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KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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EQUIVALENT SEMI-NORMS FOR NIKOL'SKII-BESOV SPACES

V.I. Burenkov, A. Senouci

Communicated by M.L. Goldman

Key words: equivalent semi-norms, Nikol'skii-Besov spaces.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. The aim of this paper is to establish the equivalence of various semi-norms involving differences for Nikol'skii-Besov spaces on an interval.

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1 Introduction

We start with recalling the definitions of Niko'skii-Besov spaces $B^l_{p,\theta}(a,b)$ and semi-normed Nikol'skii-Besov spaces $b^l_{p,\theta}(a,b)$.

Definition 1. Let $l > 0$, $k \in \mathbb{N}$, $k > l$, $1 \le p$, $\theta \le \infty$, $\alpha_1 \ge 0$, $\alpha_2 \ge k$, and $-\infty \le a < b \le +\infty$. Then $f \in b^l_{p,\theta}(a,b)$ if f is measurable on (a,b) and the following semi-norm is finite:

$$\|f\|_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{k}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.1}$$

if $1 \le \theta < \infty$ and

$$\|f\|_{b^l_{p,\theta}(a,b)} = \sup_{h\in(0,\frac{b-a}{k})} \frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l}, \tag{1.2}$$

if $\theta = \infty$.

Moreover, $B^l_{p,\theta}(a,b) = b^l_{p,\theta}(a,b) \cap L_p(a,b)$ with the norm

$$\|f\|_{B^l_{p,\theta}(a,b)} = \|f\|_{L_p(a,b)} + \|f\|_{b^l_{p,\theta}(a,b)}.$$

Here $\Delta_h^k f$ is the difference of order k of f with step h:

$$(\Delta_h^k f)(x) = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x+mh).$$

Let for $\alpha_1 \ge 0$, $\alpha_2 \ge k$,

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h,b-\alpha_2 h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.3}$$

if $1 \le \theta < \infty$ and

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \sup_{h\in(0,\frac{b-a}{\alpha_1+\alpha_2})} \|\Delta^k_h f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)} \tag{1.4}$$

if $\theta = \infty$.

Respectively,

$$\|f\|^{(1)}_{B^l_{p,\theta}(a,b)} = \|f\|_{L_p(a,b)} + \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}.$$

We shall prove the equivalence of (1.1) and (1.3), (1.2) and (1.4) respectively, for an arbitrary interval (a,b). We note the following results related to this statement.

The following theorem was proved in [5].

Theorem 1.1. *Let $l > 0$, $k \in \mathbb{N}$, $k > l$, $1 \le p$, $\theta \le \infty$, $0 < \delta \le \infty$, $s \ge 2$, $a \in \mathbb{R}$, $0 < \alpha < \infty$.*

Then there exists $c_1 > 0$, depending only on α, s, k and l, such that

$$\left(\int_0^\delta \left(\frac{\|\Delta^k_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_1 \sup_{\substack{m\in\mathbb{N}_0,\\ m\le k(s-1)-1}} \left(\int_0^{\frac{\delta}{s}} \left(\frac{\|\Delta^k_h f\|_{L_p(a+(\alpha+m)h, a+(s\alpha+m+1)h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.5}$$

for all read-valued functions If f measurable on $(a, a+(k+\alpha)\delta)$ for which the left-hand side of this inequality is finite, in partcular, for all $f \in C^\infty([a, a+(k+\alpha)\delta])$.

Corollary 1.1. *If $s = 2$ inequality (1.5) takes the form*

$$\left(\int_0^\delta \left(\frac{\|\Delta^k_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_2 \left(\int_0^{\frac{\delta}{2}} \left(\frac{\|\Delta^k_h f\|_{L_p(a+\alpha h, a+(2\alpha+k)h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}, \tag{1.6}$$

where $c_2 > 0$ depends only on α, k and l.

For differences of order one the following statement was proved in [10].

Theorem 1.2. *Let $1 \le p, \theta \le \infty$, $l > 0$, $a \in \mathbb{R}$, $0 < \delta \le \infty$, $t > 0$, $0 \le b < c$, $T > 0$, and $0 \le B < C$.*

Then there exists $c_3 > 0$, depending only on t, b, c, T, B, C, and l, such that

$$\left(\int_0^\delta \left(\frac{\|\Delta_{th} f\|_{L_p(a+bh,a+ch)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_3 \left(\int_0^{\frac{c+t}{B+T}\delta} \left(\frac{\|\Delta_{Th} f\|_{L_p(a+Bh,a+Ch)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \tag{1.7}$$

for all measurable functions $f : [a,\infty) \to \mathbb{R}$.

