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## KHARIN STANISLAV NIKOLAYEVICH

(to the 85th birthday)

On December 4, 2023 Doctor of Physical and Mathematical Sciences, Academician of the National Academy of Sciences of the Republic of Kazakhstan, member of the editorial board of the Eurasian Mathematical Journal Stanislav Nikolaevich Kharin turned 85 years old.

Stanislav Nikolayevich Kharin was born in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. For these outstanding achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research as evidenced by his scientific publications in high-ranking journals with his students in recent years.

The Editorial Board of the Eurasian Mathematical Journal, his friends and colleagues cordially congratulate Stanislav Nikolayevich on the occasion of his 85th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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# CORRECT AND COERCIVE SOLVABILITY CONDITIONS FOR A DEGENERATE HIGH ORDER DIFFERENTIAL EQUATION

**R.D. Akhmetkaliyeva, T.D. Mukasheva, K.N. Ospanov**

Communicated by Ya.T. Sultanaev

**Key words:** degenerate fifth-order differential equation, unbounded coefficient, generalized solution, correct solvability, coercive estimate.

**AMS Mathematics Subject Classification:** 35J70.

**Abstract.** In the work, we consider a fifth-order singular differential equation with variable coefficients. The singularity means, firstly, that the equation is given on the real axis $\mathbb{R} = (-\infty, \infty)$, and secondly, its coefficients are unbounded functions. We study a new degenerate case, when the intermediate coefficients of the equation grow faster than the lowest coefficient (potential), and also the potential is not sign-definite. We obtain sufficient conditions for the existence and uniqueness of the generalized solution of the equation. We also prove a coercive estimate for the solution. The coefficients of the equation are assumed to be smooth, but we do not impose any restrictions on their derivatives to prove the results. Note that the well-known stationary Kawahara equation can be reduced to the considered equation after linearization.

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## 1 Introduction

The Kawahara equation or the generalized Korteweg-de Vries equation describes the propagation of one-dimensional nonlinear waves in a dispersive medium. It was presented for the first time in the paper [7] and is written as follows

$$-\beta y_x^{(5)}(x, t) + \alpha y_x^{(3)}(x, t) + \frac{3}{2} y(x, t) y_x^{'}(x, t) + y_t^{'}(x, t) = 0, \tag{1.1}$$

where $\alpha$, $\beta$ are real numbers. Boundary value problems for equation (1.1) were studied in many works (see [4,5,16]).

In [11], a Kawahara-type equation with variable coefficients defined in a non-compact domain was studied. Conditions for the solvability of the Cauchy problem were obtained. The authors of [11] assumed that the coefficients of the equation are bounded. Related references can also be found in [11]. A natural continuation of these studies is the study of Kawahara-type equations with unbounded coefficients. The present work is devoted to this case.

Let us consider the following stationary Kawahara-type equation

$$-y^{(5)} + r_0(x) y^{(3)} + q_0(x) y y^{'} = f_0(x), \tag{1.2}$$

where $x \in \mathbb{R}$. As a result of the linearization of (1.2), we will get one of the following two differential equations

$$-y^{(5)} + r_1(x) y^{(3)} + q_1(x) y = f_1(x) \tag{1.3}$$

and

$$-y^{(5)} + r_2(x)\, y^{(3)} + q_2(x)\, y' = f_2(x)\,. \tag{1.4}$$

Equation (1.3) with a positive potential $q_1(x)$ was considered in [9] (see also references therein). It should be noted that the problem of the correctness of (1.3) with sign-variable $q_1(x)$, as well as of equation (1.4), remained open.

In the present work, we will study the following linear equation

$$L_0 y \equiv -y^{(5)} + r(x)\, y^{(3)} + q(x)\, y' + p(x)\, y = f(x)\,, \tag{1.5}$$

where $x \in \mathbb{R}$, $f(x) \in L_2(\mathbb{R})$, $r$ is a three times continuously differentiable function, $q$ is a continuously differentiable function, and $p(x)$ is a continuous but not a sign-constant function. Equation (1.5) generalizes both (1.3) and (1.4). Using some modifications of methods of [9, 10, 13], we prove sufficient conditions for the correct solvability of equation (1.5) and the fulfillment of the following the so-called maximal regularity estimate

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 + \left\|qy'\right\|_2 + \|py\|_2 \le c\|f\|_2 \tag{1.6}$$

for the solution $y$, where $c > 0$ depends only on $r$, $q$, $p$. Here $\|\cdot\|_2$ is the norm in $L_2(\mathbb{R})$.

Note that in [1, 8-15, 17], the stationary singular differential equations of the second and high orders with intermediate coefficients were studied.

