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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
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The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
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NEW 2-MICROLOCAL BESOV AND TRIEBEL–LIZORKIN SPACES VIA THE LITLLEWOOD – PALEY DECOMPOSITION

K. Saka

Communicated by D. Yang

Key words: wavelet, Besov space, Triebel–Lizorkin space, pseudo-differential operator, Calderón–Zygmund operator, atomic and molecular decomposition, 2-microlocal space, φ–transform.

AMS Mathematics Subject Classification: 42B35, 42B20, 42B25, 42C40.

Abstract. In this paper we introduce and investigate new 2-microlocal Besov and Triebel–Lizorkin spaces via the Littlewood - Paley decomposition. We establish characterizations of these function spaces by the φ–transform, the atomic and molecular decomposition and the wavelet decomposition. As applications we prove boundedness of the the Calderón–Zygmund operators and the pseudo–differential operators on the function spaces. Moreover, we give characterizations via oscillations and differences.

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1 Introduction

It is well known that function spaces have increasing applications in many areas of modern analysis, in particular, harmonic analysis and partial differential equations. The most general function spaces, probably, are the Besov spaces and the Triebel–Lizorkin spaces which cover many classical concrete function spaces such as Lebesgue spaces, Lipschitz spaces, Sobolev spaces, Hardy spaces and BMO spaces ([37], [38]).

D. Yang and W. Yuan in [41], [42] and W. Sickel, D. Yang and W. Yuan in [36], introduced a class of Besov type and Triebel–Lizorkin type spaces which generalized many classical function spaces such as Besov spaces, Triebel-Lizorkin spaces, Morrey spaces and Q-type spaces. Recently the Besov type and Triebel-Lizorkin type space with variable exponents was investigated by many authors (e.g. [43], [44]).

The 2-microloal space is due to Bony [3] in order to study the propagation of singularities of the solutions of nonlinear evolution equations. It is an appropriate instrument to describe the local regularity and the oscillatory behavior of functions near singularity (Meyer [32]). The theory has been elaborated and widely used in fractal analysis and signal processing. For systematic discussions of the concept and further references of 2-microlocal spaces, we refer to Meyer[31], [32], Levy-Vehel and Seuret [30], Jaffard ([17], [18], [19], [20]), Jaffard and Mélot [21], and Jaffard and Meyer [22].

The 2-microlocal spaces have been generalized by Jaffard as a general pointwise regularity associated with Banach or quasi-Banach spaces [19], [20]. In this paper we introduce new inhomogeneous 2-microlocal spaces based on Jaffard's idea (See [33] for the homogeneous 2-microlocal spaces) and we will investigate the properties and the characterizations of these new 2-microlocal Besov and Triebel–Lizorkin spaces which unify many classical function spaces such as the Besov type and Triebel–Lizorkin type spaces, the 2–microlocal spaces in the sense of Meyer [32], the Morrey space and the local Morrey spaces. These new function space are very similar to the classical 2-microlocal

Besov and Triebel-Lizorkin spaces studied recently by many authors ([1], [6], [8], [13], [14], [15], [16], [25], [26], [39], [40]).

The plan of the remaining sections in the paper is as follows:

In Section 2 we give the definitions of our new 2-microlocal spaces via the Littlewood-Paley decomposition and the notations which are used later and we give examples for these spaces.

In Section 3 we define corresponding sequence spaces for our function spaces. Furthermore, we give some auxiliary lemmas which are needed in later sections.

In Section 4 we will characterize our function spaces via the corresponding sequence spaces by the φ–transform in the sense of Fraizer–Jarwerth [10], the atomic and molecular decomposition and the wavelet decomposition. Moreover, we investigate the properties for these function spaces and we also study relations between our 2-microlocal spaces and the classical 2-microlocal spaces.

In Section 5, as applications, we give the conditions under which the Calderón–Zygmund operators and the pseudo–differential operators are bounded on the function spaces.

In Section 6 we give the characterizations via differences and oscillations.

Throughout the paper, we use C to denote a positive constant. But the same notation C are not necessarily the same on any two occurrences. We use the notations $i \vee j = \max\{i,\ j\}$, $i \wedge j = \min\{i,\ j\}$, and $a_+ = a \vee 0$. The symbol $X \sim Y$ means that there exist positive constants C_1 and C_2 such that $X \leq C_1 Y$ and $Y \leq C_2 X$.

2 Definitions

We consider the dyadic cubes in $\mathbb{R}^n$ of the form $Q = [0,\ 2^{-l})^n + 2^{-l}k$ for $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$, and use the notation $l(Q) = 2^{-l}$ for the side length and $x_Q = 2^{-l}k$ for the corner point. Throughout the paper, we use the notations $P,\ Q,\ R$ for the dyadic cubes of the form $[0,\ 2^{-l})^n + 2^{-l}k$ in $\mathbb{R}^n$, and when the dyadic cubes Q appear as indices, it is understood that Q runs over all dyadic cubes of this form in $\mathbb{R}^n$. We denote by $\mathcal{D}$ the set of all dyadic cubes of this form. For a dyadic cube Q and a constant $c > 1$, cQ denotes the cube of same center as Q and c times larger. We denote by χ_E the characteristic function of a set E in $\mathbb{R}^n$.

We set $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'$ its dual.

We use $\langle f,\ g\rangle$ for the standard inner product $\int f\bar{g}$ of two functions and the same notation is employed for the action of a distribution $f \in \mathcal{S}'$ on $\bar{g} \in \mathcal{S}$.
Let ϕ_0 be a Schwartz function and $\hat{\phi}_0$ its Fourier transform satisfying

(1.1) $\text{supp}\ \hat{\phi}_0 \subset \{\xi \in \mathbb{R}^n :\ |\xi| \leq 2\}$,

(1.2) $\hat{\phi}_0(\xi) = 1$ if $|\xi| \leq 1$.

We set

$\phi(x) = \phi_0(x) - 2^{-n}\phi_0(2^{-1}x)$, $\phi_0^j = 2^{jn}\phi_0(2^j x)$, $S_j f = f * \phi_0^j$ for $j \in \mathbb{N}_0$, and $\phi_j(x) = 2^{jn}\phi(2^j x)$ for $j \in \mathbb{N}$.

Then we have

(1.3) $\text{supp}\ \hat{\phi} \subset \{\xi \in \mathbb{R}^n :\ \frac{1}{2} \leq |\xi| \leq 2\}$, and

(1.4) there exist positive numbers c and a sufficiently small ϵ such that $\hat{\phi}(\xi) \geq c$ in $1 - \epsilon \leq |\xi| \leq 1 + \epsilon$.

It holds that $\sum_{j \in \mathbb{N}_0} \hat{\phi}_j = 1$. Let $f \in \mathcal{S}'$, then we have the Littlewood-Paley decomposition $f = \sum_{j \in \mathbb{N}_0} f * \phi_j$ (convergence in $\mathcal{S}'$) [36, Triebel 2.3.1(6)].

Let $s \in \mathbb{R}$. For $f \in \mathcal{S}'$, we define some sequences indexed by dyadic cubes P:

$c(B^s_{pq})(P) = (\sum_{i \geq (-\log_2 l(P)) \vee 0} ||2^{is} f * \phi_i||^q_{L^p(P)})^{1/q}$, $0 < p, q \leq \infty$,

$c(F^s_{pq})(P) = ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is}|f * \phi_i|)^q\}^{1/q}||_{L^p(P)}$,

$0 < p < \infty$, $0 < q \leq \infty$,
$c(F^s_{\infty q})(P) = l(P)^{-\frac{n}{q}} ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is}|f * \phi_i|)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

We shall use the notation E^s_{pq} with either B^s_{pq} or F^s_{pq}. We say the B-type case when $E^{s'}_{pq} = B^{s'}_{pq}$, and the F-type case when $E^{s'}_{pq} = F^{s'}_{pq}$.

Definition 1. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
The space $A^s(E^{s'}_{pq})^\sigma_{x_0}$ is defined to be the space of all $f \in \mathcal{S}'$ such that

$$||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D} \ni Q \ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D} \ni P \subset 3Q} l(P)^{-s} c(E^{s'}_{pq})(P) < \infty.$$

The following abbreviation $A^0(E^{s'}_{pq})^\sigma_{x_0} \equiv (E^{s'}_{pq})^\sigma_{x_0}$, $A^s(E^{s'}_{pq})^0_{x_0} \equiv A^s(E^{s'}_{pq})$ and $A^0(E^{s'}_{pq})^0_{x_0} \equiv E^{s'}_{pq} \equiv E^{s'}_{pq}(\mathbb{R}^n)$ will be used in the sequel. We note that the space $A^s(E^{s'}_{pq})$ is the inhomogeneous Besov type space or the inhomogeneous Triebel–Lizorkin type space in the sense of Yang–Sickel–Yuan [26] and the space $E^{s'}_{pq} \equiv E^{s'}_{pq}(\mathbb{R}^n)$ is the classical inhomogeneous Besov or inhomogeneous Triebel–Lizorkin space.

Let $f \in \mathcal{S}'$, then we define some sequences indexed by dyadic cubes P:

$c(\tilde{B}^{s'}_{pq})^\sigma_{x_0}(P) =$
$(\sum_{i \geq (-\log_2 l(P)) \vee 0} ||2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma}||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \leq \infty$,
$c(\tilde{F}^{s'}_{pq})^\sigma_{x_0}(P) =$
$||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma})^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty$, $0 < q \leq \infty$,
$c(\tilde{F}^{s'}_{\infty q})^\sigma_{x_0}(P) =$
$l(P)^{-\frac{n}{q}} ||\{\sum_{i \geq (-\log_2 l(P)) \vee 0} (2^{is'}|f * \phi_i(x)|(2^{-i} + |x_0 - x|)^{-\sigma})^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

We shall use the notation $\tilde{E}^{s'}_{pq}$ with either $\tilde{B}^{s'}_{pq}$ or $\tilde{F}^{s'}_{pq}$. We say the B-type case when $\tilde{E}^{s'}_{pq} = \tilde{B}^{s'}_{pq}$, and the F-type case when $\tilde{E}^{s'}_{pq} = \tilde{F}^{s'}_{pq}$.

Definition 2. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
The space $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ is defined to be the space of all $f \in \mathcal{S}'$ such that

$$||f||_{A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D} \ni P} l(P)^{-s} c(\tilde{E}^{s'}_{pq})^\sigma_{x_0}(P) < \infty.$$

The space $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ is the classical 2–microlocal Besov or Triebel–Lizorkin space.
We use the abbreviation $A^0(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \equiv (\tilde{E}^{s'}_{pq})^\sigma_{x_0}$.

Examples.

(i) The spaces $A^0(E^{s'}_{pq})^0_{x_0} = A^0(\tilde{E}^{s'}_{pq})^0_{x_0} = E^{s'}_{pq}(\mathbb{R}^n)$ are the inhomogeneous Besov spaces or inhomogeneous Triebel–Lizorkin spaces [37], [38].

(ii) The Besov type spaces $B^{s,\tau}_{pq}(\mathbb{R}^n)$ and the Triebel–Lizorkin type spaces $F^{s,\tau}_{pq}(\mathbb{R}^n)$ introduced by D. Yang , W. Sickel and W. Yuan [36] , are contained in our definition as
$E^{s',s}_{pq}(\mathbb{R}^n) = A^{ns}(E^{s'}_{pq})^0_{x_0} = A^{ns}(\tilde{E}^{s'}_{pq})^0_{x_0}$.

(iii) The Besov-Morrey spaces $\mathcal{N}^s_{uqp}$, and the Triebel–Lizorkin-Morrey spaces $\mathcal{E}^s_{uqp}$ studied by Y. Sawano and H. Tanaka [34], or Y. Sawano, D. Yang and W. Yuan [35] are realized in our definition as
$\mathcal{N}^s_{uqp} \subset A^{n(\frac{1}{p}-\frac{1}{u})}(B^s_{pq})^0_{x_0}$ if $0 < p \le u \le \infty$ and $0 < q \le \infty$,
$\mathcal{E}^s_{uqp} = A^{n(\frac{1}{p}-\frac{1}{u})}(F^s_{pq})^0_{x_0}$ if $0 < p \le u \le \infty$ and $0 < q \le \infty$.
The Morrey space $\mathcal{M}^u_p$ is realized as
$\mathcal{M}^u_p = A^{n(\frac{1}{p}-\frac{1}{u})}(F^0_{p2})^0_{x_0}$ if $1 < p < u < \infty$.

(iv) The $\dot{B}_\sigma$-Morrey spaces $\dot{B}_\sigma(L_{p,\lambda})$ studied by Y. Komori-Furuya et al. [28], are contained in our definition as
$\dot{B}_\sigma(L_{p,\lambda}) = A^{\lambda+\frac{n}{p}}(F^0_{p2})^\sigma_0,\ \ 1 < p < \infty$.

(v) The 2-microlocal Besov spaces $B^{s,s'}_{pq}(U)$ studied in H. Kempka [23, 24], are realized in our definition as
$B^{s,s'}_{pq}(U) = (\tilde{B}^{s+s'}_{pq})^{-s'}_{x_0}$ when $U = \{x_0\}$.

(vi) The local Morrey spaces $LM_{p,\lambda}$ introduced by V.I. Burenkov and H.V. Guliyes [6] and studied in Ts. Batbold and Y. Sawano [2] and a number of papers, are realized in our definition as
$LM_{p,\lambda} = (F^0_{p2})^{\lambda/p}_0,\ \ 1 < p < \infty$.

(vii) The spaces $C^{s,s'}_{x_0}$ studied in Y. Meyer [31], [32], are realized in our definition as
$C^{s,s'}_{x_0} = (\tilde{B}^{s+s'}_{\infty\infty})^{-s'}_{x_0} = (B^{s+s'}_{\infty\infty})^{-s'}_{x_0}$.

3 Sequence spaces

For a sequence $c = (c(R))$ with $l(R) \le 1$ we define some sequences indexed by dyadic cubes P:

$c(b^s_{pq})(P) = (\sum_{i\ge(-\log_2 l(P))\vee 0} || \sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \le \infty$,
$c(f^s_{pq})(P) = ||\{\sum_{i\ge(-\log_2 l(P))\vee 0} \big(\sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R\big)^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty,\ 0 < q \le \infty$, and
$c(f^s_{\infty q})(P) = l(P)^{-\frac{n}{q}}||\{\sum_{i\ge(-\log_2 l(P))\vee 0} \big(\sum_{l(R)=2^{-i}} 2^{is}|c(R)|\chi_R\big)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \le \infty$, with the usual modification for $q = \infty$.

The notation e^s_{pq} is used to denote either b^s_{pq} or f^s_{pq}. We say the B-type case when $e^{s'}_{pq} = b^{s'}_{pq}$, and the F-type case when $e^{s'}_{pq} = f^{s'}_{pq}$.

