Gamma := \Re_{i_0}^{n-1}$ is a (unique) degenerate principal face of the polyhedron $\Re(P)$, the conditions (necessary and sufficient) for the fulfillment of the right - hand side of this estimation ($P = \mathcal{P} + \mathcal{P}_1 < \mathcal{P}$) are given by Theorem 2.2 (first part): it means, that each pair of polynomials $(P_j, \mathcal{P}) \, (j = 1, ..., l-1)$ must satisfy the conditions of Theorem 2.2,

3) in a particular case (sufficient), the validity conditions for relation $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$ are given in the second part of Theorem 2.2.

Remark 8. From the course of the proof of Theorem 2.2, it became obvious that under the conditions of this theorem a) $P_0 < P_0 + P_1 < P$ (however, this is clear from the proof of Theorem 1.1′ also), b) the polynomials $P_2, ..., P_M$ do not affect the behavior at infinity of the polynomial P (although they can participate in the construction of the Newton polyhedron $\Re(P)$).

3 Adding lower-order terms and main result

Recall, that in Theorem 1.1′ we considered only the case, when in the studied degenerate polynomial $P = P_0 + P_1 + P_2 + ...$ at all points $\eta \in \Sigma(P_0) := \{\eta \in \mathbb{R}^{n,0} : P_0(\eta) = 0\}$ it was the first of polynomials $\{P_j\}$ that did not vanish: $P_1(\eta) \neq 0 \ \forall \eta \in \Sigma(P_0)$. Now we want to free ourselves from this restriction.

Namely, let, like to Theorem 1.1′, $\Gamma := \Re_{i_0}^{n-1}$ be the only degenerate principal face (with the outward normal μ) of the complete Newton polyhedron $\Re(P)$ of polynomial $P \in \mathbb{I}_n$ and with respect to the vector μ the polynomial P is represented as a sum of $\mu-$homogeneous polynomials

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) = \sum_{j=0}^{M} \sum_{(\mu,\alpha)=d_j} \gamma_\alpha \, \xi^\alpha, \tag{3.1}$$

where $d_0 > d_1 > ... > d_l > ... > d_M \geq 0$.

Suppose, that $P_l(\eta) \neq 0$ $(1 \leq l \leq M)$ for all $\eta \in \Sigma(P_0)$ and each polynomial P_j $(j = 1, 2..., l-1)$ vanishes at least at one point $\eta \in \Sigma(P_0)$ and put $\Re^* := \{\beta \in \Re, (\mu, \beta) \leq d_l\}$, $\mathcal{P}(\xi) := P_0(\xi) + P_l(\xi) + P_{l+1}(\xi) + ... + P_M(\xi)$, $\mathcal{P}_1(\xi) := P_1(\xi) + ... + P_{l-1}(\xi)$. If $l = 1$, then $\mathcal{P}(\xi) \equiv P(\xi)$ and it follows from Theorem 1.1$'$, that $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^*$.

A question naturally arises: suppose $l \geq 2$, and polynomial $\mathcal{P}$ satisfies the conditions of Theorem 1.1$'$. Therefore, $\xi^\nu < \mathcal{P}$ for all $\nu \in \Re^*$. Which conditions must the polynomials P_j $(j = 1, ..., l-1)$ satisfy, so that for newly introduced set $\Re^*$ the relation $\xi^\nu < P = \mathcal{P} + \mathcal{P}_1$ also holds for all $\nu \in \Re^*$?

To do this, we need to answer the following question (which, besides of numerous applications in differential equations, of course, is also of independent interest): which lower - order terms Q can be added to the polynomial $P = P_0 + P_1 + ...$, so that a) $\Re(P+Q) = \Re(P)$, b) the polynomials P and $R := P + Q$ have the same power, i. e. $P < R < P$? In this case, we will call the polynomial Q of lower - order term with respect to the polynomial P.

It is clear, that in this case our question sounds like this: what should be polynomials $P_1, P_2, ..., P_{l-1}$ so that the polynomials P and $\mathcal{P}$ had the same power, i.e., that the relation $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1 < \mathcal{P}$ held ?