Corollary 1.2. *If $b=0, c=\alpha, B=\alpha, C=\beta, 0\le\alpha<\beta, t=T=1$, inequality (1.7) takes the form*

$$\left(\int_0^\delta \left(\frac{\|\Delta_h f\|_{L_p(a,a+\alpha h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}$$

$$\le c_4 \left(\int_0^\delta \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha h,a+\beta h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}}. \tag{1.8}$$

where $c_4>0$ depends only on α,β and l.

The proof of the equivalences of (1.1) and (1.3), (1.2) and (1.4) will be based on Corollary 1.1, a general statement (Lemma 2.1) for semi-normed space, connected with application of the Banach theorem on the boundedness of an inverse operator, and the inclusion $b^l_{p,\theta}(a,b)\subset L_p(a,b)$, proved in [6].

2 Equivalent semi-norms

Theorem 2.1. *Let $l>0$, $k\in\mathbb{N}$, $k>l$, $1\le p$, $\theta\le\infty$, $\alpha_1\ge 0$, $\alpha_2\ge k$.*

Then for an arbitrary interval (a,b) the semi-norms $\|f\|_{b^l_{p,\theta}(a,b)}$ and $\|f\|^{(1)}_{b^l_{p,\theta}(a,b)}$ are equivalent. Moreover, there exists $c_5>0$ is depending only on l, k, p, θ, α_1 and α_2 such that

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \le \|f\|_{b^l_{p,\theta}(a,b)} \le c_5\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \tag{2.1}$$

for all $f\in b^l_{p,\theta}(a,b)$.

Lemma 2.1. *Let E_1, E_2 be semi-normed spaces with the corresponding semi-norms $\|.\|_{E_1}$, $\|.\|_{E_2}$, $E_1\subset E_2$ and*

$$\theta_1=\{g\in E_1 : \|g\|_{E_1}=0\}=\theta_2=\{g\in E_2 : \|g\|_{E_2}=0\}.$$

Furthermore, let the space E_1 be complete with respect to the semi-norms $\|.\|_{E_1}$ and $\|.\|_{E_1}+\|.\|_{E_2}$.

Then there exists $c_6>0$ such that

$$\|f\|_{E_2}\le c_6\|f\|_{E_1} \tag{2.2}$$

for all $f\in E_1$.

Proof. We consider the factor spaces

$$\tilde{E}_1=E_1/\theta_1, \tilde{E}_2=E_2/\theta_1 \text{ and } \tilde{E}_{12}=E_{12}/\tilde{\theta}_1,$$

where E_{12} is the space $E_1\cap E_2=E_1$, equipped with the semi-norms $\|.\|_{E_{12}}=\|.\|_{E_1}+\|.\|_{E_2}$.

By the definitions of a factors-space and corresponding semi-norm, $\tilde{E}_1$ is the set of all non-intersecting classes $\tilde{f}$ generated by elements $f\in E_1$: $\tilde{f}=\{f+g : g\in\theta_1\}$, and $\|\tilde{f}\|_{\tilde{E}_1}=\inf\limits_{h\in\tilde{f}}\|h\|_{E_1}$.

Since θ_1 is the null-set of E_1, it follows that $\|\tilde{f}\|_{\tilde{E}_1}=\|f\|_{E_1}\ \forall\ f\in\tilde{f}$.

Since $\theta_2=\theta_1$, $\tilde{E}_2=E_2/\theta_2$ and similarly $\|\tilde{f}\|_{\tilde{E}_2}=\|f\|_{E_2}\ \forall\ f\in\tilde{f}$. Finally, for each $\tilde{f}\in\tilde{E}_{12}$

$$\|\tilde{f}\|_{\tilde{E}_{12}}=\|\tilde{f}\|_{\tilde{E}_1}+\|\tilde{f}\|_{\tilde{E}_2}. \tag{2.3}$$

We note that $\tilde{E}_1$ and $\tilde{E}_{12}$ are Banach spaces. Next we consider the identity operator

$$I:\tilde{E}_{12}\to\tilde{E}_1.$$

This operator is linear, continuous, and such that $\|\tilde{f}\|_{\tilde{E}_1} \le \|\tilde{f}\|_{\tilde{E}_{12}}$. Moreover it bijectively maps $\tilde{E}_1$ onto $\tilde{E}_{12}$. By the theorem on boundedness of an inverse operator (corollary of the Banach theorem on an open map), the operator $I^{-1} : \tilde{E}_1 \to \tilde{E}_{12}$ is also continuous. Hence it is bounded, therefore there exists $M > 0$, such that

$$\|\tilde{f}\|_{\tilde{E}_{12}} \le M\|\tilde{f}\|_{\tilde{E}_1}$$

which implies inequality (2.2). □

Remark 1. For the case of normed spaces E_1 and E_2 this statement is proved in book [7] (Theorem 2, pp. 268-269).