Definition 3. Let $s,\ s',\ \sigma\ \in \mathbb{R},\ 0 < p, q \le \infty$ and $x_0 \in \mathbb{R}^n$.
We define the sequence space $a^s(e^{s'}_{pq})^\sigma_{x_0}$ to be the space of all sequences $c = (c(R))_{l(R)\le 1}$ such that

$$||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \equiv \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}c(e^{s'}_{pq})(P) < \infty.$$

We use the abbreviation $a^0(e^{s'}_{pq})^{\sigma}_{x_0} \equiv (e^{s'}_{pq})^{\sigma}_{x_0}$, $a^s(e^{s'}_{pq})^0_{x_0} \equiv a^s(e^{s'}_{pq})$ and $a^0(e^{s'}_{pq})^0_{x_0} \equiv e^{s'}_{pq} \equiv e^{s'}_{pq}(\mathbb{R}^n)$. We note that the space $a^s(e^{s'}_{pq})$ is the sequence space of the inhomogeneous Besov type space or the inhomogeneous Triebel–Lizorkin type space in the sense of Yang–Sickel–Yuan [36] and the space $e^{s'}_{pq} \equiv e^{s'}_{pq}(\mathbb{R}^n)$ is the sequence space of the classical inhomogeneous Besov or inhomogeneous Triebel–Lizorkin space.

Remark 1. It is easy that when $\sigma < 0$, we have $A^s(E^{s'}_{pq})^{\sigma}_{x_0} = \{0\}$ and $a^s(e^{s'}_{pq})^{\sigma}_{x_0} = \{0\}$ for $0 < p, q \leq \infty$ (See Proposition 4.1 below).

We define that for a sequence $(c(R))_{l(R)\leq 1}$,

$c(\tilde{b}^{s'}_{pq})^{\sigma}_{x_0}(P) =$
$(\sum_{i\geq(-\log_2 l(P))\vee 0} || \sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i} + |x_0 - x|)^{-\sigma}\chi_R||^q_{L^p(P)})^{1/q}$,
$0 < p,\ q \leq \infty$,
$c(\tilde{f}^{s'}_{pq})^{\sigma}_{x_0}(P) =$
$||\{\sum_{i\geq(-\log_2 l(P))\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i} + |x_0 - x|)^{-\sigma}\chi_R)^q\}^{1/q}||_{L^p(P)}$,
$0 < p < \infty$, $0 < q \leq \infty$,
$c(\tilde{f}^{s'}_{\infty q})^{\sigma}_{x_0}(P) = l(P)^{-\frac{n}{q}} \times$
$||\{\sum_{i\geq(-\log_2 l(P))\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}|c(R)|(2^{-i} + |x_0 - x|)^{-\sigma}\chi_R)^q\}^{1/q}||_{L^q(P)}$,
$0 < q \leq \infty$,
with the usual modification for $q = \infty$.

The notation $\tilde{e}^{s'}_{pq}$ is used to denote either $\tilde{b}^{s'}_{pq}$ or $\tilde{f}^{s'}_{pq}$. We say the B-type case when $\tilde{e}^{s'}_{pq} = \tilde{b}^{s'}_{pq}$, and the F-type case when $\tilde{e}^{s'}_{pq} = \tilde{f}^{s'}_{pq}$.

Definition 4. Let $s,\ s',\ \sigma\ \in \mathbb{R}$, $0 < p, q \leq \infty$ and $x_0 \in \mathbb{R}^n$.
We define the sequence space $a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$ to be the space of all sequences $c = (c(R))_{l(R)\leq 1}$ such that

$$||c||_{a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}} \equiv \sup_{\mathcal{D}\ni P} l(P)^{-s} c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) < \infty.$$

We use the abbreviation $a^0(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0} \equiv (\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$.

Definition 5. Let $r_1,\ r_2 \geq 0$ and $L > 0$. We say that a matrix operator $A = \{a_{QP}\}_{QP}$, indexed by dyadic cubes Q and P, is (r_1, r_2, L)-almost diagonal if the matrix $\{a_{QP}\}$ satisfies

$|a_{QP}| \leq C\left(\frac{l(Q)}{l(P)}\right)^{r_1}(1 + l(P)^{-1}|x_Q - x_P|)^{-L}$ if $l(Q) \leq l(P)$,
$|a_{QP}| \leq C\left(\frac{l(P)}{l(Q)}\right)^{r_2}(1 + l(Q)^{-1}|x_Q - x_P|)^{-L}$ if $l(Q) > l(P)$.

The results about the boundedness of almost diagonal operators in [9: Theorem 3.3], also hold in our cases.

Lemma 3.1. *Suppose that $s,\ s',\ \sigma\ \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $0 < p,\ q\ \leq \infty$. Then,*
(i) *an (r_1, r_2, L)–almost diagonal matrix operator A is bounded on $a^s(e^{s'}_{pq})^{\sigma}_{x_0}$ for $r_1 > \max(s',\ \sigma + s + s' - \frac{n}{p})$, $r_2 > J - s'$ and $L > J$ where $J = n/\min(1,\ p,\ q)$ in the case $e^{s'}_{pq} = f^{s'}_{pq}$, and $J = n/\min(1,\ p)$ in the case $e^{s'}_{pq} = b^{s'}_{pq}$, respectively,*
(ii) *an (r_1, r_2, L)-almost diagonal matrix operator A is bounded on $a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}$ for $r_1 > \max(s' + (\sigma \vee 0),\ (\sigma \vee 0) + s + s' - \frac{n}{p})$, $r_2 > J - s' + (\sigma \wedge 0)$ and $L > J$ where $J = n/\min(1,\ p,\ q)$ in the case $\tilde{e}^{s'}_{pq} = \tilde{f}^{s'}_{pq}$, and $J = n/\min(1,\ p)$ in the case $\tilde{e}^{s'}_{pq} = \tilde{b}^{s'}_{pq}$, respectively.*

Proof : (i) We may assume $\sigma \geq 0$ by Remark 1. We assume that $A = (a_{RR'})$ is (r_1, r_2, L)–almost diagonal. Let $c = (c(R)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. For dyadic cubes P and R with $R \subset P$, we write $Ac = A_0c + A_1c + A_2c$ with

$$(A_0c)(R) = \sum_{l(R)\leq l(R')\leq l(P)} a_{RR'}c(R'),$$
$$(A_1c)(R) = \sum_{l(R')<l(R)\leq l(P)} a_{RR'}c(R'),$$
$$(A_2c)(R) = \sum_{l(R)\leq l(P)<l(R')\leq 1} a_{RR'}c(R').$$

We claim that

$$||A_ic||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \leq C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}, \quad i = 0, 1, 2.$$

We will consider the case of F-type for $0 < p < \infty$, $0 < q \leq \infty$. Since A is almost diagonal, we see that for dyadic cubes P with $l(P) = 2^{-j}$,

$$\begin{aligned}
(A_0c)(f^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} \big(2^{is'}|(A_0c)(R)|\big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}\big(\sum_{i\geq k\geq j\vee 0}\sum_{l(R')=2^{-k}} |a_{RR'}||c(R')|\big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q} \times \\
&\big(\sum_{i\geq k\geq j\vee 0}\sum_{l(R')=2^{-k}} 2^{-(i-k)r_1}(1 + 2^k|x_R - x_{R'}|)^{-L}|c(R')|\big)^q\chi_R\}^{1/q}||_{L^p(P)}.
\end{aligned}$$

Using the maximal function $M_tf(x)$, $0 < t \leq 1$, defined by

$$M_tf(x) = \sup_{x\in Q}\big(\frac{1}{l(Q)^n}\int_Q |f(y)|^t\, dy\big)^{1/t}$$

(cf. [28: Lemma 7.1] or [9: Remark A.3]), we have for $L > n/t$,

$$\begin{aligned}
(A_0c)(f^{s'}_{pq})(P) &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}2^{-ir_1q}\times \\
&\Big(\sum_{i\geq k\geq j\vee 0} 2^{kr_1}2^{(k-i)+n/t}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{-i(r_1-s')q}\Big(\sum_{i\geq k\geq j\vee 0} 2^{kr_1}M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}M_t\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P),
\end{aligned}$$

where these inequalities follow from Hardy's inequality if $r_1 > s'$ and the Fefferman-Stein inequality if $0 < t < \min(p, q)$.

For the B-type case we have the same estimate for $r_1 > s'$ and $0 < t < \min(1, p)$.

Therefore, we get the estimate

$$A_0 c(e^{s'}_{pq})(P) \leq Cc(e^{s'}_{pq})(P)$$

if $r_1 > s'$, $0 < p < \infty$, $0 < q \leq \infty$, $L > J$.

In the sane way we will get the estimate for $(A_1 c)(f^{s'}_{pq})(P)$. We have that for dyadic cubes P with $l(P) = 2^{-j}$,

$$\begin{aligned}
(A_1 c)(f^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} \big(2^{is'}|(A_1 c)(R)|\big)^q \chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}\big(\sum_{i\leq k}\sum_{l(R')=2^{-k}} |a_{RR'}||c(R')|\big)^q \chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q} \times \\
&\big(\sum_{i\leq k}\sum_{l(R')=2^{-k}} 2^{-(k-i)r_2}(1+2^i|x_R - x_{R'}|)^{-L}|c(R')|\big)^q \chi_R\}^{1/q}||_{L^p(P)}.
\end{aligned}$$

Using the maximal function $M_t f(x)$ as above, we have

$$\begin{aligned}
(A_1 c)(f^{s'}_{pq})(P) &\leq C||\{\sum_{i\geq j\vee 0}\sum_{l(R)=2^{-i}} 2^{is'q}2^{ir_2 q}\times \\
&\Big(\sum_{i\leq k} 2^{-kr_2}2^{(k-i)_+ n/t} M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q \chi_R\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{i(r_2+s'-n/t)q} \times \\
&\Big(\sum_{i\leq k} 2^{-k(r_2-n/t)} M_t\big(\sum_{l(R')=2^{-k}} |c(R')|\chi_{R'}\big)\Big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q} M_t\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0} 2^{is'q}\big(\sum_{l(R')=2^{-i}} |c(R')|\chi_{R'}\big)^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P),
\end{aligned}$$

where these inequalities follow from Hardy's inequality if $r_2 + s' - n/t > 0$ and the Fefferman-Stein inequality if $0 < t < \min(p, q)$.

In the same way we get the same estimate for the B-type case that

$$(A_1 c)(b^{s'}_{pq})(P) \leq Cc(b^{s'}_{pq})(P)$$

if $r_2 + s' - n/t > 0$, $0 < t < \min(1, p)$. Therefore, we get the estimate

$$A_1 c(e^{s'}_{pq})(P) \leq Cc(e^{s'}_{pq})(P)$$

if $r_2 > J - s'$, $0 < p < \infty$, $0 < q \leq \infty$, $L > J$.

When $p = \infty$, we get the same estimate. Thus, we get

$$||A_i c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \leq C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}, \quad i = 0, 1$$

if $r_1 > s'$, $r_2 > J - s'$, $L > J$, $0 < p \le \infty$ and $0 < q \le \infty$.

Next, we will give the estimates for the A_2 case.

We note that if $L > n$,

$$\sum_{l(P)=2^{-j}} (1 + 2^j |x_R - x_P|)^{-L} < \infty$$

(cf. [4, Lemma 3.4]), and if $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, then

$$|c(R)| \le C(|x_0 - x_R| + l(R))^\sigma l(R)^{s+s'-n/p} ||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}$$

for a dyadic cube $R \subset 3Q$ and $x_0 \in Q$. Hence, we obtain, for dyadic cubes P with $l(P) = 2^{-j}$, $0 < p < \infty$ and $0 < q \le \infty$,

$$\begin{aligned}
&(A_2c)(f^{s'}_{pq})(P) = ||\{\sum_{i\ge j} \sum_{l(R)=2^{-i}} \big(2^{is'}|(A_2c)(R)|\big)^q \chi_R\big\}^{1/q}||_{L^p(P)} \\
&\le C||\{\sum_{i\ge j} \sum_{l(R)=2^{-i}} 2^{is'q} \times \\
&\big(\sum_{j\ge k\ge 0} \sum_{l(R')=2^{-k}} 2^{-(i-k)r_1}(1 + 2^k|x_R - x_{R'}|\big)^{-L} |c(R')|)^q \chi_R\big\}^{1/q}||_{L^p(P)} \\
&\le C||\{\sum_{i\ge j} 2^{-i(r_1-s')q} \times \\
&\quad \big(\sum_{j\ge k\ge 0} 2^{kr_1} 2^{-k(\sigma+s+s'-n/p)}(1 + 2^k|x_0 - x_P|)^\sigma ||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}}\big)^q\big\}^{1/q}||_{L^p(P)} \\
&\le C2^{-j(r_1-s')} 2^{-jn/p} \sum_{j\ge k\ge 0} 2^{k(r_1-\sigma-s-s'+n/p)}(1 + 2^j|x_0 - x_P|)^\sigma ||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} \\
&\le C2^{-j(r_1-s'+n/p)} 2^{j(r_1-\sigma-s-s'+n/p)}(1 + 2^j|x_0 - x_P|)^\sigma ||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} \\
&\le C2^{-js}(2^{-j} + |x_0 - x_P|)^\sigma ||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}}
\end{aligned}$$

where these inequalities follow if $r_1 > \sigma + s + s' - \frac{n}{p}$, $r_1 > s'$, $L > n$ and $\sigma \ge 0$.

In the same way for the B-type case we have the same estimate.

Hence, we have,

$$||A_2c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}$$

if $r_1 > \sigma + s + s' - n/p$, $r_1 > s'$, $0 < p < \infty$ and $0 < q \le \infty$.

We get the same estimate for the case $p = \infty$. Therefore, we obtain the desired conclusion.

(ii) We put $w_i = (2^{-i} + |x_0 - x|)^{-\sigma}$. We see that $w_i \le 2^{(i-k)+\sigma} w_k$ if $0 \le \sigma$, and $w_i \le 2^{(k-i)+\sigma} w_k$ if $0 > \sigma$. Then, using these inequalities we can prove the desired result by using the same way in the above proof of (i). □

Lemma 3.2. *Let $r_1, r_2 \in \mathbb{N}_0$, $L > n$ and $L_1 > n + r_1, L_2 > n + r_2$. Assume that for dyadic cubes P*

and R, ϕ_P *and* φ_R *are functions on* $\mathbb{R}^n$ *satisfying following properties:*

$$(2.1)\quad \int_{\mathbb{R}^n} \phi_P(x)x^\gamma dx = 0 \quad for \quad |\gamma| < r_1,$$
$$(2.2)\quad |\phi_P(x)| \le C(1 + l(P)^{-1}|x - x_P|)^{-\max(L,L_1)},$$
$$(2.3)\quad |\partial^\gamma \phi_P(x)| \le Cl(P)^{-|\gamma|}(1 + l(P)^{-1}|x - x_P|)^{-L}$$
$$for \quad 0 < |\gamma| \le r_2,$$
$$(2.4)\quad \int_{\mathbb{R}^n} \varphi_R(x)x^\gamma dx = 0 \quad for \quad |\gamma| < r_2,$$
$$(2.5)\quad |\varphi_R(x)| \le C(1 + l(R)^{-1}|x - x_R|)^{-\max(L,L_2)},$$
$$(2.6)\quad |\partial^\gamma \varphi_R(x)| \le Cl(R)^{-|\gamma|}(1 + l(R)^{-1}|x - x_R|)^{-L}$$
$$for \quad 0 < |\gamma| \le r_1,$$

where (2.1) *and* (2.6) *are void when* $r_1 = 0$, *and* (2.3) *and* (2.4) *are void when* $r_2 = 0$. *Then, we have that*

$$l(P)^{-n}|\langle \phi_P \ , \ \varphi_R\rangle| \le C\big(\tfrac{l(P)}{l(R)}\big)^{r_1}(1 + l(R)^{-1}|x_P - x_R|)^{-L}$$
$$if \quad l(P) \le l(R),$$
$$l(R)^{-n}|\langle \phi_P \ , \ \varphi_R\rangle| \le C\big(\tfrac{l(R)}{l(P)}\big)^{r_2}(1 + l(P)^{-1}|x_P - x_R|)^{-L}$$
$$if \quad l(R) < l(P).$$

Proof. We refer to [10: Corollary B.3] , [5: Lemma 6.3] or [29: Lemma 3.1]. □

Lemma 3.3. *Suppose that* $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n$ *and* $0 < p,\ q \ \le \infty$. *Let* $r_1, r_2 \in \mathbb{N}_0$ *and* $L > n$. *Assume that functions* ϕ_P *and* φ_P *satisfy* (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) *in Lemma* 3.2. *Let* J *as in Lemma* 3.1. *Then we have*
(i) *for a dyadic cube* R *and a sequence* $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$,
$\sum_{\mathcal{D}\ni P,\ l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle$ *is convergent if* $r_1 > J - n - s'$ *and* $L > J$,
(ii) *for a dyadic cube* R *and a sequence* $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$,
$\sum_{\mathcal{D}\ni P,\ l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle$ *is convergent if* $r_1 > J - n - s' - (\sigma \wedge 0)$ *and* $L > J + \sigma$.