The next proposition in a sense solves the question posed in the class of polynomials that we considered above.

Theorem 3.1. *Suppose that a degenerate polynomial P and a $\mu-$homogeneous polynomial Q of $\mu-$order $\delta_Q \in (d_1, d_o)$ satisfy the conditions of the first part of Theorem 2.2 (consequently $Q < P$). Let for each point $\eta \in \Sigma(P_0)$ and for any sequence $\{\eta^s\}$: $\eta^s \to \eta$ for $s \to \infty$ the following relation is true*

$$\psi(\eta^s) := |Q(\eta^s)|/|P_0(\eta^s)|^{(\delta_Q - d_1)/(d_0 - d_1)} \to 0. \tag{3.2}$$

Then

1) $|Q(\xi)|/[|P(\xi)| + 1] \to 0$ *as* $|\xi| \to \infty$,
2) $Q < P + Q$, $P < P + Q < P$.

Proof of statement 1). Suppose, to the contrary, that conditions of Theorem are satisfied, but there exist a sequence $\{\xi^s\}$ and a number $c_1 > 0$ such that $\xi^s \to \infty$ for $s \to \infty$ and

$$|Q(\xi^s)|/[|P(\xi^s)| + 1] \geq c_1 \;, (s = 1, 2, ...). \tag{3.3}$$

Reasoning as in the proof of Theorem 2.2 we obtain (for sufficiently large s) the following estimations for the polynomial P (see the representation (2.24))

$$P(\xi^s) = \rho_s^{\lambda_1^s(\mu,\alpha)} P_0(\rho_s^{h^s}) + \rho_s^{\lambda_1^s(\mu,\beta)} P_1(\rho_s^{h^s}) + o(\rho_s^{\lambda_1^s(\mu,\beta)}).$$

Since $\lambda_1^s \to 1$ as $s \to \infty$, from this, for a number $c_2 > 0$ and sufficiently large s we have

$$|P(\xi^s)| + 1 \geq c_2 \,[1 + |\rho_s^{d_0} P_0(\rho_s^{h^s}) + \rho_s^{d_1} P_1(\rho_s^{h^s})|]. \tag{3.4}$$

Taking into account condition (3.2), for polynomial Q and the same s we have

$$|Q(\xi^s)| = \rho_s^{\delta_Q} |Q(\rho_s^{h^s})| = \rho_s^{\delta_Q} |P_0(\rho_s^{h^s})|^{(\delta_Q - d_1)/(d_0 - d_1)} \;\; \psi(\rho_s^{h^s}). \tag{3.5}$$

Then, from (3.4) - (3.5), with some constant $c_3 > 0$ we have

$$[|Q(\xi^s)|/|P(\xi^s)| + 1] \leq c_3 \, M(\rho_s^{h^s}) \;\; \psi(\rho_s^{h^s}), \tag{3.6}$$

where
$$M(\rho_s^{h^s}) := \rho_s^{\delta_Q}\,|P_0(\rho_s^{h^s})|^{(\delta_Q-d_1)/(d_0-d_1)}]/[|\rho_s^{d_0}P_0(\rho_s^{h^s})+\rho_s^{d_1}P_1(\eta)\,|+1|].$$

Let us prove the existence of some constant $c_4>0$ for which the following inequality holds
$$M(\rho_s^{h^s}) \le c_4 \;\; (s=1,2,...). \tag{3.7}$$

We introduce the notation $x=x_s=\rho_s^{\delta_Q}$, $y=y_s=|P_0(\rho_s^{h^s})|^{1/(d_0-d_1)}$, $a=\delta_Q$, $b=\delta_Q-d_1$, $c=d_0$, $d=d_0-d_1$. Then inequality (3.7) takes the form
$$x^{\delta_Q}\,y^{\delta_Q-d_1} \le c_4[1+|x^{d_0}\,y^{d_0-d_1}+P_1(\eta)x^{c-d}|], \tag{3.8}$$
where $P_1(\eta)>0$, $x\ge 1$, $y\in[0,1]$ for sufficiently large s.

