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ON SOME SYSTEMS OF NONLINEAR INTEGRAL EQUATIONS ON THE WHOLE AXIS WITH MONOTONOUS HAMMERSTEIN-VOLTERRA TYPE OPERATORS

Kh.A. Khachatryan, H.S. Petrosyan, A.R. Hakobyan

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Abstract. The work is devoted to studying and solving some systems of nonlinear integral equations with monotonous Hammerstein-Volterra integral operators. In specific cases of matrix kernels and nonlinearities the specified systems have applications in various fields of mathematical physics and mathematical biology. Firstly, a quasilinear system of integral equations on the whole axis with monotonous nonlinearity will be investigated, and a constructive theorem of existence of a one-parameter family of componentwise nonnegative (nontrivial) bounded solutions will be proved. Then, the asymptotic behaviour of the constructed solutions will be studied at $-\infty$. Then, using the obtained results, a system of integral equations with two nonlinearities with different characteristics will be investigated. Under certain limitations on the first nonlinearity we will prove the existence of componentwise nonnegative and bounded solution for such systems. In addition, the limit of the constructed solution at $-\infty$ will be calculated, and the asymptotics of the difference between the limit and the solution will be established. At the end of this paper specific examples of matrix kernels and nonlinearities will be given for the illustration of the obtained results.

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1 Introduction

In the present paper we study the following quasilinear and essentially nonlinear integral equations with monotonous Hammerstein-Volterra operator on the whole axis $\mathbb{R} := (-\infty, +\infty)$:

$$f_i(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{f_j(t) + \omega_{ij}(t, f_j(t))\}dt, \; i = 1, 2, ..., n, \; x \in \mathbb{R}, \tag{1.1}$$

$$\varphi_i(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\varphi_j(t)) + \omega_{ij}(t, \varphi_j(t))\}dt, \; i = 1, 2, ..., n, \; x \in \mathbb{R}, \tag{1.2}$$

with respect to the unknown measurable on $\mathbb{R}$ vector-functions $f(x) = (f_1(x), ..., f_n(x))^T$ and $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$ respectively (T is the sign of transposition). In systems (1.1) and (1.2) the matrix kernel $K(x,t) = (K_{ij}(x,t))_{i,j=1}^{n \times n}$ satisfies the following conditions:

a) $K_{ij}(x,t) > 0$, $(x,t) \in \mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, $K_{ij} \in L_\infty(\mathbb{R}^2)$, $i, j = 1, 2, ..., n$, where $L_\infty(\mathbb{R}^2)$ is the space of all essentially bounded functions on the set $\mathbb{R}^2$,

$b)$ there exists a symmetric matrix $A = (a_{ij})_{i,j=1}^{n\times n}$ with positive elements a_{ij} and with a unit spectral radius such that

$b_1)$
$$\gamma_{ij}(x) := a_{ij} - \int_{-\infty}^{x} K_{ij}(x,t)dt \geq 0,\ \ \gamma_{ij}(x) \not\equiv 0,\ \ x \in \mathbb{R},$$
$$\lim_{x\to-\infty} \gamma_{ij}(x) = 0,\ \ i,j = 1,2,...,n,$$

$b_2)$
$$\int_{t}^{\infty} K_{ij}(x,t)dx \leq a_{ij},\ \ t \in \mathbb{R},\ \ i,j = 1,2,...,n,$$

$b_3)$
$$\int_{-\infty}^{0} (-x)\gamma_{ij}(x)dx < +\infty,\ \ i,j = 1,2,...,n,$$

$c)$ there exists a number $\delta_0 > 0$ such that
$$\varepsilon_{ij} := \inf_{x\in(-\infty,0]} \int_{\delta_0}^{\infty} K_{ij}(x+y,x)dy > 0,\ \ i,j = 1,2,...,n.$$

From the properties of the matrix A, by Perron's theorem (see [12]), follows the existence of a vector $\eta = (\eta_1,...,\eta_n)^T$ with positive coordinates $\eta_i,\ i = 1,2,...,n$, such that
$$A\eta = \eta. \tag{1.3}$$

The nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ satisfy the following conditions:

I) $G_j \in C[0,+\infty)$, $G_j(u)$ is a concave function on the set $[0,+\infty)$, $G_j(0) = 0$, $j = 1,2,...,n$,

II) $G_j(u)$ are increasing with respect to u on the set $[0,+\infty)$, $j = 1,2,...,n$,

III) there exists a number $\alpha > 0$, such that $G_j(\eta_j^*) = \eta_j^*$, $G_j(u) \geq u$, $u \in [0,\eta_j^*]$, where $\eta_j^* = \alpha\eta_j$, $j = 1,2,...,n$,

A) $\omega_{ij}(t,0) \equiv 0$, $t \in \mathbb{R}$, $i,j = 1,2,...,n$,

B) for every fixed $t \in \mathbb{R}$ the functions $\omega_{ij}(t,u)$, $i,j = 1,2,...,n$ monotonically increase with respect to u on the set $[0,+\infty)$,

C) there exist functions
$$\beta_{ij}(t) := \sup_{u\in[0,+\infty)} (\omega_{ij}(t,u)),\ \ i,j = 1,2,...,n,$$
such that $\beta_{ij}(t)$, $i,j = 1,2,...,n$ are monotone nondecreasing with respect to t on the set $\mathbb{R}$ and satisfy the following inequality
$$\sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right) \leq \sum_{j=1}^{n} \eta_j\gamma_{ij}(x),\ \ x \in \mathbb{R},\ \ i = 1,2,...,n, \tag{1.4}$$

D) $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ satisfy the Caratheodory condition with respect to the argument u on the set $\mathbb{R}\times[0,+\infty)$, i.e. for every fixed $u\in[0,+\infty)$ the functions $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ are measurable with respect to t on $\mathbb{R}$ and for almost every $t\in\mathbb{R}$ these functions are continuous with respect to u on the set $[0,+\infty)$.