Proof : (i) We may assume that $\sigma \ge 0$ by Remark 1.
We write $\sum_{\mathcal{D}\ni P} c(P)\langle \phi_P \ , \ \varphi_R\rangle = I = I_0 + I_1$ with

$$I_0 = \sum_{l(R)\le l(P)\le 1} c(P)\langle \phi_P \ , \ \varphi_R\rangle,$$
$$I_1 = \sum_{l(P)<l(R)} c(P)\langle \phi_P \ , \ \varphi_R\rangle$$

for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. We claim that $I_i < \infty, \quad i = 0, 1$.

For a dyadic cube R with $l(R) = 2^{-i}$ we have, by Lemma 3.2 that

$$\begin{aligned} |I_0| &\le C \sum_{i\ge j\ge 0}\ \sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P \ , \ \varphi_R\rangle| \\ &\le C \sum_{i\ge j\ge 0}\ \sum_{l(P)=2^{-j}} |c(P)|2^{-in}2^{(j-i)r_2}(1 + 2^j|x_R - x_P|)^{-L} \\ &\le C \sum_{i\ge j\ge 0} 2^{-i(r_2+n)}2^{jr_2}M_t\Big(\sum_{l(P)=2^{-j}} |c(P)|\chi_P\Big)(x), \end{aligned}$$

for $L > n/t$, $0 < t < 1$ and $x \in R$. Taking $L^1(R)$ norm and using the Fefferman-Stein inequality, we have,

$$\begin{aligned} |I_0|2^{-in} &= ||I_0||_{L^1(R)} \\ &\leq C2^{-in}||\sum_{i\geq j\geq 0} M_t(\sum_{l(P)=2^{-j}} |c(P)|\chi_P)||_{L^1(R)} \\ &\leq C2^{-in}||\sum_{i\geq j\geq 0}\sum_{l(P)=2^{-j}} |c(P)|\chi_P||_{L^1(R)} \\ &\leq C\sum_{1\geq l(P),\ R\subset P} |c(P)|2^{-2in} < \infty. \end{aligned}$$

In the same way we obtain the estimate of I_1:

$$\begin{aligned} |I_1| &\leq C\sum_{j\geq i\vee 0}\sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P\ ,\ \varphi_R\rangle| \\ &\leq C\sum_{j\geq i\vee 0}\sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}(1+2^i|x_R-x_P|)^{-L} \\ &\leq C\sum_{j\geq i\vee 0} 2^{-j(r_1+n)}2^{ir_1}\sum_{l(P)=2^{-j}} |c(P)|(1+2^i|x_R-x_P|)^{-L} \\ &\leq C\sum_{j\geq i\vee 0} 2^{-j(r_1+n-n/t+s')}2^{ir_1}2^{-in/t}M_t(\sum_{l(P)=2^{-j}} 2^{js'}|c(P)|\chi_P)(x) \end{aligned}$$

if $0 < t \leq 1,\ L > n/t$ and $x \in R$ with $l(R) = 2^{-i}$.

By using the monotonicity of l^q-norm and Hölder's inequality, we get the following result,

$$|I_1| \leq C2^{-i(n+s')}\{\sum_{j\geq i\vee 0}(M_t(\sum_{l(P)=2^{-j}} 2^{js'}|c(P)|\chi_P)(x))^q\}^{1/q}$$

if $r_1 + n - n/t + s' > 0,\ 0 < q \leq \infty$ and $x \in R$.

Taking $L^p(R)$ norm and using the Fefferman-Stein inequality, we have, for a dyadic cube R with $l(R) = 2^{-i}$ and $c \in a^s(f^{s'}_{pq})^\sigma_{x_0}$,

$$\begin{aligned} |I_1|2^{-in/p} &= ||I_1||_{L^p(R)} \leq C2^{-i(n+s')}c(f^{s'}_{pq})(R) \\ &\leq C2^{-i(n+s'+\sigma+s)}||c||_{a^s(f^{s'}_{pq})^\sigma_{x_0}} < \infty \end{aligned}$$

if $0 < t < \min(p,q),\ 0 < p < \infty,\ 0 < q \leq \infty$. In the same way we get the same estimate for the case $p = \infty$. Furthermore, we obtain the same estimate for the B-type case if $0 < t < p$, $0 < p \leq \infty,\ 0 < q \leq \infty$. Therefore, we obtain that I_1 is convergent if $r_1 > J - n - s'$ and $L > J$.

(ii) Let I_0 and I_1 be as in the proof of (i). Then by arguing as in the proof of (i), we have $I_0 < \infty$ for $L > n$. We put $w_j(P) = (2^{-j} + |x_P - x_0|)^{-\sigma}$ for a dyadic cube P with $l(P) = 2^{-j}$.

Note that

$$|c(P)| \leq Cl(P)^{s+s'-n/p}w_j(P)^{-1}||c||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}}$$

for $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$. We have, by Lemma 3.2 for a dyadic cube R with $l(R) = 2^{-i}$ and $\sigma \geq 0$,

$$
\begin{aligned}
|I_1| &\le C \sum_{j\ge i\vee 0} \sum_{l(P)=2^{-j}} |c(P)||\langle \phi_P\ ,\ \varphi_R\rangle| \\
&\le C \sum_{j\ge i\vee 0} \sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}(1+2^i|x_R-x_P|)^{-L} \\
&\le C \sum_{j\ge i\vee 0} \sum_{l(P)=2^{-j}} |c(P)|2^{-jn}2^{(i-j)r_1}w_j(P)w_j(P)^{-1}\times \\
&\quad (1+2^i|x_R-x_P|)^{-L} \\
&\le C \sum_{j\ge i\vee 0} 2^{-j(r_1+n)}2^{ir_1}2^{-i\sigma} \sum_{l(P)=2^{-j}} |c(P)|w_j(P)\times \\
&\quad (1+2^i|x_R-x_P|)^{-(L-\sigma)} \\
&\le C \sum_{j\ge i\vee 0} 2^{-j(r_1+n-n/t+s')}2^{i(r_1+\sigma-n/t)}\times \\
&\quad M_t(\sum_{l(P)=2^{-j}} 2^{js'}w_j(P)|c(P)|\chi_P)(x).
\end{aligned}
$$

By using the same way as in the proof of (i), we get

$$
\begin{aligned}
|I_1|2^{-ip/n} &\le C2^{-i(n+s'+\sigma)}c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(R) \\
&\le C2^{-i(n+s'+\sigma+s)}||c||_{a^s(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}} < \infty
\end{aligned}
$$

if $r_1 > J-n-s'$ and $L > \sigma + J$. We also obtain the same estimate for the case $\sigma < 0$. □

For a sequence $c(P)$ with $l(P)=2^{-j}$, we define the sequence $c^*(P)$ by

$$
c^*(P) = \sum_{l(R)=2^{-j}} |c(R)|(1+2^j|x_P-x_R|)^{-L}
$$

for $L > J$ where J is as in Lemma 3.1.

We define for $f \in \mathcal{S}'$, $\gamma \in \mathbb{N}_0$ and a dyadic cube P with $l(P)=2^{-j}$, the sequence $\inf_\gamma(f)(P)$ and $t_\gamma(P)$ by

$\inf_\gamma(f)(P) = \max\{\inf_{R\ni y}|\phi_j * f(y)| : R\subset P, l(R)=2^{-(\gamma+j)}\}$,

$$
t_\gamma(P) = \inf_{P\ni y}|\phi_{j-\gamma} * f(y)|.
$$

Lemma 3.4. *For $s',\ \sigma\in\mathbb{R},\ x_0\in\mathbb{R}^n,\ 0<p,q\le\infty,\ f\in\mathcal{S}'$ and a dyadic cube P with $l(P)=2^{-j}$, we have*

(i)

$$
c(e^{s'}_{pq})(P) \sim c^*(e^{s'}_{pq})(P), \quad c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \sim c^*(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P),
$$

(ii)

$$
\inf_\gamma(f)(P)\chi_P \le C2^{\gamma L}\textstyle\sum_{R\subset P, l(R)=2^{-(\gamma+j)}} t^*_\gamma(R)\chi_R.
$$

for γ sufficient large.

Proof. (i) It suffices to prove

$$
c^*(e^{s'}_{pq})(P) \le Cc(e^{s'}_{pq})(P)
$$

since $|c(P)| \le c^*(P)$.

Using the Fefferman-Stein inequality, we have

$$\begin{aligned} c^*(f^{s'}_{pq})(P) &= ||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}|c^*(R)|\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}\sum_{l(R')=2^{-i}}|c(R')| \times \\ &\quad (1+2^i|x_R-x_{R'}|)^{-L}\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}M_t(\sum_{l(R')=2^{-i}}|c(R')|\chi_{R'})\chi_R)^q\}^{1/q}||_{L^p(P)} \\ &\le C||\{\sum_{i\ge j\vee 0}(\sum_{l(R')=2^{-i}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P) \end{aligned}$$

if $0 < t < \min(p,q)$, $L > n/t$ and $0 < p < \infty, 0 < q \le \infty$. Moreover, for the $p = \infty$ case, we have the same result. For the B-type case , we obtain the same result by the same argument as above. We also obtain the same result for the other case.

(ii) Let R_0 and R in P be cubes with $l(R_0) = l(R) = 2^{-(\gamma+j)}$. It suffices to show

$$t_\gamma(R_0) \le C2^{\gamma L}t^*_\gamma(R).$$

Since

$$1 \le 2^L 2^{\gamma L}(1+2^{\gamma+j}|x_R - x_{R_0}|)^{-L},$$

we have

$$\begin{aligned} t_\gamma(R_0) &\le Ct_\gamma(R_0)2^{\gamma L}(1+2^{\gamma+j}|x_R - x_{R_0}|)^{-L} \\ &\le C2^{\gamma L}\sum_{l(R')=2^{-(\gamma+j)}}t_\gamma(R')(1+2^{\gamma+j}|x_R - x_{R'}|)^{-L} = C2^{\gamma L}t^*_\gamma(R). \end{aligned}$$

□

4 Characterizations

Remark 2. (See [11: (3.20)]). Let ϕ_0 be a Schwartz function satisfying (1.1) and (1.2) and let ϕ be a Schwartz function satisfying (1.3) and (1.4). Then there exist a Schwartz function φ_0 satisfying the same conditions (1.1) and (1.2) and a Schwartz function φ satisfying the same conditions (1.3) and (1.4) such that

$\sum_{j\in\mathbb{N}_0}\hat{\varphi}_j(\xi)\hat{\phi}_j(\xi) = 1$ for any ξ where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j \in \mathbb{N}$.

Hence we have the φ-transform [8; Lemma 2.1] for $f \in \mathcal{S}'$ such that

$$f = \sum_{l(Q)\le 1} l(Q)^{-n}\langle f,\ \varphi_Q\rangle\phi_Q,$$

where $\phi_Q(x) = \phi(l(Q)^{-1}(x - x_Q))$ and $\varphi_Q(x) = \varphi(l(Q)^{-1}(x - x_Q))$ for a dyadic cube Q with $l(Q) < 1$, and $\phi_Q(x) = \phi_0(l(Q)^{-1}(x - x_Q))$ and $\varphi_Q(x) = \varphi_0(l(Q)^{-1}(x - x_Q))$ for a dyadic cube Q with $l(Q) = 1$.

Theorem 4.1. *For* $s,\ s',\ \sigma \in \mathbb{R},\ 0 < p,\ q \le \infty,\ x_0 \in \mathbb{R}^n$ *and* $\phi_0,\ \phi \in \mathcal{S}$ *as in Remark* 2*, we have*

(i)

$$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\},$$

and

(ii)

$$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$$

Remark 3. (1) We see that $\sum_{l(Q)\le 1} c(Q)\phi_Q$ is convergent in $\mathcal{S}'$ for each sequence $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $c \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$ by Lemma 3.3.

(2) We notice that $D \equiv \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad c \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$ is independent of the choice of $\phi_0,\ \phi \in \mathcal{S}$ as in Remark 2. Indeed, suppose $\{\phi^1_0, \phi^1\}$ and $\{\phi^2_0, \phi^2\}$ are Schwartz functions as in Remark 2, and the spaces D^1 and D^2 are defined by using $\{\phi^1_0, \phi^1\}$ and $\{\phi^2_0, \phi^2\}$ in the place of $\{\phi_0, \phi\}$ respectively. We consider the φ-transform

$$\phi^1_P = \sum_{l(R)\le 1} l(R)^{-n} \langle \phi^1_P\ ,\ \ \varphi^2_R\rangle \phi^2_R.$$

Then for $D^1 \ni f = \sum_{l(P)\le 1} c(P)\phi^1_P,\ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, we have

$$f = \sum_{l(P)\le 1} c(P)\phi^1_P = \sum_{l(R)\le 1} Ac(R)\phi^2_R$$

where $A = \{l(R)^{-n}\langle \phi^1_P\ ,\ \ \varphi^2_R\rangle\}_{RP}$. From Lemma 3.1 and Lemma 3.2, we see that for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, $Ac \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. This shows that $D^1 \subset D^2$. By the same argument, we see that $D^2 \subset D^1$. That is, $D^1 = D^2$. These imply that the space D is independent of the choice of $\{\phi_0, \phi\}$. In the same way $\tilde{D} = \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}$ is independent of the choice of $\{\phi_0, \phi\}$.