To prove the inequality ((3.8) we apply the Lemma 2.4. The conditions of this lemma are satisfied, because $a=\delta_Q<d_0=c$, $c-a=d-b=d_0-\delta_Q$, $c-d=d_1$, $\sigma_1=1$, $\sigma_2=P_1(\eta)>0$.

Thus, inequality (3.7) is proved. Since $\psi(\rho_s^{h^s})\to 0$ for $s\to\infty$, the inequalities (3.6), (3.7) together contradict the assumption (3.3) and prove the first part of Theorem.

The second part of Theorem is an immediate consequence of the first part. It is only necessary to reverse the fact, that now the behavior of polynomial Q does not affect the behavior of $P+Q$ when $|\xi|\to\infty$, (i.e. $P(\xi)\to\infty$). □

Let us give an example of a pair of polynomials (P,Q) satisfying the conditions of Theorem 3.1.

Example 4. Let $n=2$, $P(\xi)=(\xi_1-\xi_2)^8+(\xi_1^2+\xi_2^2)^2$, $Q(\xi)=(\xi_1-\xi_2)^5(\xi_1+\xi_2)$.

Here $P_0(\xi)=(\xi_1-\xi_2)^8$, $P_1(\xi)=(\xi_1^2+\xi_2^2)^2$, $\Sigma(P_0)=\{\pm\eta=\pm(1/\sqrt{2},1/\sqrt{2}),\}$, $d_0=8, d_1=4$, $\Delta(\eta,P_0)=8$, $\delta_Q=6$, $\Delta(\eta,Q)=5$, $(\delta_Q-d_1)/(d_0-d_1)=1/2$.

Conditions I.1) - I.3) of Theorem 2.2 can be easily verified, and condition (3.2) of Theorem 3.1 is satisfied, since for any sequence $\{\eta^s\}:\ \eta^s\to\eta$ for $s\to\infty$ we have
$$\psi(\eta^s) := |Q(\eta^s)|/[|P_0(\eta^s)]^{1/2} = ((\eta_1^s)^2-(\eta_2^s)^2)\to\eta_1^2-\eta_2^2=0.$$

At the same time, it is obvious, that $Q\not<P_0$.

As for the pair of polynomials from the Example 2.2 for any sequence $\{\eta^s\}:\ \eta^s\to(1/\sqrt{2},1/\sqrt{2})$ as $s\to\infty$, $\psi(\eta^s)=|Q(\eta^s)/|P_0(\eta^s)|^{1/2}=(\eta_1^s)^2+(\eta_2^s)^2\to 1$, i.e. condition (3.2) is violated. Despite this, as we saw above, $P<P+Q$ and $Q<P+Q$ because the pair (P,Q) satisfies the condition of the second part of Theorem 2.2.

Now we are already in a position to turn into the question, posed at the beginning of this section. Namely, let with respect to a vector $\mu\in\mathbb{R}^n$ a (generalized) polynomial P be represented as a sum of $\mu-$homogeneous polynomials in the form (3.1). We need to describe those multi - indites $\nu\in\mathbb{N}_0^n$ for which $\xi^\nu<P$, i.e. a constant $c=c(\nu,P)>0$ exists, such that
$$|\xi^\nu|\le c\,[|P(\xi)|+1]\ \forall\xi\in\mathbb{R}^n. \tag{3.9}$$

Theorem 3.2 (main result). *Let $\Re=\Re(P)$ be the complete Newton polyhedron of a polynomial $P\in I_n$. Let all of the principal faces of $\Re$ with exception of a $(n-1)-$dimensional face $\Gamma:=\Re_{i_0}^{n-1}$ (with the outward normal μ) be non-degenerate and the face Γ be degenerate.*

Let the polynomial P be represented as a sum of $\mu-$homogeneous polynomials in form (3.1) and
$$P=P_0+P_1+...P_l+...+P_M,$$
where $P_0(\xi)=:P^{i_0,n-1}(\xi)$, P_j is a $\mu-$ homogeneous polynomial of $\mu-$order d_j $j=0,1,...,l,...,M$, $d_0>d_1>...>d_l>...>d_M\ge 0$.