The study of systems of nonlinear integral equations (1.1) and (1.2), besides purely mathematical interest, has also an important interest in different applied problems of mathematical physics and mathematical biology. In particular, for specific representations of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ such systems of nonlinear integral equations can be found in the kinetic theory of gases, radiative transfer theory, Markovian processes and in the mathematical theory of space-time epidemic spread (see [1]-[5], [10], [13], [14]).

In the case, when the kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ depend on the difference of their arguments and satisfy the supercritical condition (the spectral radius of the matrix A is greater than one) with particular restrictions on the functions $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ system (1.1) on $(-\infty,0]$ (and the corresponding system of nonlinear integral equations on $[0,+\infty)$, whose right-hand-side integrals have limits from $x\geq 0$ to $+\infty$) is studied in sufficient detail in the work [9]. In the present paper a one-parameter family of positive summable and bounded on $(-\infty,0]$ (on $[0,+\infty)$) solutions is constructed and the set of the corresponding parameters is described.

It should also be noted that in the case when $K_{ij}(x,t)=K_{ij}(x-t)$, $(x,t)\in\mathbb{R}^2$, $i,j=1,2,...,n$ the corresponding systems of convolution type nonlinear integral equations (NIE) (i.e. when the integral in the right-hand sides of (1.1) and (1.2) has the limits from $-\infty$ to $+\infty$) were studied in the works [6]-[8].

In the present paper under conditions $a)-c)$, I) - III) and A) - D) we will deal with the problems of existence of nonnegative (nontrivial) and bounded solutions of systems of nonlinear integral equations (1.1) and (1.2) and also will study the asymptotic behaviour of the constructed solutions on $-\infty$. Firstly, a constructive theorem of existence of a one-parameter family of componentwise nonnegative (nontrivial) and bounded solutions, which have finite limit values in $-\infty$ will be proved. Then, we will prove the integrability on the set $(-\infty,0]$ of the difference between the limit (at $-\infty$) and the constructed solution for every value of the corresponding parameter on the set $(0,+\infty)$ (see Theorem 2.1). Owner furthermore, by using these results, we will construct componentwise nonnegative and bounded on $\mathbb{R}$ solution $\varphi(x)=(\varphi_1(x),...,\varphi_n(x))^T$ of system of nonlinear integral equations (1.2). Additionally, we will prove the existence of

$$\lim_{x\to-\infty}\varphi_j(x)=\eta_j^*$$

and that $\eta_j^*-\varphi_j\in L_1(-\infty,0)$, $j=1,2,...,n$ (see Theorem 3.1). At the end of the work we will provide specific examples of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$, $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ that satisfy all the conditions of the proved theorem. Note that a part of those examples have applied character (they arise in specific problems of mathematical physics and biology).

2 One parameter family of solutions for system (1.1)

In the current section we will prove the following result for system of NIE (1.1):

Theorem 2.1. *Under conditions a) - c) and A) - D) system of NIE* (1.1) *has a one-parameter family of componentwise nonnegative (nontrivial) and bounded solutions* $f^\gamma(x)=(f_1^\gamma(x),...,f_n^\gamma(x))^T$, $\gamma\in(0,+\infty)$, *such that*

$$\lim_{x\to-\infty}f_j^\gamma(x)=\eta_j\gamma$$

and $\eta_j\gamma-f_j^\gamma\in L_1(-\infty,0)$, $j=1,2,...,n$, *where* η *is defined by* (1.3).

Proof. Firstly, let us consider the first auxiliary system of linear nonhomogeneous Volterra type integral equations:

$$\psi_i(x) = g_i(x) + \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j(t)dt, \ i = 1,2,...,n, \ x \in \mathbb{R} \tag{2.1}$$

with respect to an unknown summable on $\mathbb{R}$ vector-function $\psi(x) = (\psi_1(x), ..., \psi_n(x))^T$, where the vector-function $g(x) = (g_1(x), ..., g_n(x))^T$ has the following structure:

$$g_i(x) = \sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right), \ i = 1,2,...,n, \ x \in \mathbb{R}. \tag{2.2}$$

We introduce the following iterations for system (2.1):

$$\begin{gathered} \psi_i^{(m+1)}(x) = g_i(x) + \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j^{(m)}(t)dt, \\ \psi_i^{(0)}(x) = g_i(x), \ x \in \mathbb{R}, \ i = 1,2,...,n, \ m = 0,1,... . \end{gathered} \tag{2.3}$$

By mathematical induction it is not hard to verify that

1) $\psi_i^{(m)}(x)$ are measurable on $\mathbb{R}$, $i = 1,2,...,n$, $m = 0,1,2,...$,

2) $\psi_i^{(m)}(x) \uparrow$ with respect to m, $i = 1,2,...,n$, $x \in \mathbb{R}$.

We will prove that

3) $\psi_i^{(m)}(x) \le \eta_i$, $m = 0,1,2,...$, $i = 1,2,...,n$, $x \in \mathbb{R}$.

Indeed, estimate 3) for $m = 0$ directly follows from b_1), (1.3) and (1.4):

$$\psi_i^{(0)}(x) = g_i(x) \le \sum_{j=1}^{n} \eta_j\gamma_{ij}(x) \le \sum_{j=1}^{n} a_{ij}\eta_j = \eta_i, \ x \in \mathbb{R}, \ i = 1,2,...,n.$$

Assume that 3) holds for some $m \in \mathbb{N}$. Then, with consideration of b_1), (1.3), (1.4), a) and (2.2) from (2.3) we get

$$\psi_i^{(m+1)}(x) \le \sum_{j=1}^{n} \beta_{ij}(x)\left(a_{ij} - \gamma_{ij}(x)\right) + \sum_{j=1}^{n} \eta_j \int_{-\infty}^{x} K_{ij}(x,t)dt \le$$

$$\le \sum_{j=1}^{n} \eta_j\gamma_{ij}(x) + \sum_{j=1}^{n} \eta_j\left(a_{ij} - \gamma_{ij}(x)\right) = \sum_{j=1}^{n} a_{ij}\eta_j = \eta_i, \ i = 1,2,...,n, \ x \in \mathbb{R}.$$

Now we will prove that

4) $\psi_i^{(m)} \in L_1(-\infty,0)$, $i = 1,2,...,n$, $m = 0,1,2,...$.