Proof of Theorem 4.1. (i) We may assume $\sigma \ge 0$ by Remark 1. We put $D \equiv \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q :$ $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$. In order to prove $D \subset A^s(E^{s'}_{pq})^\sigma_{x_0}$ we claim for a dyadic cube P, and for $f = \sum_Q c(Q)\phi_Q \in D$,

$$c(E^{s'}_{pq})(P) \le Cc(e^{s'}_{pq})(P) \qquad \text{(a)}$$

if $0 < p, q \le \infty$. Let $(c(P)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Since $\mathcal{S}$ is closed under the convolution, we have, for $i \ge 0$,

$$\begin{aligned} |\phi_i * f(x)| &= |\sum_{l(P)\le 1} c(P)\phi_i * \phi_P(x)| \\ &= |\sum_{j=(i-1)\vee 0}^{i+1} \sum_{l(P)=2^{-j}} c(P)\phi_i * \phi_P(x)| \\ &\le C \sum_{j=(i-1)\vee 0}^{i+1} \sum_{l(P)=2^{-j}} |c(P)|(1 + 2^j|x - x_P|)^{-L} \end{aligned}$$

for a sufficiently large number L. Hence we have, using the maximal function $M_t f(x)$, $0 < t \le 1$, as in the proof of Lemma 3.1

$$
\begin{aligned}
\{\sum_{i\geq j\vee 0}(2^{is'}|\phi_i * f|)^q\}^{1/q} &\leq C\{\sum_{i\geq j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}|\phi_i * f|\chi_R)^q\}^{1/q} \\
&\leq C\{\sum_{i\geq j\vee 0}(2^{is'}\sum_{l(R)=2^{-i}}(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}|c(R')|(1+2^k|x-x_{R'}|)^{-L})\chi_R)^q\}^{1/q} \\
&\leq C\{\sum_{i\geq j\vee 0}(\sum_{l(R)=2^{-i}}M_t(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'})\chi_R)^q\}^{1/q}
\end{aligned}
$$

if $0<t\leq 1$ and $L>n/t$. Taking $L^p(P)$-norm and using the Fefferman-Stein inequality, we have for a dyadic cube P with $l(P)=2^{-j}$

$$
\begin{aligned}
c(F^{s'}_{pq})(P) &= ||\{\sum_{i\geq j\vee 0}(2^{is'}|\phi_i * f|)^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}(M_t(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'}))^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}(\sum_{k=(i-1)\vee 0}^{i+1}\sum_{l(R')=2^{-k}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)} \\
&\leq C||\{\sum_{i\geq j\vee 0}(\sum_{l(R')=2^{-i}}2^{is'}|c(R')|\chi_{R'})^q\}^{1/q}||_{L^p(P)} = Cc(f^{s'}_{pq})(P)
\end{aligned}
$$

if $0<t<\min(p,q)$ and $0<p<\infty$. For the $p=\infty$ case , we obtain the same result. In the same way for the B-type case we have the same estimate

$$c(B^{s'}_{pq})(P)\leq Cc(b^{s'}_{pq})(P) \quad \text{if} \quad 0<p\leq\infty.$$

This implies $D\subset A^s(E^{s'}_{pq})^\sigma_{x_0}$.

In order to complete the proof of Theorem 4.1 (i), we will show the inverse. We consider the φ-transform $f=\sum_{l(P)\leq 1}c(f)(P)\varphi_P$, $c(f)(P)=l(P)^{-n}\langle f\ ,\ \phi_P\rangle$ where ϕ_P and φ_P as in Remark 2. It suffices to show that $c(f)(P)\in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $f\in A^s(E^{s'}_{pq})^\sigma_{x_0}$. More precisely, we claim that for a dyadic cube P with $l(P)=2^{-j}$,

$$c(f)(e^{s'}_{pq})(P)\leq Cc(E^{s'}_{pq})(P) \tag{b}$$

where $c(f)(e^{s'}_{pq})(P)$ is a sequence defined by replacing the sequence $c(P)$ by the sequence $c(f)(P)$ in the definition of $c(e^{s'}_{pq})(P)$. For $f\in\mathcal{S}'$ and a dyadic cube P with $l(P)=2^{-j}$, we define the sequence $\sup(f)(P)$ by setting

$$\sup(f)(P)=\sup_{P\ni y}|\phi_j * f(y)|.$$

For $\gamma\in\mathbb{N}_0$ the sequences $\inf_\gamma(f)(P)$, $t_\gamma(P)$ are defined previously and for a sequence $c(P)$, we also define a sequence $c^*(P)$ previously (See Lemma 3.4). We have, from the fact in [9, Lemma A.4] that $\sup(f)^*(P)\sim\inf_\gamma(f)^*(P)$ for γ sufficiently large.

Thus, we have

$$|c(f)(P)|=l(P)^{-n}|\langle f\ ,\ \phi_P\rangle|=|\phi_j * f(x_P)|\leq\sup(f)(P)\leq\sup(f)^*(P)\sim\inf_\gamma(f)^*(P)$$

for γ sufficiently large. Therefore, from Lemma 3.4 (i) and (ii) we have

$$
\begin{aligned}
|c(f)(f^{s'}_{pq})(P)| &\le C\inf_\gamma(f)^*(f^{s'}_{pq})(P) \le C\inf_\gamma(f)(f^{s'}_{pq})(P) \\
&\le C||\{\sum_{i\ge j\vee 0}(\sum_{l(R)=2^{-i}} 2^{is'}\inf_\gamma(f)(R)\chi_R)^q\}^{1/q}||_{L^p(P)} \\
&\le C||\{\sum_{i\ge j\vee 0}(2^{is'}2^{\gamma L}\sum_{l(R')=2^{-(\gamma+i)}} t^*_\gamma(R')\chi_{R'})^q\}^{1/q}||_{L^p(P)} \\
&\le C2^{\gamma L}||\{\sum_{i\ge(j\vee 0)+\gamma}(2^{is'}2^{-\gamma s'}\sum_{l(R')=2^{-i}} t_\gamma(R')\chi_{R'})^q\}^{1/q}||_{L^p(P)} \\
&\le C2^{\gamma(L-s')}||\{\sum_{i\ge(j\vee 0)+\gamma}(2^{is'}\sum_{l(R')=2^{-i}} |\phi_{i-\gamma}*f(y)|\chi_{R'})^q\}^{1/q}||_{L^p(P)} \\
&\le C2^{\gamma(L-s')}||\{\sum_{i\ge j\vee 0}(2^{is'}2^{s'\gamma}\sum_{l(R')=2^{-(i+\gamma)}} |\phi_i*f(y)|\chi_{R'})^q\}^{1/q}||_{L^p(P)} \\
&\le C2^{\gamma L}||\{\sum_{i\ge j\vee 0}(2^{is'}|\phi_i*f(y)|)^q\}^{1/q}||_{L^p(P)} = Cc(F^{s'}_{pq})(P)
\end{aligned}
$$

if $0 < p < \infty$. For $p = \infty$, we obtain the same result. For the B-type case we can prove the same result by the same argument as above,

$$c(f)(b^{s'}_{pq})(P) \le Cc(B^{s'}_{pq})(P)$$

if $0 < p \le \infty$. Thus, we obtain

$$c(f)(e^{s'}_{pq})(P) \le Cc(E^{s'}_{pq})(P).$$

By Remark 3 (2) this implies that, $A^s(E^{s'}_{pq})^\sigma_{x_0} \subset D$, $0 < p \le \infty$. Hence, we obtain $A^s(E^{s'}_{pq})^\sigma_{x_0} = D$.

(ii) We can prove (ii) in the same way as (i). □

We have the following properties from Theorem 4.1.

Proposition 4.1. *Suppose that* s, s', $\sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *When* $\sigma < 0$, *we have* $A^s(E^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \le \infty$,

(ii) *When* $\sigma + s < 0$, *we have* $A^s(B^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \le \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p < \infty$, $0 < q \le \infty$,

(iii) When $s < 0$, *we have* $A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p, q \le \infty$, *and* $A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0} = \{0\}$, *for* $0 < p < \infty$, $0 < q \le \infty$.

Proof. These properties are shown easily. □

Proposition 4.2. *Suppose that* s, s', $\sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *When* $s \le 0$, *we have*

$A^s(B^{s'}_{pq})^\sigma_{x_0} = (B^{s'}_{pq})^{s+\sigma}_{x_0}$ *for* $0 < p, q \le \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = (F^{s'}_{pq})^{s+\sigma}_{x_0}$ *for* $0 < p < \infty$, $0 < q \le \infty$,

In particular, when $\sigma \ge 0$ *and* $\sigma + s = 0$, *we have*

$A^s(B^{s'}_{pq})^\sigma_{x_0} = B^{s'}_{pq}(\mathbb{R}^n)$ *for* $0 < p, q \le \infty$, *and* $A^s(F^{s'}_{pq})^\sigma_{x_0} = F^{s'}_{pq}(\mathbb{R}^n)$ *for* $0 < p < \infty$, $0 < q \le \infty$.

(ii) *When* $\sigma \ge 0$, *we have*

$A^s(E^{s'+\sigma}_{pq}) \subset A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \subset A^s(E^{s'}_{pq})^\sigma_{x_0}$,

and when $\sigma < 0$, *we have*

$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} \subset A^s(E^{s'+\sigma}_{pq})$.

(iii) *If $\sigma \geq 0$, then we have*

$$(E^{s'}_{\infty\infty})^{\sigma}_{x_0} = (\tilde{E}^{s'}_{\infty\infty})^{\sigma}_{x_0}.$$

Proof. The property (i) can be proved from the fact that

$$Cl(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} c(e^{s'}_{pq})(P) \geq l(Q)^{-(\sigma+s)} c(e^{s'}_{pq})(Q)$$

and

$$l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} c(e^{s'}_{pq})(P) \leq Cl(Q)^{-(\sigma+s)} \sup_{\mathcal{D}\ni P\subset 3Q} c(e^{s'}_{pq})(P),$$

if $s \leq 0$.

We obtain the property (ii) from the fact that

$$c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \leq Cc(e^{\sigma+s'}_{pq})(P),$$

and

$$l(Q)^{-\sigma} l(P)^{-s} c(e^{s'}_{pq})(P) \leq Cl(P)^{-s} c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P)$$

since $l(Q)^{-\sigma} \leq C(l(P) + |x_0 - x_P|)^{-\sigma}$ for $P \subset 3Q$ if $\sigma \geq 0$. The last half of property (ii) can be proved since

$$c(\tilde{e}^{s'}_{pq})^{\sigma}_{x_0}(P) \geq c(e^{s'+\sigma}_{pq})(P)$$

if $\sigma < 0$. To prove the property (iii), it suffices to see from property (ii),

$$(e^{s'}_{\infty\infty})^{\sigma}_{x_0} \subset (\tilde{e}^{s'}_{\infty\infty})^{\sigma}_{x_0}.$$

We consider any dyadic cube R with $l(R) = 2^{-i}$ and dyadic cubes Q_l with $x_0 \in Q_l$ and $l(Q_l) = 2^{-l}$, $i \geq l$ such that $Q_i \subset \cdots \subset Q_l \subset Q_{l-1} \subset \cdots$ and $\cup_{i\geq l} Q_l = \mathbb{R}^n$. We set $Q^0_l \equiv 3Q_l \setminus 3Q_{l+1}$, $i > l$ and $Q^0_i \equiv 3Q_i$. We divide the proof into two cases:

Case (a): $R \subset Q^0_l$, $i > l$ case. Then we have $2^{-i} + |x_0 - x_R| \geq C2^{-l}$,

Case (b): $R \subset Q^0_i$. Then we have $2^{-i} + |x_0 - x_R| \geq 2^{-i}$.

In the case (a) we have

$$\begin{aligned} 2^{is'}|c(R)|(2^{-i} + |x_0 - x_R|)^{-\sigma} &\leq C2^{is'}2^{l\sigma}|c(R)| \\ &\leq C \sup_{x_0\in Q} 2^{l\sigma} \sup_{R\subset 3Q} 2^{is'}|c(R)| < \infty. \end{aligned}$$

In the case (b) we have

$$\begin{aligned} 2^{is'}|c(R)|(2^{-i} + |x_0 - x_R|)^{-\sigma} &\leq C2^{is'}2^{i\sigma}|c(R)| \\ &\leq C \sup_{x_0\in Q} 2^{l\sigma} \sup_{R\subset 3Q} 2^{is'}|c(R)| < \infty. \end{aligned}$$

□

Proposition 4.3. *Suppose that* $s, s,' \sigma \in \mathbb{R}$, *and* $x_0 \in \mathbb{R}^n$.

When $0 < q_1 \leq q_2 \leq \infty$, $0 < p \leq \infty$, *we have*

$$A^s(B^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}, \quad A^s(\tilde{B}^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{B}^{s'}_{pq_2})^{\sigma}_{x_0},$$

and when $0 < q_1 \leq q_2 \leq \infty$, $0 < p < \infty$, *we have*

$$A^s(F^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}, \quad A^s(\tilde{F}^{s'}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{F}^{s'}_{pq_2})^{\sigma}_{x_0}.$$

Proof. These inclusions are corollaries of the monotonicity of the l^p-norm. □

Proposition 4.4. *Suppose that* $s, s,' \sigma \in \mathbb{R}$, $0 < \epsilon$ *and* $x_0 \in \mathbb{R}^n$. *We have*

(i) $A^s(B^{s'+\epsilon}_{pq_1})^{\sigma-\epsilon}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p \le \infty,\ 0 < q_1,\ q_2 \le \infty$, *and*
$A^s(F^{s'+\epsilon}_{pq_1})^{\sigma-\epsilon}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p < \infty,\ 0 < q_1,\ q_2 \le \infty$, *and*

(ii) $A^{s+\epsilon}(E^{s'}_{pq})^{\sigma-\epsilon}_{x_0} \subset A^s(E^{s'}_{pq})^{\sigma}_{x_0}$ *for* $0 < p,\ q \le \infty$, *and*

(iii) $A^{s-\epsilon}(B^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(B^{s'}_{pq_2})^{\sigma}_{x_0}$, *and* $A^{s-\epsilon}(\tilde{B}^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{B}^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p,\ q_1,\ q_2 \le \infty$, *and*
$A^{s-\epsilon}(F^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(F^{s'}_{pq_2})^{\sigma}_{x_0}$, *and* $A^{s-\epsilon}(\tilde{F}^{s'+\epsilon}_{pq_1})^{\sigma}_{x_0} \subset A^s(\tilde{F}^{s'}_{pq_2})^{\sigma}_{x_0}$ *for* $0 < p < \infty,\ 0 < q_1,\ q_2 \le \infty$.

Proof. (ii) is obvious. (i) and (iii) are corollaries of Hölder's inequality and the monotonicity of the l^p-norm. □

Proposition 4.5. *Suppose that* $s,\ s,'\ \sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

(i) *If* $0 < p_2 \le p_1 \le \infty$ *and* $0 < q \le \infty$, *then*
$A^{s+\frac{n}{p_1}}(B^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(B^{s'}_{p_2q})^{\sigma}_{x_0},\ A^{s+\frac{n}{p_1}}(\tilde{B}^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(\tilde{B}^{s'}_{p_2q})^{\sigma}_{x_0}$,
and, if $0 < p_2 \le p_1 < \infty$ *and* $0 < q \le \infty$, *then*
$A^{s+\frac{n}{p_1}}(F^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(F^{s'}_{p_2q})^{\sigma}_{x_0},\ A^{s+\frac{n}{p_1}}(\tilde{F}^{s'}_{p_1q})^{\sigma}_{x_0} \subset A^{s+\frac{n}{p_2}}(\tilde{F}^{s'}_{p_2q})^{\sigma}_{x_0}$.

(ii) *If* $0 < q \le \infty,\ 0 < p \le \infty,\ \frac{n}{p} < s$, *then*

$A^s(E^{s'}_{pq})^{\sigma}_{x_0} = (E^{s+s'-\frac{n}{p}}_{\infty\infty})^{\sigma}_{x_0}$ *and* $A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0} = (\tilde{E}^{s+s'-\frac{n}{p}}_{\infty\infty})^{\sigma}_{x_0}$.
In particular, if $0 \le \sigma,\ 0 < q \le \infty,\ 0 < p \le \infty,\ \frac{n}{p} < s$, *then*

$A^s(E^{s'}_{pq})^{\sigma}_{x_0} = A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0}$.