Suppose, that $P_l(\eta)\ne 0$ for all $\eta\in\Sigma(P_0):=\{\xi\in\mathbb{R}^{n,0}\ |\xi,\mu|=1\}$ and each polynomial $P_j\in\mathfrak{M}:=\{P_1,P_2,...,P_{l-1}\}$ vanishes at least at one point $\eta\in\Sigma(P_0)$.

Let $\mathcal{P}(\xi) := P_0(\xi) + P_l(\xi)$, $\mathcal{P}_1(\xi) := P_1(\xi) + ... + P_{l-1}(\xi)$, $p(\xi) := P_{l+1}(\xi) + ... + P_M(\xi)$, $\Re^* := \{\beta \in \Re, (\mu, \beta) \leq d_l\}$ *and suppose, that* $\Re(\mathcal{P}) = \Re(P)$.

Then

a) *if* $\nu \notin \Re^*$, *inequality* (3.9) *cannot hold,*

b) *inequality* (3.9) *holds for any multi-index* $\nu \in \Re^*$ *if each of polynomials* $P_j \in \mathfrak{M}$ *satisfies one of the following conditions*

b.1) *for the pair of* ($\mu-$*homogeneous) polynomials* (P_j, P_0) $(1 \leq j \leq l-1)$ *the assumptions of Theorem* 2.1 *are satisfied,*

b.2) *for the pair of polynomials* $(P_j, \mathcal{P})$ $(1 \leq j \leq l-1)$ *the assumptions* $I) - II)$ *of Theorem* 2.2 *are satisfied,*

b.3) *for the pair of polynomials* $(P_j, \mathcal{P})$ $(1 \leq j \leq l-1)$ *the assumptions of Theorem* 3.1 *are satisfied.*

Remark 9 Before proceeding to the proof of the theorem, we note that

1) conditions b.2) and b.3) should be set not for a pair of polynomials $(P_j, \mathcal{P})$ but for a pair $(P_j, \mathcal{P} + p)$. On one hand, the notation $(P_j, \mathcal{P})$ simplifies writing and reasoning, on the other hand, it is legitimate, since by Remark 8, the polynomial $p(\xi)$ does not affect the behaviour of polynomials P and $\mathcal{P}$ at infinity,

2) from condition II.2) of Theorem 2.1, condition I.3) of Theorem 2.2 and (3.2) of Theorem 3.1 it follows that in all b.1) - b.3) cases of this theorem the polynomials $P_j \in \mathfrak{M}$ $(j = 1, ..., l-1)$ must vanish at all points $\eta \in \Sigma(P_0)$.

Proof of Theorem 3.2. Bearing in mind that for the polynomial $\mathcal{P}$ estimate (3.9) is valid for all $\nu \in \Re^*$, it is sufficient for us to prove that $\mathcal{P} < P = \mathcal{P} + \mathcal{P}_1$.

Firstly, let us add to the polynomial $\mathcal{P}$ those polynomials from $\mathfrak{M}$ that (together with the polynomial P_0) satisfy condition b.1) of Theorem (i.e.conditions of Theorem 2.1). Let these be polynomials $\mathfrak{M}_1 = \{P_{i_1}, P_{i_2}, ..., P_{i_{k_1}}\} \subset \mathfrak{M}$ $(1 \leq i_j \leq l-1,\ j = 1, ..., k_1)$, $k_1 \leq l-1$ i.e $P_{i_j} < P_0$ $j = 1, ..., k_1)$.

Since $d_{i_j} < d_0$ $(j = 1, ..., k_1)$, by Lemma 2.1 $P_{i_j}(\xi) = o(|P_0(\xi)|)$ for $|P_0(\xi)| \to \infty$ i.e. $|P_{i_1}(\xi)| + |P_{i_2}(\xi)| + ..., |P_{i_k}(\xi)| = o(|P_0(\xi)|)$ for $|P_0(\xi)| \to \infty$.