Indeed, in the case when $m = 0$ inclusion 4) follows from the definition of $g_i(x)$, $i = 1,2,...,n$, with consideration of (1.4), conditions b_1) and b_3). Let $\psi_i^{(m)} \in L_1(-\infty,0)$, $i = 1,2,...,n$ for some natural

m. Then, considering (1.4), a), b), c) and (2.2), from (2.3) for every $\delta<0$ by Fubini's theorem (see [11]) we have

$$0\leq\int\limits_{\delta}^{0}\psi_i^{(m+1)}(x)dx\leq\sum_{j=1}^{n}\int\limits_{\delta}^{0}\eta_j\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{\delta}^{0}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j^{(m)}(t)dtdx=$$

$$=\sum_{j=1}^{n}\eta_j\int\limits_{\delta}^{0}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{\delta}^{0}\int\limits_{-\infty}^{\delta}K_{ij}(x,t)\psi_j^{(m)}(t)dtdx+\sum_{j=1}^{n}\int\limits_{\delta}^{0}\int\limits_{\delta}^{x}K_{ij}(x,t)\psi_j^{(m)}(t)dtdx\leq$$

$$\leq\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{-\infty}^{\delta}\psi_j^{(m)}(t)\int\limits_{\delta}^{0}K_{ij}(x,t)dxdt+\sum_{j=1}^{n}\int\limits_{\delta}^{0}\psi_j^{(m)}(t)\int\limits_{t}^{0}K_{ij}(x,t)dxdt\leq$$

$$\leq\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}\gamma_{ij}(x)dx+\sum_{j=1}^{n}a_{ij}\int\limits_{-\infty}^{0}\psi_j^{(m)}(t)dt<+\infty.$$

By passing to the limit as $\delta\to-\infty$, we conclude that $\psi_i^{(m+1)}\in L_1(-\infty,0)$. Now let $t\leq 0$ be an arbitrary number. We multiply both sides of (2.3) by η_i, $i=1,2,...,n$ and taking into account conditions a), b), c), (1.4) and also the proven inclusions 1)-4), we integrate both sides of the obtained equality by $x\in(-\infty,t]$, then we add the equations for $i=1,2,...,n$. As a result we obtain

$$\sum_{i=1}^{n}\eta_i\int\limits_{-\infty}^{t}\psi_i^{(m+1)}(x)dx\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{x}K_{ij}(x,y)\psi_j^{(m+1)}(y)dydx=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{0}K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)d\tau dx=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{t}K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{-\delta_0}\int\limits_{-\infty}^{t}K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{t}K_{ij}(x,x+\tau)\psi_j^{(m+1)}(x+\tau)dxd\tau=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{-\delta_0}\int\limits_{-\infty}^{t+\tau}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{t+\tau}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{-\delta_0}\int\limits_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{t}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\int\limits_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{-\delta_0}\int\limits_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)\psi_j^{(m+1)}(z)dzd\tau+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t}\psi_j^{(m+1)}(z)\int\limits_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int\limits_{-\infty}^{t-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int\limits_{t-\delta_0}^{-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t}\psi_j^{(m+1)}(z)\int\limits_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int\limits_{-\infty}^{-\delta_0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t}\psi_j^{(m+1)}(z)\int\limits_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int\limits_{-\infty}^{0}K_{ij}(z-\tau,z)d\tau dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{-\delta_0}^{0}K_{ij}(z-\tau,z)d\tau dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)\int\limits_{z}^{\infty}K_{ij}(y,z)dydz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}a_{ij}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz\sum_{i=1}^{n}a_{ji}\eta_i+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t-\delta_0}\psi_j^{(m+1)}(z)dz+$$

$$+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz,$$

from which it follows that

$$\begin{aligned}\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\\+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dydz.\end{aligned}\tag{2.4}$$

Observe that

$$\begin{aligned}a_{ij}-\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dy\geq\int\limits_{z}^{\infty}K_{ij}(y,z)dy-\int\limits_{z}^{z+\delta_0}K_{ij}(y,z)dy=\\=\int\limits_{z+\delta_0}^{\infty}K_{ij}(y,z)dy=\int\limits_{\delta_0}^{\infty}K_{ij}(z+u,z)du\geq\varepsilon_{ij}\text{ for }z\leq 0,\ i,j=1,2,...,n\ .\end{aligned}\tag{2.5}$$

Considering (2.4) and (2.5), we obtain

$$\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq$$

$$\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)\left(\sum_{i=1}^{n}a_{ji}\eta_i-\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\right)dz=$$

$$=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx+\sum_{j=1}^{n}\eta_j\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz-\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz,$$

which is the same as

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dz\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{t}\gamma_{ij}(x)dx. \tag{2.6}$$

Let $p<0$ be an arbitrary number. We integrate both sides of (2.6) with respect to t from p to 0. Then, according to b_1), b_3) and Fubini's theorem from (2.6) we obtain

$$\begin{gathered}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{p}^{0}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dzdt\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{t}\gamma_{ij}(x)dxdt=\\=\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty.\end{gathered}\tag{2.7}$$

By passing to the limit as $p\to-\infty$, we obtain

$$0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\infty}^{0}\int\limits_{t-\delta_0}^{t}\psi_j^{(m+1)}(z)dzdt\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx$$

or

$$\begin{gathered}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\infty}^{0}\int\limits_{-\delta_0}^{0}\psi_j^{(m+1)}(t+\tau)d\tau dt\leq\\\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty,\ m=0,1,2,\dots.\end{gathered}\tag{2.8}$$