Let

$$L_0 y = -y^{(5)} + r(x)\, y^{(3)} + q(x)\, y' + p(x)\, y$$

be a differential operator defined on the set $C_0^{(5)}(\mathbb{R})$ of all five times continuously differentiable functions with compact support. Due to the conditions imposed on the functions $r(x)$, $q(x)$ and $p(x)$, the operator $L_0$ can be closed by the norm of $L_2(\mathbb{R})$. We denote its closure by $L$.

**Definition 1.** A function $y \in D(L)$ satisfying the equality $Ly = f$ is called a solution to the differential equation (1.5).

Let $g$ and $h \neq 0$ be real-valued continuous functions. We introduce the following notations:

$$\alpha_{g,h,j}(x) = \left(\int_0^x g^2(t)\, dt\right)^{\frac{1}{2}} \left(\int_x^{+\infty} t^{2j} h^{-2}(t)\, dt\right)^{\frac{1}{2}} \quad (x > 0),$$

$$\beta_{g,h,j}(\tau) = \left(\int_\tau^0 g^2(t)\, dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\tau} t^{2j} h^{-2}(t)\, dt\right)^{\frac{1}{2}} \quad (\tau < 0),$$

$$\gamma_{g,h,j} = \max\left(\sup_{x>0} \alpha_{g,h,j}(x)\,,\ \sup_{\tau<0} \beta_{g,h,j}(\tau)\right) \quad (j = 1,\, 2)\,.$$

**Lemma 1.1.** [3]. *If the functions $g$, $h$ satisfy the condition $\gamma_{g,h,j} < \infty$ $(j = 1, 2)$, then for each $y \in C_0^{(j+1)}(\mathbb{R})$ the following inequality*

$$\int_{-\infty}^{+\infty} |g(x)\, y(x)|^2\, dx \le \frac{2}{j}\gamma_{g,h,j} \int_{-\infty}^{+\infty} \left|h(x)\, y^{(j+1)}(x)\right|^2 dx \tag{1.7}$$

*holds.*

## 2 Auxiliary statements

We consider the differential operator $l_0 y = -y^{(5)} + r(x)\, y^{(3)}$ defined on the set $C_0^{(5)}(\mathbb{R})$. By $l$ we denote the closure of $l_0$ in $L_2(\mathbb{R})$. Consider the following equation

$$ly = -y^{(5)} + r(x)\, y^{(3)} = h(x), \tag{2.1}$$

where $h \in L_2(\mathbb{R})$.

**Definition 2.** A function $y \in D(l)$ satisfying the equality $ly = f$ is called a solution to differential equation (2.1).

**Lemma 2.1.** *If the function $r(x)$ is three times continuously differentiable and*

$$r \geq 1, \gamma_{1,\sqrt{r},2} < \infty,$$

*then for each $f \in L_2(\mathbb{R})$, there exists a unique solution $y$ of equation (2.1). Moreover, for the solution $y$ the following estimate holds*

$$\left\| \sqrt{r} y^{(3)} \right\|_2 + \|y\|_2 \leq \left(\gamma_{1,\sqrt{r},2} + 1\right) \|ly\|_2 . \tag{2.2}$$

*Proof.* Let $y \in C_0^{(5)}(\mathbb{R})$. Integrating by parts we get

$$\left(l_0 y, y^{(3)}\right) = -\int_{\mathbb{R}} y^{(5)} \bar{y}^{(3)} dx + \int_{\mathbb{R}} r \left|y^{(3)}\right|^2 dx = \left\|y^{(4)}\right\|_2^2 + \left\|\sqrt{r} y^{(3)}\right\|_2^2 \geq \left\|\sqrt{r} y^{(3)}\right\|_2^2 .$$

By condition $r \geq 1$ and Hölder's inequality, we have

$$\left|\left(l_0 y,\, y^{(3)}\right)\right| \leq \left(\int_{\mathbb{R}} \left|\frac{1}{\sqrt{r}} ly\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left|\sqrt{r} y^{(3)}\right|^2 dx\right)^{\frac{1}{2}} .$$

It follows from the last two inequalities that

$$\left\|\sqrt{r} y^{(3)}\right\|_2 \leq \left\| \frac{1}{\sqrt{r}} l_0 y \right\|_2 . \tag{2.3}$$

By inequality (1.7) in Lemma 1.1, (2.3) implies that

$$\|y\|_2 \leq \gamma_{1,\sqrt{r},2} \left\|\sqrt{r} y^{(3)}\right\|_2 \leq \gamma_{1,\sqrt{r},2} \|l_0 y\|_2 . \tag{2.4}$$

From inequalities (2.3) and (2.4) we obtain that

$$\|y\|_2 + \left\|\sqrt{r} y^{(3)}\right\|_2 \leq \left(\gamma_{1,\sqrt{r},2} + 1\right) \|l_0 y\|_2 , y \in C_0^{(5)}(R) . \tag{2.5}$$