Remark 8. implies that $P_0 < \mathcal{P} = P_0 + P_l$, hence $|P_{i_1}(\xi)| + |P_{i_2}(\xi)| + ..., |P_{i_{k_1}}(\xi)| = o(|\mathcal{P}(\xi)|)$ for $|P_0(\xi)| \to \infty$. Thus, there exists a constant $c > 0$ such that, for sufficiently large $|P_0(\xi)|$, the inequality

$$|\mathcal{P}(\xi)| \leq c\,[1 + |\mathcal{P}(\xi) + P_{i_1}(\xi) + P_{i_2}(\xi) + ..., P_{i_{k_1}}(\xi)|] \tag{3.10}$$

holds. If $|P_0(\xi)|$ is bounded for $|\xi| \to \infty$, then the polynomials $\{P_{i_j}\}$ are also bounded on this sequence (recall that $P_{i_j} < P_0$ $(j = 1, ..., k_1)$). On the other hand, since $\mathcal{P} \in \mathbb{I}_n$ hence $\mathcal{P}(\xi) \to \infty$, and inequality (3.10) (perhaps with a different constant) is obvious. As a result, we get that $\mathcal{P} < \mathcal{P} + P_{i_1} + P_{i_2} + ... + P_{i_{k_1}} < \mathcal{P}$. It means, that further, when comparing the polynomials $\mathcal{P}$ and P, it suffices to compare the polynomials $\mathcal{P}^1 := \mathcal{P} + P_{i_1} + P_{i_2} + ... + P_{i_{k_1}}$ and P.

If $k_1 = l-1$, i.e $\mathcal{P}^1(\xi) := \mathcal{P}(\xi) + \mathcal{P}_1(\xi) = P(\xi)$ $\forall \xi \in \mathbb{R}^n$, then this proves Theorem.

Consider the case when $k_1 < l-1$, i.e. $\mathfrak{M}_1 \neq \mathfrak{M}$.

Let us first consider those polynomials $P_j \in \mathfrak{M} \setminus \mathfrak{M}_1$ that satisfy condition b.3). Let these be polynomials $\mathfrak{M}_3 := \{P_{k_1+i_1}, P_{k_1+i_2}, ..., P_{k_1+k_2}$ $(1 \leq i_j \leq l-1,\ j = 1, ..., k_2), k_1 + k_2 \leq l-1\}$ i.e $|P_{i_j}(\xi)|/[\,P(\xi)| + 1] \to 0$ as $|\xi| \to \infty$ and $P < P + P_{i_j} < P$ for all $j = k_1 + 1, ..., k_1 + k_2$.

Arguing as in the previous case, we find that $\mathcal{P} < \mathcal{P}^2 := \mathcal{P}^1 + P_{k_1+i_1} + P_{k_1+i_2} + ... + P_{i_{k_1}+k_2} < \mathcal{P}$, i.e. further, when comparing the polynomials $\mathcal{P}$ and P, it suffices to compare the polynomials $\mathcal{P}^2$ and P.

Finally, to the polynomial $\mathcal{P}^2$ we add the remaining polynomials from $\mathfrak{M}$ that satisfy condition b.2) of Theorem (i.e. conditions of Theorem 2.2). Let these be polynomials $\mathfrak{M}_2 :=$

$\{P_{k_1+k_2+i_1}, P_{k_1+k_2+i_2}, ..., P_{k_1+k_2+k_3}\}$, $k_1+k_2+k_3 = l-1$. Then $\mathcal{P}^2(\xi)+ P_{k_1+k_2+i_1}(\xi)+P_{k_1+k_2+i_2}(\xi)+ ... + P_{k_1+k_2+k_3}(\xi) = P(\xi)$ for all $\xi \in \mathcal{R}^n$.