By changing the order of integration in (2.8), we have

$$\begin{gathered}0\leq\sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{0}\psi_j^{(m+1)}(t+\tau)dtd\tau\leq\\\leq\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx,\ m=0,1,2,\dots,\end{gathered}$$

from which it follows that

$$0 \le \sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\delta_0}^{0}\int\limits_{-\infty}^{-\delta_0}\psi_j^{(m+1)}(y)dyd\tau \le$$

$$\le \sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx,\ m=0,1,2,...$$

or

$$\begin{aligned} 0 \le \sum_{j=1}^{n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i\int\limits_{-\infty}^{-\delta_0}\psi_j^{(m+1)}(y)dy \le \\ \le \frac{1}{\delta_0}\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx,\ m=0,1,2,...\ . \end{aligned} \tag{2.9}$$

Due to 1)-3) we have

$$0 \le \int\limits_{-\delta_0}^{0}\psi_j^{(m+1)}(y)dy \le \eta_j\delta_0,\ j=1,2,...,n,\ m=0,1,2,...\ . \tag{2.10}$$

We denote

$$\mu := \min_{1\le j\le n}\sum_{i=1}^{n}\varepsilon_{ij}\eta_i. \tag{2.11}$$

Then, from (2.9), in particular, it follows that

$$\begin{aligned} 0 \le \int\limits_{-\infty}^{-\delta_0}\psi_j^{(m+1)}(y)dy \le \frac{1}{\mu\delta_0}\cdot\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx, \\ m=0,1,2,...,\ j=1,2,...,n. \end{aligned} \tag{2.12}$$

Therefore, inequalities (2.10) and (2.12) entail the following two-sided estimate

$$\begin{aligned} 0 \le \int\limits_{-\infty}^{0}\psi_j^{(m+1)}(y)dy \le (\max_{1\le j\le n}\eta_j)\delta_0 + \frac{1}{\mu\delta_0}\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx < +\infty, \\ j=1,2,...,n,\ m=0,1,2,...\ . \end{aligned} \tag{2.13}$$

From 1)-4) and (2.11) it follows that the sequence of measurable on $\mathbb{R}$ vector-functions $\psi^{(m)}(x) = \left(\psi_1^{(m)}(x),...,\psi_n^{(m)}(x)\right)^T$, $m=0,1,2,...$ has a pointwise limit when $m\to\infty$:

$$\lim_{m\to\infty}\psi^{(m)}(x) = \psi(x),$$

additionally, the limit vector-function $\psi(x) = (\psi_1(x),...,\psi_n(x))^T$ according to B. Levi's theorem (see [11]) satisfies system (2.1). Once again using 1)-4) and (2.11), we can state that

$$g_i(x) \le \psi_i(x) \le \eta_i,\ x\in\mathbb{R},\ i=1,2,...,n, \tag{2.14}$$

$$0\leq\int\limits_{-\infty}^{0}\psi_j(x)dx\leq(\max_{1\leq j\leq n}\eta_j)\delta_0+\frac{1}{\mu\delta_0}\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty,\quad i=1,2,\dots\,. \tag{2.15}$$

We now consider the second auxiliary linear nonhomogeneous system of integral equations on $\mathbb{R}$:

$$\psi_i^*(x)=g_i^*(x)+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j^*(t)dt,\ \ x\in\mathbb{R},\ \ i=1,2,...,n \tag{2.16}$$

with respect to the unknown vector function $\psi^*(x)=(\psi_1^*(x),...,\psi_n^*(x))^T$, where

$$g_i^*(x)=\sum_{j=1}^{n}\eta_j\gamma_{ij}(x),\ \ i=1,2,...,n,\ \ x\in\mathbb{R}. \tag{2.17}$$

Repeating the same reasoning as for system (2.1), wherein taking $\psi_i(x)$, $i=1,2,...,n$ as the zero approximation, we can prove that system of integral equations (2.16) has a componentwise nonnegative and bounded solution $\psi^*(x)=(\psi_1^*(x),...,\psi_n^*(x))^T$, and, besides that

$$g_i(x)\leq\psi_i(x)\leq\psi_i^*(x)\leq\eta_i,\ \ x\in\mathbb{R},\ \ i=1,2,...,n, \tag{2.18}$$

$$0\leq\int\limits_{-\infty}^{0}\psi_j^*(x)dx\leq(\max_{1\leq j\leq n}\eta_j)\delta_0+\frac{1}{\mu\delta_0}\sum_{i=1}^{n}\eta_i\sum_{i=1}^{n}\eta_j\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx<+\infty,\quad i=1,2,\dots\,. \tag{2.19}$$

On the other hand, note that system of integral equations (2.16) also has a trivial solution $\eta=(\eta_1,...,\eta_n)^T$. Indeed, considering b_1), (2.17) and (1.3), we obtain

$$g_i^*(x)+\sum_{j=1}^{n}\eta_j\int\limits_{-\infty}^{x}K_{ij}(x,t)dt=\sum_{j=1}^{n}\eta_j(a_{ij}-\gamma_{ij}(x))+\sum_{j=1}^{n}\eta_j\gamma_{ij}(x)=\sum_{j=1}^{n}a_{ij}\eta_j=\eta_i,\quad i=1,2,...,n.$$

From (2.18) and (2.19) it follows that $\psi_i^*(x)\not\equiv\eta_i$, $x\in\mathbb{R}$, $i=1,2,...,n$. Therefore,

$$\Phi_i(x):=\eta_i-\psi_i^*(x)\geq0,\ \ \Phi_i(x)\not\equiv0,\ \ x\in\mathbb{R},\ \ i=1,2,...,n$$

and also satisfies the homogeneous system of integral equations

$$\Phi_i(x)=\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\Phi_j(t)dt,\ \ x\in\mathbb{R},\ \ i=1,2,...,n. \tag{2.20}$$