Now we show that estimate (2.5) holds for any $y \in D(l)$. For $y \in D(l)$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq C_0^{(5)}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ $(n \to \infty)$. By (2.5),

$$\|y_n\|_2 + \left\|\sqrt{r} y_n^{(3)}\right\|_2 \leq \left(\gamma_{1,\sqrt{r},2} + 1\right) \|l_0 y_n\|_2 \;\; (n = 1,\, 2,\, ...). \tag{2.6}$$

Therefore, the inequality

$$\|y_n - y_m\|_2 + \left\|\sqrt{r}\left(y_n^{(3)} - y_m^{(3)}\right)\right\|_2 \leq \left[\gamma_{1,\sqrt{r},2} + 1\right] \|l_0 y_n - l_0 y_m\|_2 \to 0$$

holds for any natural numbers $n$ and $m$. Let $\dot{W}^3_{2,r}(\mathbb{R})$ be the closure of $C_0^{(3)}(\mathbb{R})$ with respect to the norm $\|y\|_W = \left\|\sqrt{r}y^{(3)}\right\|_2 + \|y\|_2$. The sequence $\{y_n\}_{n=1}^{\infty}$ converges to $y \in \dot{W}^3_{2,r}(\mathbb{R})$. Passing to the limit in (2.6), we obtain that estimate (2.2) holds for $y \in D(l)$.

Inequality (2.2) shows that there exists the inverse operator $l^{-1}$ to the operator $l$. By (2.2) and Definition 2, the solution of equation (2.1) is unique (if exists).

Now we will show the solvability of equation (2.1). If we denote $y^{(3)} = z$ and $\mathfrak{I}z = -z'' + r(x)z$, then equation (2.1) takes the following form

$$\mathfrak{I}z = -z'' + r(x)z = h(x).$$

It follows from estimate (2.2) that $D(\mathfrak{I}) \subseteq L_2(\mathbb{R})$. By Definition 2, it suffices to show that $R(\Im) = L_2(\mathbb{R})$. Assume the contrary: $R(\Im) \neq L_2(\mathbb{R})$. Due to inequality (2.2), the set $R(\Im)$ is closed, so there exists a nonzero element $v \in L_2(\mathbb{R})$ and $v \perp R(\Im)$ [18]. It is easy to check that

$$\Im^* v = -v'' + r(x)v = 0,$$

where $\Im^*$ is the adjoint operator to $\Im$. By condition $r \geq 1$ and known properties of the Sturm-Liouville equation, $v \notin L_2(\mathbb{R})$. We obtain a contradiction that shows $R(\Im) = L_2(\mathbb{R})$. □

**Remark 1.** The condition $r \geq 1$ in Lemma 2.1 can be replaced by the inequality $r \geq \delta > 0$. To check of this fact, it is enough to make the substitution $x = \frac{1}{\sqrt{\delta}}t$ in (2.1), where the condition $r \geq \delta > 0$ is fulfilled.

**Lemma 2.2.** *Let a function $r$ satisfy the conditions of Lemma* 2.1 *and*

$$C^{-1} \leq \frac{r(x)}{r(\eta)} \leq C, \forall x, \eta \in \mathbb{R} : |x - \eta| \leq 1, \tag{2.7}$$

*where $C > 1$. Then the following estimate*

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 \leq C_1 \|ly\|_2 \tag{2.8}$$

*holds for the solution $y$ of equation* (2.1)*. Here $C_1$ depends only on $r$.*

*Proof.* Let $y$ be the solution of equation (2.1). By estimate (2.2), we have $y^{(3)} \in L_2(\mathbb{R})$. If we denote $y^{(3)} = z$, then equation (2.1) takes the following form:

$$\mathfrak{I}z = -z'' + r(x)z = h(x).$$

It is known that the solution $z$ of last equation satisfies the following inequality

$$\left\|z^{(2)}\right\|_2 + \|rz\|_2 \leq C_1 \|f\|_2,$$

if condition $r \geq 1$ in Lemma 2.1 and (2.7) are fulfilled [8]. Since $y^{(3)} = z$, we obtain estimate (2.8) for the solution $y$ of equation (2.1). □

## 3 Main result

**Theorem 3.1.** *Let a function $r(x)$ satisfy the conditions of Lemma* 2.2*, and functions $q(x)$ and $p(x)$ be such that*

$$\gamma_{q,r,1} < \infty, \gamma_{p,r,2} < \infty. \tag{3.1}$$

*Then for any $f \in L_2(\mathbb{R})$ there exists a solution $y$ of equation* (1.5) *and it is unique. Furthermore, the following estimate holds for the solution $y$:*

$$\left\|y^{(5)}\right\|_2 + \left\|ry^{(3)}\right\|_2 + \|qy'\|_2 + \|py\|_2 \leq C_2\|f\|_2, \tag{3.2}$$