Proof of Theorem 2.1. Step 1. The inequality

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} \le \|f\|_{b^l_{p,\theta}(a,b)}$$

being trivial, it is required to prove the right-hand side inequality of (2.1).

Assume that $-\infty < a < b < \infty$.

By applying Minkowski's inequality, we obtain for any $1 \le p,\ \theta \le \infty$(if $\theta = \infty$,then integrals should be replaced by appropriate supremums)

$$\begin{aligned}
\|f\|_{b^l_{p,\theta}(a,b)} \le & \left(\int_{\frac{b-a}{\alpha_1+\alpha_2}}^{\frac{b-a}{k}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,b-kh))}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& \le \left(\int_{\frac{b-a}{\alpha_1+\alpha_2}}^{\frac{b-a}{k}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a,a+\alpha_1 h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h,b-\alpha_2 h)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(b-\alpha_2 h,b-kh)}}{h^l}\right)^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} \\
& \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since

$$\|\Delta_h^k f\|_{L_p(a,b-kh)} \le 2^k\|f\|_{L_p(a,b)}$$

we have that for some $c_7 > 0$ independent of f

$$I_1 \le c_7\|f\|_{L_p(a,b)}.$$

Let in Corollary 1.1, $\delta = \frac{b-a}{(\alpha_1+\alpha_2)}$. Then, since

$$(a + \alpha_1 h, a + (2\alpha_1 + k)h) \subset (a + \alpha_1 h, b - \alpha_2 h)$$

for $0 \le h \le \frac{b-a}{2(\alpha_1+\alpha_2)}$, we obtain

$$\begin{aligned} I_2 &\le c_1 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} \\ &\le c_8 \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \end{aligned}$$

where $c_8 > 0$ depends only on α_1, α_2, k, l..

Next, $I_3 = \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}$.

To estimate I_4 we first note that

$$\|\Delta_h^k f\|_{L_p(b-\alpha_2 h, b-kh)} = \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x+mh) \right\|_{L_p(b-\alpha_2 h, b-kh)}$$

$$= \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(b-kh+a-y+mh) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(a+b-(y+(k-m)h)) \right\|_{L_{p(a,a+(\alpha_2-k)h)}}$$

$$= \left\| \sum_{s=0}^{k} (-1)^{s} \binom{k}{k-s} f(a+b-(y+sh)) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(a+b-(y+(k-s)h)) \right\|_{L_p(a, a+(\alpha_2-k)h)}$$

$$= \left\| \Delta_h^k g \right\|_{L_p(a, a+(\alpha_2-k)h)},$$

where $g(x) = f(a+b-x)$. (We changed the variable $x = b-kh+a-y$ and the summation index $m = k-s$). Consequently

$$I_4 = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k g\|_{L_p(a, a+(\alpha_2-k)h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}}.$$

By Corollary 1.1, with $\delta = \frac{b-a}{\alpha_1+\alpha_2}$, we obtain

$$I_4 \le c_9 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k g\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}},$$

where $c_9 > 0$ depends only on α_1, α_2, k, l..

By changing the variable $y = b-kh+a-x$ similarly to the above we get

$$\|\Delta_h^k g\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}$$

$$= \|\Delta_h^k f(a+b-y)\|_{L_p(a+(\alpha_2-k)h, a+(2\alpha_2-k)h)}$$

$$= \|\Delta_h^k f\|_{L_p(b-2\alpha_2 h, b-\alpha_2 h)}.$$

Therefore,

$$I_4 \;\le\; c_9 \left(\int_0^{\frac{b-a}{2(\alpha_1+\alpha_2)}} \left(\frac{\|\Delta_h^k f\|_{L_p(b-2\alpha_2 h, b-\alpha_2 h)}}{h^l} \right)^{\theta} \frac{dh}{h} \right)^{\frac{1}{\theta}}.$$

Since

$$(b-2\alpha_2 h, b-\alpha_2 h) \subset (a+\alpha_1 h, b-\alpha_2 h)$$

for $0 \le h \le \frac{b-a}{2(\alpha_1+\alpha_2)}$, we have

$$\begin{aligned} I_4 \;&\le\; c_9 \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)}}{h^l} \right)^{\theta} \frac{dh}{h} \right)^{\frac{1}{\theta}} \\ &=\; c_7 \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}. \end{aligned}$$

Finally, we get

$$\|f\|_{b^l_{p,\theta}(a,b)} \le c_{10}(\|f\|_{L_p(a,b)} + \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}) \tag{2.4}$$

where $c_{10} = \max(c_7, 1, c_8, c_9)$.