We now prove that there exists

$$\lim_{x\to-\infty}\Phi_i(x)=\eta_i,\ \ i=1,2,...,n.$$

Indeed, for negative values of x from (2.16) due to a) and b_1) we conclude that

$$0\leq\psi_i^*(x)\leq\sum_{j=1}^{n}\eta_j\gamma_{ij}(x)+\sum_{j=1}^{n}\sup_{(x,t)\in\mathbb{R}^2}(K_{ij}(x,t))\cdot\int\limits_{-\infty}^{x}\psi_j^*(t)dt\to0,\ \text{when}\ x\to-\infty,$$

from which we obtain that there exists $\lim\limits_{x\to-\infty}\psi_i^*(x)=0$, $i=1,2,...,n$. Therefore, there exists $\lim\limits_{x\to-\infty}\Phi_i(x)=\eta_i$, $i=1,2,...,n$. Since $\psi_i^*\in L_1(-\infty,0)$, $i=1,2,...,n$, hence $\eta_i-\Phi_i\in L_1(-\infty,0)$, $i=1,2,...,n$.

Finally, we consider the following family of successive approximations for the system (1.1):

$$f_{i,\gamma}^{(m+1)}(x)=\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\{f_{j,\gamma}^{(m)}(t)+\omega_{ij}(t,f_{j,\gamma}^{(m)}(t))\}dt,$$
$$f_{i,\gamma}^{(0)}(x)=\gamma\Phi_i(x),\ m=0,1,2,...,\ i=1,2,...,n,\ x\in\mathbb{R}, \tag{2.21}$$

where $\gamma\in(0,+\infty)$ is an arbitrary parameter.

By using mathematical induction it is not hard to verify that for every $\gamma\in(0,+\infty)$

Γ_1)
$$f_{i,\gamma}^{(m)}(x)\text{ are measurable on }\mathbb{R},\ i=1,2,...,n,\ m=0,1,2,..., \tag{2.22}$$

Γ_2)
$$f_{i,\gamma}^{(m)}(x)\uparrow\text{ with respect to }m,\ i=1,2,...,n,\ x\in\mathbb{R}. \tag{2.23}$$

We will now prove that

Γ_3)
$$f_{i,\gamma}^{(m)}(x)\le\gamma\Phi_i(x)+\psi_i(x),\ i=1,2,...,n,\ x\in\mathbb{R}. \tag{2.24}$$

For $m=0$ the given inequality directly follows from the definition of the zero approximation with consideration of nonnegativity of the functions $\{\psi_i(x)\}_{i=1}^n$ on $\mathbb{R}$. Assume that (2.24) holds for some $m\in\mathbb{N}$. Then, taking into account (2.1), (2.20) and A)-C), from (2.21) we get

$$f_{i,\gamma}^{(m+1)}(x)\le\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\{\gamma\Phi_j(t)+\psi_j(t)+\omega_{ij}(t,\gamma\Phi_j(t)+\psi_j(t))\}dt\le$$
$$\le\gamma\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\Phi_j(t)dt+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j(t)dt+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\beta_{ij}(t)dt\le$$
$$\le\gamma\Phi_i(x)+\sum_{j=1}^{n}\int\limits_{-\infty}^{x}K_{ij}(x,t)\psi_j(t)dt+\sum_{j=1}^{n}\beta_{ij}(x)(a_{ij}-\gamma_{ij}(x))=\gamma\Phi_i(x)+\psi_i(x),$$
$$i=1,2,...,n,\ x\in\mathbb{R}.$$

Let us prove that

Γ_4) If $\gamma_1,\gamma_2\in(0,+\infty)$ are arbitrary parameters and $\gamma_1>\gamma_2$, then

$$f_{i,\gamma_1}^{(m)}(x)-f_{i,\gamma_2}^{(m)}(x)\ge(\gamma_1-\gamma_2)\Phi_i(x),\ x\in\mathbb{R},\ i=1,2,...,n,\ m=0,1,2,... \tag{2.25}$$

Indeed, in the case of $m=0$ inequalities (2.25) are transformed to equalities by the definition of the zero approximation in iterations (2.21). Let (2.25) hold for some natural m. Then, from (2.21) due to conditions B) and (2.20) we will obtain

$$f_{i,\gamma_1}^{(m+1)}(x) - f_{i,\gamma_2}^{(m+1)}(x) =$$

$$= \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{f_{j,\gamma_1}^{(m)}(t) - f_{j,\gamma_2}^{(m)}(t) + \omega_{ij}(t, f_{j,\gamma_1}^{(m)}(t)) - \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t))\}dt \geq$$

$$\geq (\gamma_1-\gamma_2) \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{\Phi_j(t) + \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t) + (\gamma_1-\gamma_2)\Phi_j(t)) - \omega_{ij}(t, f_{j,\gamma_2}^{(m)}(t))\}dt \geq$$

$$\geq (\gamma_1-\gamma_2) \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\Phi_j(t)dt = (\gamma_1-\gamma_2)\Phi_i(x), \ i=1,2,...,n, \ x \in \mathbb{R}.$$

So, from Γ_1) - Γ_4) it follows that the sequence of measurable vector functions $f_\gamma^{(m)}(x) = (f_{1,\gamma}^{(m)}(x), ..., f_{n,\gamma}^{(m)}(x))^T$, $m = 0,1,2,...$, for every $\gamma \in (0,+\infty)$ has a pointwise limit when $m \to \infty$: $\lim_{m\to\infty} f_\gamma^{(m)}(x) = f^\gamma(x) = (f_1^\gamma(x), ..., f_n^\gamma(x))^T$, moreover,

$$\gamma\Phi_j(x) \leq f_j^\gamma(x) \leq \gamma\Phi_j(x) + \psi_j(x), \ j=1,2,...,n, \ x \in \mathbb{R}, \tag{2.26}$$

$$f_j^{\gamma_1}(x) - f_j^{\gamma_2}(x) \geq (\gamma_1-\gamma_2)\Phi_j(x), \ j=1,2,...,n, \ x \in \mathbb{R}, \tag{2.27}$$

where $\gamma_1, \gamma_2 \in (0,+\infty)$, $\gamma_1 > \gamma_2$ are arbitrary parameters. Considering conditions D), *b*) according to B. Levi's theorem for every $\gamma \in (0,+\infty)$ the vector function $f^\gamma(x) = (f_1^\gamma(x), ..., f_n^\gamma(x))^T$ satisfies system of NIE (1.1).