*where $C_2 > 0$ depends only on $r$, $q$, $p$.*

*Proof.* Let us make the substitution $t = \frac{x}{a}$ in equation (1.5), where $a$ is a positive number. If we denote $y\,(at) = \tilde{y}\,(t)$, $r\,(at) = \tilde{r}\,(t)$, $p\,(at) = \tilde{p}\,(t)$, $q\,(at) = \tilde{q}\,(t)$, $a^5 f\,(at) = \tilde{f}\,(at)$, then (1.5) takes the following form

$$\tilde{L}_a\tilde{y} = -\tilde{y}^{(5)} + a^2\tilde{r}\tilde{y}^{(3)} + a^4\tilde{q}\tilde{y}' + a^5\tilde{p}\tilde{y} = \tilde{f}. \tag{3.3}$$

Let $l_a$ be the closure in $L_2(\mathbb{R})$ of the differential operator

$$l_{0a}\tilde{y} = -\tilde{y}^{(5)} + a^2\tilde{r}\tilde{y}^{(3)}, \tag{3.4}$$

defined in $C_0^{(5)}(\mathbb{R})$. Since $a^2\tilde{r}\,(t) \geq \delta_0$ for some $\delta_0 > 0$ , by Lemmas 2.1 and 2.2 and Remark 1, the operator $l_a$ is continuously invertible and for any $\tilde{y} \in D\,(l_a)$ the following estimate holds

$$\left\|\tilde{y}^{(5)}\right\|_2 + \left\|a^2\tilde{r}\tilde{y}^{(3)}\right\|_2 \leq C_{l_a}\left\|l_a\tilde{y}\right\|_2, \tag{3.5}$$

where $C_{l_a}$ depends only on $\tilde{r}$ and $a$. It is easy to check that $\gamma_{\tilde{q},\tilde{r},1} = a^{-2}\gamma_{q,r,1}$ and $\gamma_{\tilde{p},\tilde{r},2} = a^{-3}\gamma_{p,r,2}$. By (3.5), condition (3.1) and Lemma 1.1, we have

$$\left\|a^4\tilde{q}\tilde{y}'\right\|_2 \leq 2a^4\gamma_{q,r,1}C_{l_a}\left\|l_a\tilde{y}\right\|_2 \tag{3.6}$$

and

$$\left\|a^5\tilde{p}\tilde{y}\right\|_2 \leq a^4\gamma_{p,r,2}C_{l_a}\left\|l_a\tilde{y}\right\|_2. \tag{3.7}$$

If we choose a number $a$ such that $a \leq \frac{1}{\sqrt[4]{2(2\gamma_{q,r,1}+\gamma_{p,r,2})C_{l_a}}}$, then by (3.6) and (3.7), we obtain

$$\left\|a^4\tilde{q}\tilde{y}'\right\|_2 + \left\|a^5\tilde{p}\tilde{y}\right\|_2 \leq \frac{1}{2}\|l_a\tilde{y}\|_2. \tag{3.8}$$

Then, according to the well-known theorem on small perturbation of a linear operator (see, for example, [6]), there exists the inverse $(\tilde{L}_a)^{-1}$ to the operator $\tilde{L}_a = l_a + a^4\tilde{q}E + a^5\tilde{p}E$ ($E$ is the unit operator) and $(\tilde{L}_a)^{-1}$ is defined on all $L_2(\mathbb{R})$. By Definition 1, for each $f \in L_2(\mathbb{R})$ there exists a solution $\tilde{y}$ of equation (3.3) and it is unique.

From (3.5) and (3.8) we have

$$\left\|\tilde{y}^{(5)}\right\|_2 + \left\|a^2\tilde{r}\tilde{y}^{(3)}\right\|_2 + \left\|a^4\tilde{q}\tilde{y}'\right\|_2 + \left\|a^5\tilde{p}\tilde{y}\right\|_2 \leq \left(C_{l_a} + \frac{1}{2}\right)\|l_a\tilde{y}\|_2. \tag{3.9}$$

By (3.8),

$$\|l_a\tilde{y}\|_2 \leq 2\left\|\tilde{L}_a\tilde{y}\right\|_2.$$

Then (3.9) implies that

$$\left\|\tilde{y}^{(5)}\right\|_2 + \left\|a^2\tilde{r}\tilde{y}^{(3)}\right\|_2 + \left\|a^4\tilde{q}\tilde{y}'\right\|_2 + \left\|a^5\tilde{p}\tilde{y}\right\|_2 \leq (2C_{l_a} + 1)\|\tilde{f}\|_2.$$

Taking into account that $x = at$ in the last inequality and passing to the variable $x$, we obtain that estimate (3.2) holds for the solution $y$ of equation (1.5). □

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