As a result of the previous two cases we have that $\mathcal{P} < \mathcal{P}^2 < \mathcal{P}$. From Theorem 2.2 it follows that $\mathcal{P} < \mathcal{P}+ P_{k_1+k_2+i_1}+P_{k_1+k_2+i_2}+...+P_{k_1+k_2+k_3}$. Hence $\mathcal{P}^2 < \mathcal{P}^2+ P_{k_1+k_2+i_1}+P_{k_1+k_2+i_2}+...+P_{k_1+k_2+k_3} = P$. So we obtain that $\mathcal{P} < P < \mathcal{P}$. $\square$

References

[1] A.D. Aleksandrov, *Internal geometry of convex surfaces.* (1948), OGIZ, Moscow; German transl. (1955), Akademic - Verlag, Berlin.

[2] G.C. Barozzi, *Sul prodotto di polinomi quasi-ellittici.* Bolletino dell'Unione Matematica Italiana. 20 (1965), no. 3, 169 - 176.

[3] D. Calvo, *Multianizotropic Gevrey classes and Caushy problem.* Ph. D. Thesis in Mathematics, Universita degli Studi di Pisa (2000).

[4] L. Cattabriga, *Su una classi di polinomi ipoellitici.* Rendiconti del Seminario Matematico della Universita Padova, 36 (1966), 285 - 309.

[5] L. Cattabriga, *Uno spazio funzionale connesso ad una classe di operatori ipoellittici.* Symposia Mat, Inst.Naz. Alta Mat., Bologna, 7 (1971), 547 - 558.

[6] A. Corli, *Un teorema di rappresentazione per certe classi generelizzate di Gevrey.* Boll.Un. Mat. It., Serie, 6, 4 -C (1985), no. 1, 245 - 257.

[7] J. Friberg, *Multi quasielliptic polynomials. Annali della Scuola Normale Superiore di Pisa,* 21 (1967), no. 3, 239 - 260.

[8] H.G. Ghazaryan, *On almost hypoelliptic polynomials increasing at infinity.* Journal of Contemporary Math. Analysis, 46 (2011), no. 6, 13 - 30.

[9] M. Gevre, *Sur la nature analitique des solutions des equations aux derivatives partielles.* Ann. Ec. Norm. Sup., Paris, 35 (1918), 129 - 190.

[10] S.G. Gindikin, *Energy estimates connected with the Newton polyhedron.* Trudy Moskov. Mat. Obshch., 31 (1974), 189 - 236; English trans. in Trans. Moscow Math. Soc. 31 (1974).

[11] L. Gȧrding, *Linear hyperbolic partial differential equations with constant coefficients.* Acta Math. 85 (1951), 1 - 62.

[12] L. Hörmander, *The analysis of linear partial differential operators. I, II* (1983), Springer - Verlag.

[13] V.Ya. Ivri, *Well posedness in Gevrey class of the Cauchy problem for non - strictly hyperbolic equation.* Math. Sb. 96 (1975), no. 138, 390 - 413 (in Russian).

[14] V.Ya. Ivri, *Linear hyperbolic equations.* Partial Differential Equations, Springer Link, 33 (2001).

[15] K. Kajitani, *Cauchy problem for non-strictly hyperbolic systems.* Pull. Res. Inst. Math. Sci. 15 (1979), no. 2, 519-550.

[16] G.G. Kazaryan, *Comparison of powers of polynomials and their hypoellipticity.* Trudy Steklov Inst. Math., 150 (1979), 151 - 167. English transl. in Proc. Steklov Inst. Math., 150 (1981), no. 4, 151 - 167.

[17] G.G. Kazaryan, *Estimates of differential operators and hypoelliptic operators.* Proc. Steklov Inst. Math. 140 (1976), 130 - 161.

[18] G.G. Kazaryan, *On adding low вЂ“ order terms to differential polynomials.* Izv. Akad. Nauk Armenian SSR, Math. 9 (1974), no. 6, 473 вЂ“ 485 (in Russian).