Since $\lim_{x\to-\infty} \psi_i^*(x) = 0$, $i = 1,2,...,n$, from (2.18) it follows that

$$\lim_{x\to-\infty} \psi_i(x) = 0, \ i=1,2,...,n. \tag{2.28}$$

From (2.15), (2.26) and (2.28) directly follows that

$$\lim_{x\to-\infty} \{f_i^\gamma(x) - \gamma\Phi_i(x)\} = 0, \ i=1,2,...,n, \gamma \in (0,+\infty), \tag{2.29}$$

$$0 \leq f_i^\gamma - \gamma\Phi_i \in L_1(-\infty,0), \ i=1,2,...,n, \ \gamma \in (0,+\infty). \tag{2.30}$$

Since $\lim_{x\to-\infty} (\eta_i - \Phi_i(x)) = 0$, $\eta_i - \Phi_i \in L_1(-\infty,0)$, $i=1,2,...,n$, hence there exists $\lim_{x\to-\infty} f_i^\gamma(x) = \gamma\eta_i$, and from the estimate

$$0 \leq |\gamma\eta_i - f_i^\gamma(x)| \leq \gamma(\eta_i - \Phi_i(x)) + f_i^\gamma(x) - \gamma\Phi_i(x) \in L_1(-\infty,0), \ i=1,2,...,n$$

it follows that $\gamma\eta_i - f_i^\gamma \in L_1(-\infty,0)$, $i=1,2,...,n$, $\gamma \in (0,+\infty)$. □

3 Solvability of system of NIE (1.2). Examples

In the current section with the use of the results of Theorem 2.1 and some geometrical inequalities for concave functions, we will deal with the problem of solvability for system of NIE (1.2).

Theorem 3.1. *Under conditions a) - c), I) - III) and A) - D) system of NIE* (1.2) *has componentwise nonnegative (nontrivial) and bounded on* $\mathbb{R}$ *solution* $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$*, such that*

$$\lim_{x\to-\infty} \varphi_j(x) = \eta_j^*$$

and $\eta_j^* - \varphi_j \in L_1(-\infty,0)$*,* $j=1,2,...,n$ *where* $\eta*$ *is defined in III).*

Proof. Due to Theorem 2.1 for the number $\gamma^* = \alpha$ corresponds a solution $f^{\gamma^*}(x) = (f_1^{\gamma^*}(x), ..., f_n^{\gamma^*}(x))^T$ of system (1.1) with the properties

$$\alpha\Phi_j(x) \leq f_j^{\gamma^*}(x) \leq \alpha\Phi_j(x) + \psi_j(x),\ j = 1, 2, ..., n,\ x \in \mathbb{R}, , \tag{3.1}$$

$$\lim_{x\to-\infty} f_j^{\gamma^*}(x) = \alpha \cdot \eta_j = \eta_j^*,\ \eta_j^* - f_j^{\gamma^*} \in L_1(-\infty, 0),\ j = 1, 2, ..., n. \tag{3.2}$$

Consider the following iterations for system (1.2):

$$\begin{gathered}\varphi_i^{(m+1)}(x) = \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\varphi_j^{(m)}(t)) + \omega_{ij}(t, \varphi_j^{(m)}(t))\}dt,\\ \varphi_i^{(0)}(x) = f_i^{\gamma^*}(x),\ m = 0, 1, 2, ...,\ i = 1, 2, ..., n,\ x \in \mathbb{R}.\end{gathered} \tag{3.3}$$

Using I), II), B), D) and *a*) with induction on m it is easy to check that

E_1) $\varphi_i^{(m)}(x)$ are measurable with respect to x on $\mathbb{R}$, $m = 0, 1, 2, ...$, $i = 1, 2, ..., n$,

E_2) $\varphi_i^{(m)}(x) \uparrow$ with respect to m, $x \in \mathbb{R}$, $i = 1, 2, ..., n$.

Below we will prove that

E_3) $\varphi_i^{(m)}(x) \leq \eta_i^* + \psi_i(x)$, $x \in \mathbb{R}$, $m = 0, 1, 2, ...$, $i = 1, 2, ..., n$.

In the case when $m = 0$ inequalities E_3) directly folow from (3.1) and III), by taking into account the estimates $\Phi_i(x) \leq \eta_i$, $i = 1, 2, ..., n$, $x \in \mathbb{R}$. Assume that E_3) holds for some natural m. Then, using the following inequalities

$$G_j(\eta_j^* + u) \leq \eta_j^* + u,\ u \geq 0,\ j = 1, 2, ..., n$$

(which follow from the concaveness of the functions $\{G_j(u)\}_{j=1}^n$ (see Fig. 1.)), and also *C*), III), II), (1.3), (2.1) and (2.14), from (3.3) we have

$$\varphi_i^{(m+1)}(x) \leq \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\{G_j(\eta_j^* + \psi_j(t)) + \omega_{ij}(t, \eta_j^* + \psi_j(t))\}dt \leq$$

$$\leq \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)(\eta_j^* + \psi_j(t) + \beta_{ij}(t))dt \leq \sum_{j=1}^{n} \eta_j^*(a_{ij} - \gamma_{ij}(x)) +$$

$$+ \sum_{j=1}^{n} \int_{-\infty}^{x} K_{ij}(x,t)\psi_j(t)dt + g_i(x) \leq \eta_i^* + \psi_i(x),\ i = 1, 2, ..., n,\ x \in \mathbb{R}.$$