Inequality (2.4) immediately implies that $\|f\|_{B^l_{p,\theta}(a,b)}$ is equivalent to $\|f\|^{(1)}_{B^l_{p,\theta}(a,b)}$.

Step 2. Let $E_2 = b^l_{p,\theta}(a,b)$ and E_1 be the set of all function f measurable on (a,b) for which

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} < \infty.$$

If $f \in E_1$ then by the result in [3] it follows that $f \in L_p(a,b)$. Hence, by inequality (2.4) $f \in E_2$, so $E_1 \subset E_2$.

By Lemma 2.1 it follows that

$$\|f\|_{b^l_{p,\theta}(a,b)} \le c_{11} \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \tag{2.5}$$

where $c_{11} = c_{11}(a,b,p,\theta,l,k,\alpha_1,\alpha_2) > 0$ is independent of the function f.

Step 3. In fact, it follows that one can assume that in the inequality c_9 is also independent of a and b. Namely,

$$c_{11}(a,b,p,\theta,l,k,\alpha_1,\alpha_2) = c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2). \tag{2.6}$$

To prove this we note that, if $-\infty < a < b < +\infty$, then

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = (b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)} \tag{2.7}$$

where $g(y) = f(a + y(b-a))$, $y \in (0,1)$. In particular, if $\alpha_1 = 0$ and $\alpha_2 = k$,

$$\|f\|_{b^l_{p,\theta}(a,b)} = (b-a)^{\frac{1}{p}-l} \|g\|_{b^l_{p,\theta}(0,1)}. \tag{2.8}$$

Indeed by substituting $y = \frac{x-a}{b-a}$ we get

$$\|\Delta_h^k f\|_{L_p(a+\alpha_1 h, b-\alpha_2 h)} = \left(\int\limits_{a+\alpha_1 h}^{b-\alpha_2 h} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} f(x+mh) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\frac{\alpha_1 h}{b-a}}^{1-\frac{\alpha_2 h}{b-a}} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} f(a + y(b-a) + mh) \right|^p (b-a) dy \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\frac{\alpha_1 h}{b-a}}^{1-\frac{\alpha_2 h}{b-a}} \left| \sum_{k=0}^{m} (-1)^{k-m} \binom{k}{m} g\left(y + \frac{mh}{b-a} \right) \right|^p dy \right)^{\frac{1}{p}} (b-a)^{\frac{1}{p}}$$

$$\left\| \Delta^k_{\frac{h}{b-a}} g \right\|_{L_p(\frac{\alpha_1 h}{b-a}, 1-\frac{\alpha_2 h}{b-a})} (b-a)^{\frac{1}{p}}.$$

Hence, by substituting $t = \frac{h}{b-a}$, we get

$$\|f\|^{(1)}_{b^l_{p,\theta}(a,b)} = \left(\int_0^{\frac{b-a}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_{\frac{h}{b-a}} g\|_{L_p(\frac{\alpha_1 h}{b-a}, 1-\frac{\alpha_2 h}{b-a})}}{h^l} \right)^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} (b-a)^{\frac{1}{p}}$$

$$\left(\int_0^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\|\Delta^k_t g\|_{L_p(\alpha_1 t, 1-\alpha_2 t)}}{(t(b-a))^l} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} (b-a)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)}.$$

By (2.8), (2.5) with $a = 0, b = 1$ and (2.7), we obtain

$$\begin{aligned} \|f\|_{b^l_{p,\theta}(a,b)} &= (b-a)^{\frac{1}{p}-l} \|g\|_{b^l_{p,\theta}(0,1)} \\ &\le c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2)(b-a)^{\frac{1}{p}-l} \|g\|^{(1)}_{b^l_{p,\theta}(0,1)} \\ &= c_{11}(0,1,p,\theta,l,k,\alpha_1,\alpha_2) \|f\|^{(1)}_{b^l_{p,\theta}(a,b)}, \end{aligned}$$

which is inequality (2.5) with c_9 defined by (2.6) for the case of a finite interval.

Inequality (2.5) also holds with the same c_{11} for infinite interval which follows by passing in (2.5) to the limit.

□

Remark 2. It may be of interest to consider a similar problem for the Nikol'skii-Besov-Morrey spaces in the definition of which L_p-norms are replaced by the norms in Morrey spaces M_p^λ. The required information on Morrey spaces and Nikol'skii-Besov-Morrey spaces can be found in [2], [3], [4], [8] and [9].

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