(iii) *If* $0 < p_1,\ p_2,\ q \le \infty$, *then*
$A^{\frac{n}{p_1}}(E^{s'}_{p_1\infty})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(E^{s'}_{p_2\infty})^{\sigma}_{x_0} = (E^{s'}_{\infty\infty})^{\sigma}_{x_0}$,
$A^{\frac{n}{p_1}}(\tilde{E}^{s'}_{p_1\infty})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(\tilde{E}^{s'}_{p_2\infty})^{\sigma}_{x_0} = (\tilde{E}^{s'}_{\infty\infty})^{\sigma}_{x_0}$,
$A^{\frac{n}{p_1}}(F^{s'}_{p_1q})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(F^{s'}_{p_2q})^{\sigma}_{x_0},\ A^{\frac{n}{p_1}}(\tilde{F}^{s'}_{p_1q})^{\sigma}_{x_0} = A^{\frac{n}{p_2}}(\tilde{F}^{s'}_{p_2q})^{\sigma}_{x_0}$.

Proof. The properties (i) are corollaries of Hölder's inequality. We will prove the properties (ii). We see that

$$a^{s+\frac{n}{p}}(e^{s'}_{pq})^{\sigma}_{x_0} \subset (e^{s'+s}_{\infty\infty})^{\sigma}_{x_0},$$

since

$$l(P)^{-(s+\frac{n}{p})}c(e^{s'}_{pq})(P) \ge l(P)^{-(s'+s)}|c(P)|.$$

Hence in order to prove (ii), it suffices to prove

$$(e^{s'+s}_{\infty\infty})^{\sigma}_{x_0} \subset a^{s+\frac{n}{p}}(e^{s'}_{pq})^{\sigma}_{x_0}.$$

Since

$$c(\dot{e}^{s'}_{pq})(P) \le C(e^{s'+s}_{\infty\infty})(P) \times l(P)^{s+\frac{n}{p}}$$

if $s > 0$ and $0 < q < \infty$, we get the desired result. Similarly, for the other case, we can prove.

The first part of properties (iii) is obtained in the same way in the proof of (ii) and the last part is just [10: Corollary 5.7]. □

Proposition 4.6. (Embedding) *Let* $s,\ s',\ \sigma \in \mathbb{R},\ 0 < p,\ q \le \infty$ *and* $x_0 \in \mathbb{R}^n$. *We have*

(i) $A^s(E^{s_1'}_{p\xi})^\sigma_{x_0} \subset A^s(E^{s_2'}_{p\eta})^\sigma_{x_0}$, $A^s(\tilde{E}^{s_1'}_{p\xi})^\sigma_{x_0} \subset A^s(\tilde{E}^{s_2'}_{p\eta})^\sigma_{x_0}$, *for* $s_1' > s_2'$ *and* $0 < \xi, \eta \leq \infty$,

(ii) $A^s(B^{s_1'}_{p_1 q})^\sigma_{x_0} \subset A^s(B^{s_2'}_{p_2 q})^\sigma_{x_0}$, $A^s(\tilde{B}^{s_1'}_{p_1 q})^\sigma_{x_0} \subset A^s(\tilde{B}^{s_2'}_{p_2 q})^\sigma_{x_0}$, *for* $s_1' - s_2' = n(\frac{1}{p_1} - \frac{1}{p_2})$ *and* $0 < p_1 \leq p_2 \leq \infty$,

$A^s(F^{s_1'}_{p_1\xi})^\sigma_{x_0} \subset A^s(F^{s_2'}_{p_2\eta})^\sigma_{x_0}$, $A^s(\tilde{F}^{s_1'}_{p_1\xi})^\sigma_{x_0} \subset A^s(\tilde{F}^{s_2'}_{p_2\eta})^\sigma_{x_0}$, *for* $s_1' - s_2' = n(\frac{1}{p_1} - \frac{1}{p_2})$ *and* $0 < p_1 < p_2 < \infty$, $0 < \xi, \eta \leq \infty$,

(iii) $A^s(B^{s'}_{pq})^\sigma_{x_0} \subset A^s(F^{s'}_{pq})^\sigma_{x_0}$, $A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0} \subset A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0}$, *for* $0 < q \leq p \leq \infty$,

$A^s(F^{s'}_{pq})^\sigma_{x_0} \subset A^s(B^{s'}_{pq})^\sigma_{x_0}$, $A^s(\tilde{F}^{s'}_{pq})^\sigma_{x_0} \subset A^s(\tilde{B}^{s'}_{pq})^\sigma_{x_0}$, *for* $0 < p \leq q \leq \infty$.

Proof. The embedding properties (i) and the first embedding of (ii) are corollaries of Hölder's inequality and the monotonicity property of the l^p-norm. For the second embedding of (ii), see [37; Proposition 2.5] (cf. [38; Theorem 2.7.1]). (iii) is a corollary of Minkowski's inequality (cf. Triebel[38: 2.3.2 Proposition 2]).

□

Remark 4. Let $0 < p, q \leq \infty$, $s, \sigma \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $s' > n(\frac{1}{p} - 1)_+$. If $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$, then f is locally integrable (and locally L^p integrable). Indeed, we consider the Littlewood-Paley decomposition

$$f = \sum_{i \geq 0} f * \phi_i.$$

It suffices to show that $\sum_{i\geq 0} f * \phi_i$ is locally integrable and locally L^p integrable. We may consider any dyadic cube P with $l(P) \geq 1$. Then we have if $1 \leq p < \infty$,

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} \\ &\leq C||\{\sum_{i\geq 0}(2^{is'}|f * \phi_i|)^q\}^{1/q}||_{L^p(P)} \leq Cc(F^{s'}_{pq})(P) < \infty \end{aligned}$$

by using Hölder inequality if $1 \leq q \leq \infty$ and the monotonicity property of the l^p-norm if $0 < q \leq 1$. In the same way we have

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} \leq C\sum_{i\geq 0}||f * \phi_i||_{L^p(P)} \\ &\leq C\{\sum_{i\geq 0}(2^{is'}||f * \phi_i||_{L^p(P)})^q\}^{1/q} \leq Cc(B^{s'}_{pq})(P) < \infty. \end{aligned}$$

If $0 < p \leq 1$, in the same way we have

$$\begin{aligned} ||\sum_{i\geq 0} f * \phi_i||_{L^p(P)} &\leq C||\sum_{i\geq 0} f * \phi_i||_{L^1(P)} \\ &\leq C\{\sum_{i\geq 0}(2^{i(s'-n(\frac{1}{p}-1))}||f * \phi_i||_{L^1(P)})^q\}^{1/q} \\ &= Cc(B^{s'-n(\frac{1}{p}-1)}_{1q})(P) \leq Cc(B^{s'}_{pq})(P) < \infty, \end{aligned}$$

where we use Proposition 4.6 in the last inequality. Similarly, by using the fact that $c(B^{s'}_{p\ p\vee q})(P) \leq c(F^{s'}_{pq})(P)$ we have the same estimate for the F-type case if $0 < p \leq 1$. Therefore, we obtain the desired result for $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$. But we note that it holds for $0 < p < \infty$ in the F-type case and for $0 < p \le \infty$ in the B-type case. We note that it holds an analogous result for $f \in A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ with the weight $w_i = (2^{-i} + |x_0 - x|)^{-\sigma}$.

We recall the definitions of smooth atoms and molecules.

Definition 6. Let $r_1,\ r_2 \in \mathbb{N}_0, L > n$. A family of functions $m = (m_Q)$ indexed by dyadic cubes Q with $l(Q) \le 1$ is called a family of $(r_1, r_2, L)-$ smooth molecules if

(3.1) $|m_Q(x)| \le C(1 + l(Q)^{-1}|x - x_Q|)^{-\max(L,L_2)}$ for some $L_2 > n + r_2$ when $l(Q) < 1$,

(3.2) $|\partial^\gamma m_Q(x)| \le Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L}$ for $0 < |\gamma| \le r_1$, when $l(Q) < 1$ and

(3.3) $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$ when $l(Q) < 1$,

where (3.2) is void when $r_1 = 0$, and (3.3) is void when $r_2 = 0$,

(3.4) $|\partial^\gamma m_Q(x)| \le Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L}, \quad |\gamma| \le r_1$ when $l(Q) = 1$,

(3.5) we do not assume the vanishing moment condition (3.3) when $l(Q) = 1$.

A family of functions $a = (a_Q)$ indexed by dyadic cubes Q with $l(Q) \le 1$ is called a family of $(r_1, r_2)-$ smooth atoms if

(3.6) $\operatorname{supp} a_Q \subset 3Q$ for each dyadic cube Q when $l(Q) \le 1$,

(3.7) $|\partial^\gamma a_Q(x)| \le Cl(Q)^{-|\gamma|}$ for $|\gamma| \le r_1$ when $l(Q) \le 1$, and

(3.8) $\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ for $|\gamma| < r_2$ when $l(Q) < 1$,

where (3.8) is void when $r_2 = 0$,

(3.9) we do not assume the vanishing moment condition (3.8) when $l(Q) = 1$.

Theorem 4.2. *Let $s,\ s',\ \sigma \in \mathbb{R}$, $0 < p,\ q \le \infty$ and $x_0 \in \mathbb{R}^n$. Let $r_1, r_2 \in \mathbb{N}_0,\ \ J$ as in Lemma* 3.1 *and $L > n$.*

(i) *We assume that r_1, r_2 and L satisfy the following condition:*

(4.1) $r_1 > \max(s',\ \sigma + s + s' - \frac{n}{p})$,

(4.2) $r_2 > J - n - s'$,

(4.3) $L > J$.

Then we have

$$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)m_Q :$$

$$(r_1, r_2, L)- \text{ smooth molecules } (m_Q), \quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$$

$$= \quad \{f = \sum_{l(Q)\le 1} c(Q)a_Q :$$

$$(r_1, r_2)- \text{ smooth atoms } (a_Q), \quad (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}.$$

(ii) *We assume that r_1, r_2 and L satisfy*

$(4.1)'$ $r_1 > \max(s' + (\sigma \vee 0),\ (\sigma \vee 0) + s + s' - \frac{n}{p})$,

$(4.2)'$ $r_2 > J - n - s' - (\sigma \wedge 0)$,

$(4.3)'$ $L > J + \sigma$

Then we have

$$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{l(Q)\le 1} c(Q)m_Q :$$

$$(r_1, r_2, L)- \text{ smooth molecules } (m_Q), \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}$$

$$= \quad \{f = \sum_{l(Q)\le 1} c(Q)a_Q :$$

$$(r_1, r_2)- \text{ smooth atoms } (a_Q), \quad (c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$$

Remark 5. From Lemma 3.3, we remark that $f = \sum_{l(Q)\le 1} c(Q)m_Q$ and $f = \sum_{l(Q)\le 1} c(Q)a_Q$ are convergent in $\mathcal{S}'$ for each $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$.

Proof of Theorem 4.2. (i) We may assume $\sigma \ge 0$ by Remark 1. We put
$A \equiv \{f = \sum_{l(Q)\le 1} c(Q)a_Q : \ (r_1, r_2)-$smooth atoms (a_Q), $(c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$,
$M \equiv \{f = \sum_{l(Q)\le 1} c(Q)m_Q : \ (r_1, r_2, L)-$smooth molecules (m_Q),
$(c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$.

Since an $(r_1, r_2)-$ atom is an $(r_1, r_2, L)-$ molecule, it is easy to see that $A \subset M$. Let $M \ni f = \sum_{l(Q)\le 1} c(Q)m_Q$ and we consider the φ-transform

$$m_Q = \sum_{l(P)\le 1} l(P)^{-n}\langle m_Q\ ,\ \varphi_P\rangle \phi_P,$$

where ϕ_P and φ_P as in Remark 2. Then we have

$$f = \sum_{l(Q)\le 1} c(Q)m_Q = \sum_{l(P)\le 1} (Ac)(P)\phi_P,$$

where $A = \{l(P)^{-n}\langle m_Q\ ,\ \varphi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that A is $(r_1,\ r_2+n,\ L)-$ almost diagonal and $Ac \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, if we put $D \equiv \{f = \sum_{l(Q)\le 1} c(Q)\phi_Q : \ \ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$, then we see that $M \subset D$. From Theorem 4.1 we see $D = A^s(E^{s'}_{pq})^\sigma_{x_0}$. Hence, we obtain $A \subset M \subset A^s(E^{s'}_{pq})^\sigma_{x_0}$.

Using the argument similar to the proof of [10: Theorem 4.1] (cf. [4: Theorem 5.9] or [5: Theorem 5.8]), for $D \ni f = \sum_{l(Q)\le 1} c(Q)\phi_Q$, $\ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$, we see that there exist a family of $(r_1, r_2)-$ atoms $\{a_Q\}$ and a sequence of coefficients $\{c'(Q)\} \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ such that $f = \sum_{l(Q)\le 1} c(Q)\phi_Q = \sum_{l(Q)\le 1} c'(Q)a_Q$. Hence, we see that $D \subset A$. Therefore, we have $A^s(\dot{E}^{s'}_{pq})^\sigma_{x_0} = M = A$. We can prove (ii) by the same way in (i). □

We recall the definition of smooth wavelets.

Definition 7. Let $r \in \mathbb{N}_0$ and $L > n$. A family of $\{\psi_0,\ \psi^{(i)}\}$ is called $(r,\ L)-$ smooth wavelets if $\{\psi_0(x-k)\ (k \in \mathbb{Z}^n),\ 2^{nj/2}\psi^{(i)}(2^j x - k)\ (i = 1, \cdots, 2^n - 1, j \in \mathbb{N}_0,\ \ k \in \mathbb{Z}^n)\}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$, and $\psi^{(i)}$ satisfies (5.1), (5.2) and (5.3), and a scaling function ψ_0 satisfies (5.4)

(5.1) $|\psi^{(i)}(x)| \le C(1+|x|)^{-\max(L,L_0)}$ for some $L_0 > n + r$,
(5.2) $|\partial^\gamma \psi^{(i)}(x)| \le C(1+|x|)^{-L}$ for $0 < |\gamma| \le r$,
(5.3) $\int_{\mathbb{R}^n} \psi^{(i)}(x)x^\gamma dx = 0$ for $|\gamma| < r$
where (5.2) and (5.3) are void when $r = 0$.
(5.4) $|\partial^\gamma \psi_0(x)| \le C(1+|x|)^{-L}$ for $\ |\gamma| \le r$,
but ψ_0 does not satisfy the vanishing moment condition (5.3). We will forget to write the index i of the wavelet, which is of no consequence.

We put $\psi_{0,k}(x) = \psi_0(x-k)$, $k \in \mathbb{Z}^n$, $\psi_Q(x) = \psi(l(Q)^{-1}(x - x_Q))$ for a dyadic cube Q with $l(Q) \le 1$.

Theorem 4.3. *Let* $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n$ *and* $0 < p,\ q \le \infty$.
(i) *For a family of* $(r, L)-$ *smooth wavelets* $\{\psi_0,\ \psi\}$ *satisfying*
(6.1) $r > \max(s',\ \sigma + s + s' - \frac{n}{p},\ J - n - s')$ *and*
(6.2) $L > J$, *where* J *as in Lemma* 3.1,
we have

$A^s(E^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k) \in a^s(e^{s'}_{pq})^\sigma_{x_0},$
$(c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\},$

where $(c_k)_{k\in\mathbb{Z}^n} \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ *means that* $(c^0(Q))_{l(Q)\le 1} \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ *such as* $c^0(Q) = c_k$ *if* $Q = Q_{0,k} = [0,1)^n + k,\ k \in \mathbb{Z}^n$ *and* $c^0(Q) = 0$ *if* $l(Q) < 1$.