So, from E_1)-E_3) we conclude that the sequence of measurable vector functions $\varphi^{(m)}(x) = (\varphi_1^{(m)}(x), ..., \varphi_n^{(m)}(x))^T$, $m = 0, 1, 2, ...$ has a pointwise limit when $m \to \infty$: $\lim\limits_{m\to\infty} \varphi^{(m)}(x) = \varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$, moreover,

$$f_i^{\gamma^*}(x) \leq \varphi_i(x) \leq \eta_i^* + \psi_i(x),\ i = 1, 2, ..., n,\ x \in \mathbb{R}. \tag{3.4}$$

Using conditions I) and D) due to B. Levi's theorem we obtain that $\varphi(x) = (\varphi_1(x), ..., \varphi_n(x))^T$ is a solution to system of NIE (1.2). From (3.4), (3.2), (2.15) and (2.28) it follows that $\lim\limits_{x\to-\infty} \varphi_i(x) = \eta_i^*$ and $\eta_i^* - \varphi_i \in L_1(-\infty, 0)$, $i = 1, 2, ..., n$. $\square$

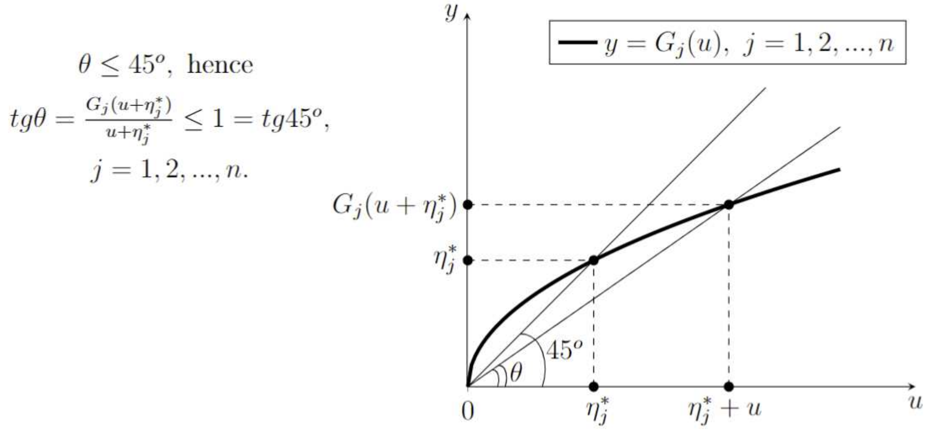

Figure 1:

At the end we will present specific examples of monotonous kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$ and nonlinearities $\{G_j(u)\}_{j=1}^{n}$, $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$ that satisfy the conditions of the proven Theorems 2.1 and 3.1.

Firstly, we will give examples of matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$. Let functions $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$ be defined and continuous on the set $\mathbb{R}$ and satisfy the following conditions

$$F_1)\ \ 0 < \rho_{ij} := \inf_{x\in\mathbb{R}} \lambda_{ij}(x) \leq \lambda_{ij}(x) \leq 1,\ \ \lambda_{ij}(x) \not\equiv 1,\ \ x \in \mathbb{R},\ \ i,j = 1,2,...,n,$$

$$F_2)\ \ \lim_{x\to-\infty} \lambda_{ij}(x) = 1,\ \ x(1-\lambda_{ij}(x)) \in L_1(-\infty,0),\ \ i,j = 1,2,...,n.$$

Also, let functions $\{\overset{\circ}{K}_{ij}(x)\}_{i,j=1}^{n\times n}$ be continuous on $\mathbb{R}$ and satisfy the following conditions:

$$H_1)\ \ \overset{\circ}{K}_{ij}(x) > 0,\ \ x \in \mathbb{R},\ \ \overset{\circ}{K}_{ij}(-t) = \overset{\circ}{K}_{ij}(t),\ \ t \geq 0,\ \ i,j = 1,2,...,n,$$

$$H_2)\ \ \overset{\circ}{K}_{ij} \in L_\infty(\mathbb{R}),\ \ a_{ij} = \int_0^\infty \overset{\circ}{K}_{ij}(x)dx,\ \ i,j = 1,2,...,n.$$

Then we can choose the following classes of matrix functions as matrix kernels $\{K_{ij}(x,t)\}_{i,j=1}^{n\times n}$:

$$W_1)\ \ K_{ij}(x,t) = \lambda_{ij}(x) \cdot \overset{\circ}{K}_{ij}(x-t),\ \ (x,t) \in \mathbb{R}^2,\ \ i,j = 1,2,...,n,$$

$$W_2)\ \ K_{ij}(x,t) = \frac{\lambda_{ij}(t)+\lambda_{ij}(x)}{2} \cdot \overset{\circ}{K}_{ij}(x-t),\ \ (x,t) \in \mathbb{R}^2,\ \ i,j = 1,2,...,n,$$

$$W_3)\ \ K_{ij}(x,t) = \lambda_{ij}(x+t) \cdot \overset{\circ}{K}_{ij}(x-t),\ \ (x,t) \in \mathbb{R}^2,\ \ i,j = 1,2,...,n.$$

Let us take a look at example $W_3)$. Condition $a)$ directly follows from $F_1)$ and $H_1), H_2)$. We will now verify condition $b)$. We have

$$\gamma_{ij}(x) = a_{ij} - \int_{-\infty}^{x} \lambda_{ij}(x+t)\overset{\circ}{K}_{ij}(x-t)dt \geq a_{ij} - \int_{-\infty}^{x} \overset{\circ}{K}_{ij}(x-t)dt = 0, i,j = 1,2,...,n, x \in \mathbb{R}.$$