[19] G.G. Kazaryan, *L_p-estimates for mixed derivatives in terms of differential polynomials.* Proc. Steklov Inst. Math., 105 (1969), 66 - 76.

[20] G.G. Kazaryan, V.N. Margaryan, *Criteria for hypoellipticity in terms of power and strength of operators.* Proc. Steklov Inst. Math. 150 (1981), 135 - 150.

[21] G.G. Kazaryan, *On the comparison of differential operators and on differential operators of constant strength.* Proc. Steklov Inst. Math. 131 (1974), 94 - 118.

[22] G.G. Kazaryan, *On a family of hypoelliptic polynomials.* Izv. Akad. Nauk Armenian SSR, Math. 9 (1974), no. 3, 189 - 211 (in Russian).

[23] A.G. Khovanskii, *Newton polyhedra (algebra and geometry).* Amer. Math. Soc. Transl. 153 (1992), no. 2, 1 - 15.

[24] A.N. Kolmogorov, S.V. Fomin, *Elements of the theory of functions and functional analysis.* Dover Publ., Inc., Mineola (1999), New York.

[25] P. Kythe, *Fundamental solutions for differential operators and applications.* Birkhäuser, Boston, Basel, Berlin, 2012.

[26] E. Larsson, *Generalized hyperbolicity.* Ark. Mat. 7 (1967), 11 - 32.

[27] V.N. Margaryan, *On the presentation of $\lambda-$homogeneous polynomials of two variables.* Collection of works of the 10th annual scientific conference of the Russian - Armenian University, (2015), 19 - 23.

[28] V.N. Margaryan, *Adding lower order terms that preserve the hypoellipticity of the operator.* Izv. NAS Armenii, Math., 15 (1980), no. 6, 443 - 460 (in Russian).

[29] V.N. Margaryan, H.G. Ghazaryan, *On a class of weakly hyperbolic operators.* Journal of Contemporary Math. Analysis, 53 (2018), no. 6, 53 - 69 (in Russian).

[30] V.N. Margaryan, H.G. Ghazaryan, *On the Cauchy problem in the multianisotropic Gevrey classes for hyperbolic equations.* Journal of Contemporary Math. Analysis. 50 (2015), no. 3, 36 - 46.

[31] V.N. Margaryan, H.G. Ghazaryan, *On fundamental solutions of a class of weak hyperbolic operators.* Eurasian Math. J. 9 (2018), no. 2, 54 - 67.

[32] V.N. Margaryan, G. H. Hakobyan, *On Gevrey type solutions of hypoelliptic equations.* Journal of Contemporary Math. Analysis, 31 (1996), no. 2, 33 - 47.

[33] V. P. Mikchailov, *The behaviour of a class of polynomials at infinity.* Proc. Steklov Inst. Math.m 91 (1967), 59 - 81 (in Russian).

[34] S. Mizohata, *On the Cachy problem.* Notes and Reports on Mathematics in Science and Enginerings. Acad press Inc. Orlando, FL Science Press, Beijing (1985), 3.

[35] I.G. Petrovsky, *Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen.* Mat. Sb., 2 (1937), 815 - 870.

[36] B. Pini, *Osservazioni sulla ipoellittisità.* Boll. Unione Mat. Ital., ser. III, 18 (1963), no. 4, 420 - 433.

[37] B. Pini, *Sulla classe di Gevrey della soluzone di certe equazioni ipoellittiche.* Boll.Unione Mat. Ital., ser. III, 18 (1963), no. 3, 260 - 269.

[38] L. Rodino, *Linear partial differential operators in Gevrey spaces.* Word Scientific(1993), Singapore.

Haik Gegamovich Ghazaryan
Department of mathematics and mathematical modelling
Russian - Armenian University
123 Ovsep Emin St.
0051 Yerevan, Armenia.
and
Institute of Mathematics the National Academy of Sciences of Armenia
24/5 Marshal Baghramyan ave
0019 Yerevan, Armenia.
E- mail: haikghazaryan@mail.ru

Received: 27.08.2022