(ii) *For a family of* $(r, L)-$ *smooth wavelets* $\{\psi_0,\ \psi\}$ *satisfying*
(6.1)′ $r > \max(s' + (\sigma \vee 0),\ (\sigma \vee 0) + s + s' - \frac{n}{p},\ J - n - s' - (\sigma \wedge 0))$ *and*
(6.2)′ $L > J + \sigma$
we have

$A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0} = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0},$
$(c(Q)) \in a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}\}.$

Remark 6. We see that by Lemma 3.3, $\sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k}$ and $\sum_{l(Q)\le 1} c(Q)\psi_Q$ are convergent in $\mathcal{S}'$ for $(c_k),\ (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ or $a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}$.

Proof of Theorem 4.3. (i) We may assume $\sigma \ge 0$ by Remark 1. We put $W = \{f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q : (c_k), (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}\}$.

Let $W \ni f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q$ and we consider the φ–transform

$$\psi_{0,k} = \sum_{l(P)\le 1} l(P)^{-n}\langle \psi_{0,k}\ ,\ \varphi_P\rangle \phi_P$$

$$\psi_Q = \sum_{l(P)\le 1} l(P)^{-n}\langle \psi_Q\ ,\ \varphi_P\rangle \phi_P$$

where ϕ_P and φ_P as in Remark 2. Then we have

$$f = \sum_{l(P)\le 1} (B_1 c_k)(P)\phi_P + \sum_{l(P)\le 1} (A_1 c)(P)\phi_P$$

where $B_1 = \{l(P)^{-n}\langle \psi_{0,k}\ ,\ \varphi_P\rangle\}_{Pk}$ and $A_1 = \{l(P)^{-n}\langle \psi_Q\ ,\ \varphi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that B_1 and A_1 are almost diagonal and $B_1 c_k,\ A_1 c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c_k,\ c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, by Theorem 4.1, we see that $W \subset D = A^s(E^{s'}_{pq})^\sigma_{x_0}$ where D is as in the proof of Theorem 4.1.

Conversely, let $D \ni f = \sum_{l(Q)\le 1} c(Q)\phi_Q$ and we consider the wavelet expansion

$$\phi_Q = \sum_{k\in\mathbb{Z}^n} \langle \phi_Q\ ,\ \psi_{0,k}\rangle \psi_{0,k} + \sum_{l(P)\le 1} l(P)^{-n}\langle \phi_Q\ ,\ \psi_P\rangle \psi_P.$$

Then we have

$$f = \sum_{k\in\mathbb{Z}^n} (B_2 c)(k)\psi_{0,k} + \sum_{l(Q)\le 1} (A_2 c)(Q)\phi_Q$$

where $B_2 = \{\langle \phi_Q\ ,\ \psi_{0,k}\rangle\}_{kQ}$ and $A_2 = \{l(P)^{-n}\langle \phi_Q\ ,\ \psi_P\rangle\}_{PQ}$. Lemma 3.1 and Lemma 3.2 yield that B_2 and A_2 are almost diagonal and $B_2 c\ A_2 c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ for $c \in a^s(e^{s'}_{pq})^\sigma_{x_0}$. Hence, by Theorem 4.1, we see that $A^s(\dot{E}^{s'}_{pq})^\sigma_{x_0} = D \subset W$.

We can prove (ii) by the same way in (i). Hence we obtain the result of Theorem 4.3. □

Remark 7. (1) we see that Theorem 4.3 is independent of the choice of smooth wavelets $\{\psi_0,\ \psi^{(i)}\}$ (see Remark 3 (2)).

(2) For $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$ or $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ the parings $\langle f \ , \ \psi_{0,k}\rangle$ and $\langle f \ , \ \psi_Q\rangle$ are well-defined. More explicitly, we see that for any $\{\phi_Q, \ \ \varphi_Q\}$ as in Remark 2,

$$\langle f \ , \ \psi_{0,k}\rangle = \sum_{l(P)\le 1} l(P)^{-n}\langle f \ , \ \phi_P\rangle\langle \psi_{0,k} \ , \ \varphi_P\rangle \equiv \sum_{l(P)\le 1} c(f)(P)\langle \psi_{0,k} \ , \ \varphi_P\rangle$$

and

$$\langle f \ , \ \psi_Q\rangle = \sum_{l(P)\le 1} l(P)^{-n}\langle f \ , \ \phi_P\rangle\langle \psi_Q \ , \ \varphi_P\rangle \equiv \sum_{l(P)\le 1} c(f)(P)\langle \psi_Q \ , \ \varphi_P\rangle$$

are convergent by Lemma 3.3 and (b) in the proof of Theorem 4.1. Thus, for $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$ or $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ we have a wavelet expansion $f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q$ in $\mathcal{S}'$ and its representation is unique in $\mathcal{S}'$, that is, $c_k = \langle f \ , \ \psi_{0,k}\rangle$ and $c(Q) = l(Q)^{-n}\langle f \ , \ \psi_Q\rangle$. Hence, we have that by Lemma 3.1, Lemma 3.2 and (b) in the proof of Theorem 4.1,

$$\begin{aligned}
||(c_k)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} &= ||\langle f \ , \ \psi_{0,k}\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ ||\sum_{l(P)\le 1} c(f)(P)\langle \psi_{0,k} \ , \ \varphi_P\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ C||c(f)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}
\end{aligned}$$

and

$$\begin{aligned}
||(c(Q))||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} &= ||l(Q)^{-n}\langle f \ , \ \psi_Q\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ C||\sum_{l(P)\le 1} c(f)(P)l(Q)^{-n}\langle \psi_Q \ , \ \varphi_P\rangle||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ C||c(f)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}.
\end{aligned}$$

Conversely, we consider the φ-transform

$$\psi_{0,k} = \sum_P l(P)^{-n}\langle \psi_{0,k} \ , \ \varphi_P\rangle\phi_P$$

and

$$\psi_Q = \sum_P l(P)^{-n}\langle \psi_Q \ , \ \varphi_P\rangle\phi_P.$$

Then we have

$$f = \sum_{k\in\mathbb{Z}^n} c_k\psi_{0,k} + \sum_Q c(Q)\psi_Q = \sum_{k\in\mathbb{Z}^n} (Bc_k)(P)\phi_P + \sum_Q Ac(P)\phi_P$$

where $B = \{l(P)^{-n}\langle \psi_{0,k} \ , \ \varphi_P\rangle\}$ and $A = \{l(P)^{-n}\langle \psi_Q \ , \ \varphi_P\rangle\}$. Hence we have by Lemma 3.1, Lemma 3.2 and (a) in the proof of Theorem 4.1,

$$\begin{aligned}
||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} &\le C||(Bc_k) + (Ac)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ C||Bc_k||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + C||Ac||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\
&\le \ C||c_k||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + C||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}.
\end{aligned}$$

Therefore, we have

$$||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \sim ||(c_k)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + ||(c(Q))||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}.$$

Similarly, we also obtain

$$||f||_{A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}} \sim ||(c_k)||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}} + ||(c(Q))||_{a^s(\tilde{e}^{s'}_{pq})^\sigma_{x_0}}.$$

5 Applications

Definition 8. Let $\mathcal{T}$ be the space of Schwartz test functions (C^∞-functions with compact support) and $\mathcal{T}'$ its dual. For arbitrary $r_1,\ r_2 \in \mathbb{N}_0$ the Calderón–Zygmund operator T with an exponent $\epsilon > 0$ is a continuous linear operator $\mathcal{T} \to \mathcal{T}'$ such that its kernel K off the diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ satisfies

(7.1) $|\partial_1^\gamma K(x,y)| \le C|x-y|^{-(n+|\gamma|)}$ for $|\gamma| \le r_1$,

(7.2) $|K(x,\ y) - K(x,\ y')| \le C|y-y'|^{r_2+\epsilon}|x-y|^{-(n+r_2+\epsilon)}$ if $2|y'-y| \le |x-y|$,

(7.3) $|\partial_1^\gamma K(x,\ y) - \partial_1^\gamma K(x,\ y')| \le C|y-y'|^{\epsilon}|x-y|^{-(n+|\gamma|+\epsilon)}$
if $2|y'-y| \le |x-y|$ for $0 < |\gamma| \le r_1$
(where this statement is void when $r_1 = 0$),
$|\partial_1^\gamma K(x,\ y) - \partial_1^\gamma K(x',\ y)| \le C|x'-x|^{\epsilon}|x-y|^{-(n+|\gamma|+\epsilon)}$
if $2|x'-x| \le |x-y|$ for $|\gamma| \le r_1$,

(where the subindex 1 stands for derivatives in the first variable)

(7.4) T is bounded on $L^2(\mathbb{R}^n)$.

We obtain the following theorem.

Theorem 5.1. *Let $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n,\ 0 < p,\ q \le \infty,\ r_1,\ r_2 \in \mathbb{N}_0$ and J as in Lemma* 3.1.

(i) *The Calderón–Zygmund operator T with an exponent $\epsilon > J - n$ satisfying $T(x^\gamma) = 0$ for $|\gamma| \le r_1$ and $T^*(x^\gamma) = 0$ for $|\gamma| < r_2$, is bounded on $A^s(E^{s'}_{pq})^\sigma_{x_0}$ if r_1 and r_2 satisfy* (4.1) *and* (4.2) *as in Theorem* 4.2 *respectively.*

(ii) *The Calderón–Zygmund operator T with an exponent $\epsilon > J - n + \sigma$ satisfying $T(x^\gamma) = 0$ for $|\gamma| \le r_1$ and $T^*(x^\gamma) = 0$ for $|\gamma| < r_2$, is bounded on $A^s(\tilde{E}^{s'}_{pq})^\sigma_{x_0}$ if r_1 and r_2 satisfy* (4.1)' *and* (4.2)' *as in Theorem* 4.2 *respectively.*

Proof. The proof is similar to ones of [12].

(i) We may assume $\sigma \ge 0$ by Remark 1. Let $f \in A^s(E^{s'}_{pq})^\sigma_{x_0}$. Then we consider a wavelet expansion $f = \sum_k c_k\psi_{0,k} + \sum_{l(Q)\le 1} c(Q)\psi_Q :\ (c_k),\ (c(Q)) \in a^s(e^{s'}_{pq})^\sigma_{x_0}$ from Theorem 4.3. We may suppose that smooth wavelets $\{\psi_0\ \psi\}$ are compactly supported by Remark 7 (1). Then there exists a positive constant c such that supp $\psi_{0,k} \subset cQ_{0,k}$ where $Q_{0,k} = [0,1)^n + k$ and supp $\psi_Q \subset cQ$ for every dyadic cube Q with $l(Q) = 2^{-l} \le 1$.

We claim that $Tf = \sum_k c_k(T\psi_{0,k}) + \sum_{l(Q)\le 1} c(Q)(T\psi_Q) \equiv \sum_k c_k m_k + \sum_{l(Q)\le 1} c(Q)m_Q$ is convergent in $\mathcal{S}'$ and $||Tf||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} \le C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}$.

More precisely , we will show that m_k and m_Q satisfy following properties:

(8.1) $|m_k(x)| \le C(1 + l(Q)^{-1}|x - x_k|)^{-L}$ with $L > J$,

(8.2) $|m_Q(x)| \le C(1 + l(Q)^{-1}|x - x_Q|)^{-(n+r_2+\epsilon)}$,

(8.3) $|\partial^\gamma m_Q(x)| \le Cl(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-(n+\epsilon)}$ for $0 < |\gamma| \le r_1$, and

(8.4) $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$.

From the assumption $T^*x^\gamma = 0$ for $|\gamma| < r_2$ we have $\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$, that is, (8.4) holds.

We choose a suitable large constant C_0. From Fraizer–Torres–Weiss [12: Corollary 2.14], when $|x - x_Q| < 2C_0 2^{-l}$, we have

$$|m_k(x)| \le ||m_k||_\infty \le C\sum_{|\beta|\le 1} ||\partial^\beta\psi_{0,k}||_\infty \le C \le C(1 + |x - x_k|)^{-L}$$

and

$$|\partial^\gamma m_Q(x)| \le ||\partial^\gamma m_Q||_\infty \le C\sum_{|\alpha|\le|\gamma|+1} 2^{l(|\gamma|-|\alpha|)}2^{l|\alpha|}||\partial^\alpha\psi_Q||_\infty$$

$$\leq C2^{l|\gamma|} \leq Cl(Q)^{-|\gamma|}(1+l(Q)^{-1}|x-x_Q|)^{-L}$$

for any $L \geq 0$ and $|\gamma| \leq r_1$. When $|x-x_Q| \geq 2C_0 2^{-l}$, using (7.1) and (7.2) in Definition 8, we obtain

$$\begin{aligned} |m_k(x)| = |\int_{\mathbb{R}^n} K(x,y)\psi_{0,k}(y)dy| &\leq C\int_{\mathbb{R}^n} |K(x,y)||\psi_{0,k}(y)|dy \\ \leq \ C\int_{|y-x_k|\leq C_0} |x-y|^{-n}(1+|y-x_k|)^{-L}dy &\leq C(1+|x-x_k|)^{-(L+n)}. \end{aligned}$$

Moreover, using (7.3) in Definition 8 for $0 < |\gamma| \leq r_1$, we have

$$\begin{aligned} |\partial^\gamma m_Q(x)| &\leq C\int_{|y-x_Q|\leq C_0 2^{-l}} |\partial_1^\gamma K(x,y) - \partial_1^\gamma K(x,x_Q)||\psi_Q(y)|dy \\ &\leq \ C\int_{|y-x_Q|\leq C_0 2^{-l}} |y-x_Q|^\epsilon |x-x_Q|^{-(n+|\gamma|+\epsilon)}dy \\ &\leq \ C2^{-l(n+\epsilon)}|x-x_Q|^{-(n+|\gamma|+\epsilon)} \leq C2^{l|\gamma|}(1+2^l|x-x_Q|)^{-(n+\epsilon)}. \end{aligned}$$

Therefore, we obtain (8.1), (8,2) and (8.3). Hence by Lemma 3.3, $Tf = \sum_k c_k m_k + \sum_Q c(Q)m_Q$ is convergent in $\mathcal{S}'$ from (8.1), (8.2), (8.3) and (8.4). For the wavelet expansion

$$m_k = \sum_k \langle m_k\ ,\ \psi_{0,k}\rangle\psi_{0,k} + \sum_P l(P)^{-n}\langle m_k\ ,\ \psi_P\rangle\psi_P,$$

$$m_Q = \sum_k \langle m_Q\ ,\ \psi_{0,k}\rangle\psi_{0,k} + \sum_P l(P)^{-n}\langle m_Q\ ,\ \psi_P\rangle\psi_P,$$

we have

$$\begin{aligned} Tf = \sum_k c_k m_k + \sum_{l(Q)\leq 1} c(Q)m_Q = \\ \sum_k ((B_1 c_k) + (B_2 c_k))\psi_{0,k} + \sum_{l(P)\leq 1} ((A_1 c) + (A_2 c))(P)\psi_P \end{aligned}$$

where $B_1 = \{\langle m_k\ ,\ \psi_{0,k'}\rangle\}_{k'k},\ B_2 = \{\langle m_Q\ ,\ \psi_{0,k'}\rangle\}_{k'Q}$,
$A_1 = \{l(P)^{-n}\langle m_k\ ,\ \psi_P\rangle\}_{Pk},\ A_2 = \{l(P)^{-n}\langle m_Q\ ,\ \psi_P\rangle\}_{PQ}$. By Lemma 3.1, Lemma 3.2, (8.1), (8,2), (8,3) and (8.4) the operators $B_1,\ B_2,\ A_1,\ A_2$ are bounded on $a^s(e^{s'}_{pq})^\sigma_{x_0}$ if r_1 and r_2 satisfy (4.1) and (4.2) respectively. By Remark 7 (2), it follows that

$$\begin{aligned} ||Tf||_{A^s(E^{s'}_{pq})^\sigma_{x_0}} &\sim ||(B_1 c_k + B_2 c_k)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + ||(A_1 c + A_2 c)||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} \\ &\leq C(||c_k||_{a^s(e^{s'}_{pq})^\sigma_{x_0}} + ||c||_{a^s(e^{s'}_{pq})^\sigma_{x_0}}) \sim C||f||_{A^s(E^{s'}_{pq})^\sigma_{x_0}}. \end{aligned}$$

Similarly, we obtain (ii). □

Definition 9. Let $\mu \in \mathbb{R}$. A smooth function a defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to the class $S^\mu_{1,1}(\mathbb{R}^n)$ if a satisfies the following differential inequalities that for all $\alpha,\ \beta \in \mathbb{N}^n_0$,

$$\sup_{x,\xi}(1+|\xi|)^{-\mu-|\alpha|+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\ \xi)| < \infty.$$

$a(x,D)$ is the corresponding pseudo-differential operator such that

$$a(x,D)f(x) = \int_{\mathbb{R}^n} e^{ix\xi}a(x,\xi)\hat{f}(\xi)\ d\xi$$

for $f \in \mathcal{S}$.

Theorem 5.2. *Let $s,\ s',\ \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^n,\ 0 < p,\ q \le \infty$. Let $\mu \in \mathbb{R}$, J as in Lemma 3.1 and $a \in S^{\mu}_{1,1}(\mathbb{R}^n)$.*

(i) $a(x,D)$ is a continuous linear mapping from $A^s(E^{s'}_{pq})^{\sigma}_{x_0}$ to $A^s(E^{s'-\mu}_{pq})^{\sigma}_{x_0}$ if $s' > J-n+\mu$ or $a(x,\xi) = a(\xi)$.

(ii) $a(x,D)$ is a continuous linear mapping from $A^s(\tilde{E}^{s'}_{pq})^{\sigma}_{x_0}$ to $A^s(\tilde{E}^{s'-\mu}_{pq})^{\sigma}_{x_0}$ if $s' > J-n+\mu+\sigma\wedge 0$ or $a(x,\xi) = a(\xi)$.

Proof. (i) We may assume $\sigma \ge 0$ by Remark 1. We write $T \equiv a(x,D)$. Let $f \in A^s(E^{s'}_{pq})^{\sigma}_{x_0}$. By Theorem 4.1, we consider the φ-transform $f = \sum_P c(P)\phi_P$ where $c(P) = c(f)(P) = l(P)^{-n}\langle f,\ \varphi_P\rangle$ and $\phi_P,\ \varphi_P$ as in Remark 2. Then we see $(c(P)) \in a^s(e^{s'}_{pq})^{\sigma}_{x_0}$. We write that $Tf = \sum_P c(P)m_P$ where $m_P = T\phi_P$. We see for a dyadic cube P with $l(P) = 2^{-j}$

$$m_P = \int e^{ix\xi} a(x,\xi)\hat{\phi}_P(\xi)\ d\xi.$$

Then we have, using a change of variables,

$$m_P(x) = \int e^{i(x-x_P)(2^j\xi)} a(x, 2^j\xi)\hat{\phi}(\xi)\ d\xi.$$

By the fact that $(1-\triangle_\xi)^L(e^{ix\xi}) = (1+|x|^2)^L e^{ix\xi}$ for the Laplacian $\triangle$ and using an integration by parts, we obtain for $\gamma \in \mathbb{N}_0^n$ and $l(P) < 1$,

$$\begin{aligned}
&\partial_x^\gamma m_P(x)\\
&= \int (1-\triangle_\xi)^L (e^{i2^j(x-x_P)\xi})(1+(2^j|x-x_P|)^2)^{-L} \times \\
&\quad \sum_{\delta\le\gamma} (2^j i\xi)^\delta \partial_x^{\gamma-\delta} a(x,2^j\xi)\hat{\phi}(\xi)\ d\xi\\
&= C(1+(2^j|x-x_P|)^2)^{-L} \int e^{i2^j(x-x_P)\xi}(1-\triangle_\xi)^L \times \\
&\quad \sum_{\delta\le\gamma} (2^j i\xi)^\delta \partial_x^{\gamma-\delta} a(x,2^j\xi)\hat{\phi}(\xi)\ d\xi.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|\partial_x^\gamma m_P(x)|\\
&\le C(1+2^j|x-x_P|)^{-2L} \int \sum_{\delta\le\gamma}\ \sum_{|\alpha+\beta+\tau|\le 2L,\alpha\le\delta} \times \\
&\quad 2^{j|\delta|}2^{j|\beta|}|\partial_\xi^\alpha(\xi)^\delta||\partial_\xi^\beta \partial_x^{\gamma-\delta} a(x,2^j\xi)||\partial_\xi^\tau \hat{\phi}(\xi)|\ d\xi\\
&\le C(1+2^j|x-x_P|)^{-2L} \int \sum_{\delta\le\gamma}\ \sum_{|\alpha+\beta+\tau|\le 2L,\alpha\le\delta} \times \\
&\quad 2^{j|\delta|}2^{j|\beta|}|\xi|^{|\delta|-|\alpha|}(1+2^j|\xi|)^{\mu+|\gamma|-|\delta|-|\beta|}|\partial_\xi^\tau \hat{\phi}(\xi)|\ d\xi\\
&\le C2^{j\mu}2^{j|\gamma|}(1+2^j|x-x_P|)^{-2L}
\end{aligned}$$

and similarly, for P with $l(P)=1$,

$$
\begin{aligned}
&|\partial_x^\gamma m_P(x)| \\
&\le C(1+|x-x_P|)^{-2L} \times \\
&\qquad \int \sum_{|\alpha+\beta+\tau|\le 2L} (1+|\xi|)^{\mu+|\gamma|-|\alpha|-|\beta|} |\partial_\xi^\tau \hat{\phi}_0(\xi)|\, d\xi \\
&\le C(1+|x-x_P|)^{-2L}.
\end{aligned}
$$

Hence, $m_P(x)$ satisfies

$$|2^{-j\mu}\partial^\gamma m_P(x)| \le C2^{j|\gamma|}(1+2^j|x-x_P|)^{-2L}$$

for P with $l(P)\le 1$, any $\gamma\in\mathbb{N}_0$ and any $L\ge 0$. We choose a suitable large L. For the $\varphi-$ transform

$$2^{-j\mu}m_P = \sum_{l(R)\le 1} l(R)^{-n}\langle 2^{-j\mu}m_P\ ,\ \varphi_R\rangle\phi_R,$$

we have

$$Tf = \sum_{l(P)\le 1} 2^{j\mu}c(P)(2^{-j\mu}m_P) = \sum_{l(R)\le 1} A(2^{j\mu}c)(R)\phi_R,$$

where $A=\{l(R)^{-n}\langle 2^{-j\mu}m_P\ ,\ \varphi_R\rangle\}_{RP}$. From Lemma 3.1 and Lemma 3.2, A is bounded on $a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma$ if $s'>J-n+\mu$ or $a(x,\xi)=a(\xi)$. We remark that in the case $s'>J-n+\mu$, we do not assume the vanishing moment condition for m_P. But in the case $a(x,\xi)=a(\xi)$, we have the vanishing moment condition for m_P, indeed, for any P with $l(P)<1$, $\int x^\gamma m_P(x)\,dx = C\partial^\gamma\hat{m}_P(0) = C\partial^\gamma(\hat{\phi}_P)\cdot a)(0)=0$ for any $\gamma\in\mathbb{N}_0$. From (a) and (b) in the proof of Theorem 4.1, it follows that

$$
\begin{aligned}
&||Tf||_{A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma} \le C||A(2^{j\mu}c)||_{a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma} \\
&\le\ C||2^{j\mu}c||_{a^s(e_{pq}^{s'-\mu})_{x_0}^\sigma} \le C||c||_{a^s(e_{pq}^{s'})_{x_0}^\sigma} \le C||f||_{A^s(E_{pq}^{s'})_{x_0}^\sigma}.
\end{aligned}
$$

(ii) Similarly, we can prove for this case. □

Corollary . *Let $s,\ s',\ \sigma\in\mathbb{R}$, $x_0\in\mathbb{R}^n$, $0<p,\ q\le\infty$.*

(i) *Let $\mu\in\mathbb{R}$. Then the Bessel potential $(1-\triangle)^{\mu/2}$ is a continuous isomorphisms from $A^s(E_{pq}^{s'})_{x_0}^\sigma$ onto $A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma$, and from $A^s(\tilde{E}_{pq}^{s'})_{x_0}^\sigma$ onto $A^s(\tilde{E}_{pq}^{s'-\mu})_{x_0}^\sigma$.*

(ii) *Let $\gamma\in\mathbb{N}_0^n$. Then the differential operator ∂^γ is continuous from $A^s(E_{pq}^{s'})_{x_0}^\sigma$ to $A^s(E_{pq}^{s'-|\gamma|})_{x_0}^\sigma$, and from $A^s(\tilde{E}_{pq}^{s'})_{x_0}^\sigma$ to $A^s(\tilde{E}_{pq}^{s'-|\gamma|})_{x_0}^\sigma$.*

Proof. These are immediate corollaries of Theorem 5.2. To finish the proof of (i) we need to show the mapping is surjective and one to one. For $h\in A^s(E_{pq}^{s'-\mu})_{x_0}^\sigma$, we set $f=(1-\triangle)^{-\mu/2}h$. Then $h=(1-\triangle)^{\mu/2}f$. □

6 Characterizations via differences and oscillations

Definition 10. Let $k\in\mathbb{N}_0$. We define the differences of functions

$\triangle_u^1 f(x) = f(x+u)-f(x)$ and $\triangle^{k+1}=\triangle^1\triangle^k$.

We set

$d_i^k f(y) = \frac{1}{|B_i(y)|} \int_{k|u|\leq 2^{-i}} |\triangle_u^k f(y)|\ du$

where $B_i(x)$ is the ball with a center x and a radius 2^{-i}, and $|B_i(x)|$ means its volume. It is obvious that $|d_i^k f(y)| \leq C \sup_{k|u|\leq 2^{-i}} |\triangle_u^k f(y)|$.

We define the oscillation of locally L^p integrable functions f $(0 < p \leq \infty)$ by

$$\mathrm{osc}_p^k f(x,i) = \inf(\frac{1}{|B_i(x)|} \int_{B_i(x)} |f(y) - P(y)|^p\ dy)^{1/p}$$

with the suitable modification for $p = \infty$, where the infimum is taken over all polynomials $P(x) \in \mathcal{P}_k$, the space of all polynomials with deg $\leq k$ on $\mathbb{R}^n$. By $P_B f$ for a ball B we denote the unique polynomial in $\mathcal{P}_k$ such that $\int_B (f(x) - P_B f(x)) x^\alpha\ dx = 0$ for all $|\alpha| \leq k$. We see that $||P_B f||_{L^\infty(B)} \leq \frac{1}{|B|} \int_B |f(x)|\ dx$ and $P_B f = f$ for $f \in \mathcal{P}_k$. We put

$$\Omega_p^k f(x,i) = (\frac{1}{|B_i(x)|} \int_{B_i(x)} |f(y) - P_{B_i(x)} f(y)|^p\ dy)^{1/p}.$$

Then we see $\mathrm{osc}_p^k f(x,i) \sim \Omega_p^k f(x,i)$ if $1 \leq p \leq \infty$ (cf. [19]).

Lemma 6.1. (i) *Let $s \in \mathbb{R}$, $\sigma \geq 0$ and let $k \in \mathbb{N}$, $k > s' > 0$, $1 \leq p \leq \infty$, $0 < q \leq \infty$ and let f be locally L^p integrable.*

Then we have

$$\begin{aligned} &\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\ &\quad (\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\ &\leq C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} \times \\ &\quad (\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} ||\mathrm{osc}_p^{k-1} f(x,i)||_{L^p(P)})^q)^{1/q}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times \\ &\quad (\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} ||\mathrm{osc}_p^{k-1} f(x,i)||_{L^p(P)})^q)^{1/q} \\ &\leq C \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s} \times \\ &\quad (||f||_{L^p(P)} + (\sum_{i\geq(-\log_2 l(P))\vee 0} (2^{is'} ||d_i^k f||_{L^p(P)})^q)^{1/q}). \end{aligned}$$

(ii) *Let $s \in \mathbb{R}$, $\sigma \geq 0$ and let $k \in \mathbb{N}$, $k > s' > 0$, $1 \leq p < \infty$, $0 < q \leq \infty$ and let f be locally L^p integrable . Then we have*

$$\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times$$
$$||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}|\triangle_u^k f|)^q)^{1/q}||_{L^p(P)}$$
$$\leq C\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times$$
$$||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\mathrm{osc}_p^{k-1}f(x,i))^q)^{1/q}||_{L^p(P)},$$

and

$$\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times$$
$$||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\mathrm{osc}_p^{k-1}f(x,i))^q)^{1/q}||_{L^p(P)}$$
$$\leq C\sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}\times$$
$$(||f||_{L^p(P)}+||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}d_i^k f)^q)^{1/q})||_{L^p(P)}.$$

Proof. We will see that for $k|u|\leq 2^{-i}$,

$|\triangle_u^k f(x)|\leq C(\sum_{e=0}^k|f(x+eu)-P_{B_i(x+eu)}f(x+eu)|)$
$\leq C\sum_{e=0}^k\sum_{l\geq i}\Omega_p^{k-1}f(x+eu,l)$.

We consider a sequence for $i<\cdots<m\to\infty$,

$$B_i(x+eu)\supset\cdots\supset B_m(x+eu)\supset\cdots\to x+eu.$$

Then we have

$$\frac{1}{|B_m|}\int_{B_m}|f-P_{B_i}f|\,dy\leq\frac{1}{|B_m|}\int_{B_m}|f-P_{B_m}f|\,dy$$
$$+\ \frac{1}{|B_m|}\sum_{l=i+1}^m\int_{B_m}|P_{B_l}f-P_{B_{l-1}}f|\,dy$$
$$\leq\ \frac{1}{|B_m|}\int_{B_m}|f-P_{B_m}f|\,dy+C\sum_{l=i+1}^m\frac{1}{|B_l|}\int_{B_l}|f-P_{B_{l-1}}f|\,dy$$
$$\leq\ \frac{1}{|B_m|}\int_{B_m}|f-P_{B_m}f|\,dy+C\sum_{l=i}^m\frac{1}{|B_l|}\int_{B_l}|f-P_{B_l}f|\,dy.$$

Hence, we have

$$|f(x+eu)-P_{B_i}(x+eu)|=\lim_{m\to\infty}\frac{1}{|B_m|}\int_{B_m}|f-P_{B_i}f|\,dy$$
$$\leq\ \lim_{m\to\infty}\frac{1}{|B_m|}\int_{B_m}|f-P_{B_m}f|\,dy+C\sum_{l=i}^\infty\frac{1}{|B_l|}\int_{B_l}|f-P_{B_l}f|\,dy$$
$$\leq\ C\sum_{l=i}^\infty\Omega_p^{k-1}f(x+eu,l).$$

Therefore, we have for a dyadic cube P with $l(P) = 2^{-j}$,

$$
\begin{aligned}
&(\sum_{i\geq j\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}\sum_{l\geq i}||\Omega_p^{k-1}f(x,l)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}||\Omega_p^{k-1}f(x,i)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(\sum_{i\geq j\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1}f(x,i)||_{L^p(3P)})^q)^{1/q}\\
&\leq C(|x_0-x_P|+2^{-j})^{\sigma}2^{-js}\sup_{\mathcal{D}\ni Q\ni x_0}l(Q)^{-\sigma}\sup_{\mathcal{D}\ni P\subset 3Q}l(P)^{-s}\times\\
&\quad(\sum_{i\geq j\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1}f(x,i)||_{L^p(P)})^q)^{1/q}
\end{aligned}
$$

by using Hardy's inequality if $s' > 0$. This completes the proof of the first half of (i).

Next, we will prove the last half of (i).

We consider a function $\theta \in \mathcal{S}$ such that supp $\theta \subset \{k|u| \leq 1\}$ and $\int \theta(u)\, du = 1$. We put

$$
h_i(x) = \int (f(x) - \triangle_u^k f(x))\theta_i(u)\, du
$$

where $\theta_i(u) = 2^{ni}\theta(2^i u)$. We claim that

$$
\mathrm{osc}_p^{k-1} f(x,i) \leq C(\tfrac{1}{|B_i(x)|}\int_{B_i(x)} |d_i^k f(y)|^p\, dy)^{1/p} + C\mathrm{osc}_p^{k-1}h_i(x,\ i).
$$

We see that

$$
\begin{aligned}
&\mathrm{osc}_p^{k-1}f(x,i)\sim\Omega_p^{k-1}f(x,i)=\\
&\quad(\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-P_{B_i(x)}f(y)|^p\,dy)^{1/p}\\
&\leq (\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-h_i(y)|^p\,dy)^{1/p}\\
&+ (\frac{1}{|B_i(x)|}\int_{B_i(x)}|h_i(y)-P_{B_i(x)}h_i(y)|^p\,dy)^{1/p}\\
&+ (\frac{1}{|B_i(x)|}\int_{B_i(x)}|P_{B_i(x)}h_i(y)-P_{B_i(x)}f(y)|^p\,dy)^{1/p}\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|f(y)-h_i(y)|^p\,dy)^{1/p}\\
&+ C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|h_i(y)-P_{B_i(x)}h_i(y)|^p\,dy)^{1/p}\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}(\int_{k|u|\leq 2^{-i}}|\triangle_u^k f(y)||\theta_i(u)|\,du)^p\,dy)^{1/p}\\
&\quad+C\Omega_p^{k-1}h_i(x,i)\\
&\leq C(\frac{1}{|B_i(x)|}\int_{B_i(x)}|d_i^k f(y)|^p\,dy)^{1/p}+C\mathrm{osc}_p^{k-1}h_i(x,i).
\end{aligned}
$$

Next, we will estimate $\mathrm{osc}_p^{k-1} h_i(x,i)$. We consider the $(k-1)$th Taylor polynomial $q(x)$ of h_i at x. Then we have

$$\begin{aligned}
&h_i(y) - q(y) \\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!} \partial^\beta h_i(x+t(y-x))(x-y)^\beta (1-t)^{k-1}\, dt \\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!} \int \sum_{m=1}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) (-1)^{k-m} \partial^\beta f(x+t(y-x)+mu) \times \\
&\quad \theta_l(u)\, du (x-y)^\beta (1-t)^{k-1}\, dt \\
&= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!} \int \sum_{m=1}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) (-1)^{k-m} m^k f(x+t(y-x)+m2^{-i}u) \times \\
&\quad \partial^\beta \theta(u)\, du (x-y)^\beta (1-t)^{k-1}\, dt.
\end{aligned}$$

Hence, we see by using Minkowski's inequality

$$\begin{aligned}
&||\mathrm{osc}_p^{k-1} h_i(x,i)||_{L^p(P)} \le ||(\frac{1}{|B_i(x)|} \int_{B_i(x)} |h_i(y) - q(y)|^p\, dy)^{1/p}||_{L^p(P)} \\
&\le C(\int_P \frac{1}{|B_i(x)|} \int_{B_i(x)} (\int_0^1 \int_{k|u|\le 1} \sum_{m=1}^{k} |f(x+t(y-x)+m2^{-i}u)| \times \\
&\quad |\partial^\beta \theta(u)| du |x-y|^k (1-t)^{k-1}\, dt)^p\, dydx)^{1/p} \\
&\le C \int_0^1 \int_{k|u|\le 1} \sum_{m=1}^{k} (\frac{1}{|B_i(x)|} \int_{B_i(0)} \int_P |f(x+ty+m2^{-i}u)|^p\, dxdy)^{1/p} \times \\
&\quad 2^{-ik}(1-t)^{k-1}\, dudt \\
&\le C \int_0^1 \int_{k|u|\le 1} \sum_{m=1}^{k} (\frac{1}{|B_i(x)|} \int_{B_i(0)} \int_{P+ty+m2^{-i}u} |f(x)|^p\, dxdy)^{1/p} \times \\
&\quad 2^{-ik}(1-t)^{k-1}\, dudt \\
&\le C 2^{-ik} (\int_{5P} |f(x)|^p\, dx)^{1/p} \le C 2^{-ik} ||f||_{L^p(5P)}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&||(\frac{1}{|B_i(x)|} \int_{B_i(x)} |d_i^k f(y)|^p\, dy)^{1/p}||_{L^p(P)} \\
&\le C(\int_P (\frac{1}{|B_i(x)|} \int_{B_i(0)} |d_i^k f(x+y)|^p\, dydx)^{1/p} \\
&\le C(\frac{1}{|B_i(x)|} \int_{B_i(0)} \int_{P+y} |d_i^k f(x)|^p\, dxdy)^{1/p} \\
&\le C(\int_{3P} |d_i^k f(x)|^p\, dx)^{1/p} \le C||d_i^k f||_{L^p(3P)}.
\end{aligned}$$

Thus, we have for a dyadic cube P with $l(P) = 2^{-j}$

$$(\sum_{i \ge j \vee 0} (2^{is'} ||\mathrm{osc}_p^{k-1} f(x,i)||_{L^p(P)})^q)^{1/q}$$

$$\leq C(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(3P)})^q)^{1/q} + C(\sum_{i\geq j\vee 0} 2^{-i(k-s')q})^{1/q}||f||_{L^p(5P)}$$

$$\leq C(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(3P)})^q)^{1/q} + C||f||_{L^p(5P)}$$

$$\leq C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}(\sum_{i\geq j\vee 0}(2^{is'}||d_i^k f||_{L^p(P)})^q)^{1/q}$$

$$+C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{\mathcal{D}\ni Q\ni x_0} l(Q)^{-\sigma} \sup_{\mathcal{D}\ni P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}$$

if $k > s'$. The proof of (i) is complete. In the same way we can prove (ii). □

Theorem 6.1. (i) *Let $s',\ s,\ \sigma \in \mathbb{R}$ with $0 < s',\ \ 0 \leq \sigma$, and let $x_0 \in \mathbb{R}^n$, $1 \leq p \leq \infty,\ 0 < q \leq \infty$. Let $k \in \mathbb{N}$ with $k > s' > 0$. Then we have following equivalences for $f \in \mathcal{S}'$*

$$\begin{aligned}
&||f||_{A^s(B^{s'}_{pq})^\sigma_{x_0}} + \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q})\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}||\mathrm{osc}_p^{k-1} f||_{L^p(P)})^q)^{1/q})\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & (\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}||d_i^k f||_{L^p(P)})^q)^{1/q}).
\end{aligned}$$

(ii) *Let $s,\ s',\ \sigma \in \mathbb{R}$ with $0 < s',\ \ 0 \leq \sigma,\ \ x_0 \in \mathbb{R}^n$, $1 \leq p < \infty,\ 1 \leq q \leq \infty$. Let $k \in \mathbb{N}$ with $k > s' > 0$. Then we have following equivalences for $f \in \mathcal{S}'$*

$$\begin{aligned}
&||f||_{A^s(F^{s'}_{pq})^\sigma_{x_0}} + \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'} \sup_{k|u|\leq 2^{-i}} |\triangle_u^k f|)^q)^{1/q}||_{L^p(P)})\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'}\mathrm{osc}_p^{k-1} f)^q)^{1/q}||_{L^p(P)})\\
\sim\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(||f||_{L^p(P)}\\
+\ & ||(\sum_{i\geq(-\log_2 l(P))\vee 0}(2^{is'} d_i^k f)^q)^{1/q}||_{L^p(P)}).
\end{aligned}$$

Proof. (i) It suffices to prove the first part of (i) by Lemma 6.1. We consider the Littlewood-Paley decomposition $f = S_i f + \sum_{l>i} f * \phi_l$. Then we have for $k|u| \leq 2^{-i}$ and a dyadic cube P with $l(P) = 2^{-j}$, $i \geq j$

$$\begin{aligned} ||\triangle_u^k f||_{L^p(P)} &\leq ||\triangle_u^k (f - S_i f)||_{L^p(P)} + ||\triangle_u^k S_i f||_{L^p(P)} \\ &\leq \ C \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)} + C||\triangle_u^k S_i f||_{L^p(P)}. \end{aligned}$$

We will estimate $||\triangle_u^k S_i f||_{L^p(P)}$. Note the following formula

$$\triangle_u^k S_i f(x) = \int_{-\infty}^{\infty} \sum_{|\nu|=k} \frac{k!}{\nu!} u^\nu \partial^\nu S_i f(x + \xi u) N_k(\xi) \ \ d\xi$$

where N_k is the B-spline of order k (e.g. See [27]). Therefor we have for $k|u| \leq 2^{-i}$

$||\triangle_u^k S_i f||_{L^p(P)} \leq C \sum_{|\nu|=k} |u|^k ||\partial^\nu S_i f||_{L^p(2P)}.$

Next, we will estimate $||\partial^\nu S_i f||_{L^p(2P)}$:

$$\begin{aligned} &||\partial^\nu S_i f||_{L^p(2P)} = || \int f(x - 2^{-i} y) \ \partial^\nu \phi_0(y) \ dy||_{L^p(2P)} \\ &\leq \ C \int (\int_{2P+2^{-i}y} |f(x)|^p \ dx)^{1/p} |\partial^\nu \phi_0(y)| \ dy \\ &\leq \ C 2^{-js} \int (|x_0 - x_P| + 2^{-j}(1 + |y|))^\sigma |\partial^\nu \phi_0(y)| \ dy \\ &\times \ \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)} \\ &\leq \ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &||\triangle_u^k f||_{L^p(P)} \\ &\leq C \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)} \\ &+ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} 2^{-ik} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

Moreover, we obtain by using Hardy's inequality if $s' > 0$

$$\begin{aligned} &(\sum_{i \geq j \vee 0} (2^{is'} \sup_{k|u| \leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\ &\leq \ C(\sum_{i \geq j \vee 0} (2^{is'} \sum_{l>i} ||\triangle_u^k (f * \phi_l)||_{L^p(P)})^q)^{1/q} \\ &+ \ C(\sum_{i \geq j \vee 0} (2^{-i(k-s')} (|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \\ &\times \ \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)})^q)^{1/q} \\ &\leq \ C(\sum_{i > j \vee 0} (2^{is'} ||\triangle_u^k (f * \phi_i)||_{L^p(P)})^q)^{1/q} \\ &+ \ C(|x_0 - x_P| + 2^{-j})^\sigma 2^{-js} \sup_{x_0 \in Q} l(Q)^{-\sigma} \sup_{P \subset 3Q} l(P)^{-s} ||f||_{L^p(P)}. \end{aligned}$$

This implies that

$$\begin{aligned}
&\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i\geq j\vee 0}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q}\\
\leq\ & C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i>j\vee 0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q}\\
+\ & C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}\\
\leq\ & C||f||_{A^s(B^{s'}_{pq})^\sigma_{x_0}} + C\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}.
\end{aligned}$$

We will show the converse statement. It is easy to see that there exist $\phi^m\in\mathcal{S}\ \ m=1,\cdots n$ such that $\phi=\sum_{m=1}^n\triangle^k_{ce_m}\phi^m$ for enough small c where $e_1,\cdots,e_n$ are the canonical basis vectors in $\mathbb{R}^n$. Then we have for $i\in\mathbb{N}$

$$f*\phi_i=\sum_{m=1}^n f*\triangle^k_{c2^{-i}e_m}\phi_i^m=\sum_{m=1}^n\triangle^k_{c2^{-i}e_m}f*\phi_i^m.$$

Therefore, we have for a dyadic cube P with $l(P)=2^{-j}$ and $i\geq j$

$$\begin{aligned}
&||f*\phi_i||_{L^p(P)}\\
\leq\ & C||\sum_{m=1}^n\triangle^k_{c2^{-i}e_m}f*\phi_i^m||_{L^p(P)}\\
\leq\ & C\int\sum_{m=1}^n(\int_{P+2^{-i}y}|\triangle^k_{c2^{-i}e_m}f(x)|^p\ dx)^{1/p}|\phi^m(y)|\ dy\\
\leq\ & C\int\sum_{m=1}^n(\int_{P+2^{-i}y}\sup_{k|u|\leq 2^{-i}}|\triangle^k_u f(x)|^p\ dx)^{1/p}|\phi^m(y)|\ dy.
\end{aligned}$$

Hence, we have if $l(P)<1$

$$\begin{aligned}
&(\sum_{i\geq j}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q}\\
\leq\ & C(|x_0-x_P|+2^{-j})^\sigma 2^{-js}\times\\
& \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i\geq j}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q}
\end{aligned}$$

and if $l(P)\geq 1$

$$\begin{aligned}
&(\sum_{i\geq 0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q}\\
\leq\ & (\sum_{i>0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q}+||f*\phi_0||_{L^p(P)}\\
\leq\ & C(|x_0-x_P|+2^{-j})^\sigma 2^{-js}\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}\times\\
& (\sum_{i>0}(2^{is'}\sup_{k|u|\leq 2^{-i}}||\triangle_u^k f||_{L^p(P)})^q)^{1/q}\\
+\ & C(|x_0-x_P|+2^{-j})^\sigma 2^{-js}\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}(\sum_{i\geq 0}(2^{is'}||f*\phi_i||_{L^p(P)})^q)^{1/q} \\
\leq\ & C \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s} \times \\
& (\sum_{i\geq 0}(2^{is'} \sup_{k|u|\leq 2^{-i}} ||\triangle_u^k f||_{L^p(P)})^q)^{1/q} \\
+\ & \sup_{x_0\in Q} l(Q)^{-\sigma} \sup_{P\subset 3Q} l(P)^{-s}||f||_{L^p(P)}.
\end{aligned}$$

This completes the proof of Theorem 6.1 (i). In the same way we can prove (ii).

□

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Koichi Saka
Department of Mathematics
Akita University,
010-8502 Akita, Japan
E-mail: sakakoichi@gmail.com

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