On the other hand, considering equations $H_2)$, $F_2)$, $F_1)$ and $H_1)$, we obtain

$$\int\limits_{-\infty}^{0}(-x)\gamma_{ij}(x)dx=\int\limits_{-\infty}^{0}(-x)\int\limits_{-\infty}^{x}(1-\lambda_{ij}(x+t))\overset{\circ}{K}_{ij}(x-t)dtdx=$$

$$=\int\limits_{-\infty}^{0}(-x)\int\limits_{0}^{\infty}(1-\lambda_{ij}(2x-y))\overset{\circ}{K}_{ij}(y)dydx=\int\limits_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int\limits_{-\infty}^{0}(-x)(1-\lambda_{ij}(2x-y))dxdy=$$

$$=\frac{1}{2}\int\limits_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int\limits_{-\infty}^{-y}\left(\frac{-t-y}{2}\right)(1-\lambda_{ij}(t))dtdy\le\frac{1}{4}\int\limits_{0}^{\infty}\overset{\circ}{K}_{ij}(y)\int\limits_{-\infty}^{0}(-t-y)\,(1-\lambda_{ij}(t))dtdy\le$$

$$\le\frac{1}{4}\int\limits_{0}^{\infty}\overset{\circ}{K}_{ij}(y)dy\int\limits_{-\infty}^{0}(-t)\,(1-\lambda_{ij}(t))dt=\frac{a_{ij}}{4}\int\limits_{-\infty}^{0}(-t)\,(1-\lambda_{ij}(t))dt<+\infty,\ i,j=1,2,...,n.$$

Now, let us verify that $\lim\limits_{x\to-\infty}\gamma_{ij}(x)=0,\ i,j=1,2,...,n.$ Due to conditions $a)$, $F_1)$, and $F_2)$, $H_2)$ we have

$$0\le\gamma_{ij}(x)=\int\limits_{-\infty}^{x}\overset{\circ}{K}_{ij}(x-t)(1-\lambda_{ij}(x+t))dt\le M\int\limits_{-\infty}^{x}(1-\lambda_{ij}(x+t))dt=$$

$$=M\int\limits_{-\infty}^{2x}(1-\lambda_{ij}(y))dy\to 0,\quad\text{when } x\to-\infty,\ \text{where } M:=\max_{1\le i,j\le n}(\sup_{\tau\in\mathbb{R}}\overset{\circ}{K}_{ij}(\tau)).$$

Finally, let us verify condition $c)$. Due to $F_1)$ and $H_2)$ we obtain

$$\int\limits_{\delta_0}^{\infty}K_{ij}(x+y,x)dy=\int\limits_{\delta_0}^{\infty}\lambda_{ij}(2x+y)\overset{\circ}{K}_{ij}(y)dy\ge\rho_{ij}\cdot\widetilde{a}_{ij},\ \text{where } \widetilde{a}_{ij}=\int\limits_{\delta_0}^{\infty}\overset{\circ}{K}_{ij}(y)dy,$$
$$i,j=1,2,...,n,\ x\in\mathbb{R}.$$

Therefore $\varepsilon_{ij}\ge\rho_{ij}\cdot\widetilde{a}_{ij}>0, i,j=1,2,...,n$. Let us now give examples of nonlinearities $\{G_j(u)\}_{j=1}^{n}$ and $\{\omega_{ij}(u)\}_{i,j=1}^{n\times n}$.

Examples of $\{G_j(u)\}_{j=1}^{n}$:

$Q_1)$ $G_j(u)=\left(\eta_j^*\right)^{\frac{p-1}{p}}\sqrt[p]{u},\ j=1,2,...,n,$ where $p\ge 2$ is a natural number, $u\in[0,+\infty)$,

$Q_2)$ $G_j(u)=\dfrac{\eta_j^*}{1-e^{-\eta_j^*}}\left(1-e^{-u}\right),\ j=1,2,...,n,\ u\in[0,+\infty)$,

$Q_3)$ $G_j(u)=\dfrac{1}{2}\left(\sqrt[p]{u}\left(\eta_j^*\right)^{\frac{p-1}{p}}+\dfrac{\eta_j^*}{1-e^{-\eta_j^*}}\left(1-e^{-u}\right)\right),\ j=1,2,...,n,\ u\in[0,+\infty)$.

Examples of $\{\omega_{ij}(t,u)\}_{i,j=1}^{n\times n}$:

$V_1)$ $\omega_{ij}(t,u)=\beta_{ij}(t)(1-e^{-u}), u\in[0,+\infty),\ t\in\mathbb{R}, i,j=1,2,...,n,$

$V_2)$ $\omega_{ij}(t,u)=\beta_{ij}(t)\frac{u}{u+1},\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$

$V_3)$ $\omega_{ij}(t,u)=\beta_{ij}(t)\cdot th(u),\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$

where

$$th(u):=\frac{e^u-e^{-u}}{e^u+e^{-u}}.$$

Note that in all examples $V_1)-V_3)$ it is assumed that $\beta_{ij}\in C(\mathbb{R}),\ i,j=1,2,...,n$. Let us verify conditions I)-III) on the example $Q_2)$. Firstly, it is obvious that $G_j\in C[0,+\infty)$, $G_j(0)=0$, $j=1,2,...,n$. Since $G_j''(u)=-\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}<0,\ u\in[0,+\infty)$, $j=1,2,...,n$, therefore, the functions $\{G_j(u)\}_{j=1}^n$ are concave. $G_j'(u)=\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}>0$, $u\in[0,+\infty)$, $j=1,2,...,n$, $G_j(u)\uparrow$ with respect to u on $[0,+\infty)$, $j=1,2,...,n$. Obviously, $G_j(\eta_j^*)=\eta_j^*$, $j=1,2,...,n$. It remains to show that $G_j(u)\geq u$, $u\in[0,\eta_j^*]$, $j=1,2,...,n$. Let us consider the following functions on the segment $[0,\eta_j^*]$:

$$\chi_j(u)=\frac{\eta_j^*}{1-e^{-\eta_j^*}}(1-e^{-u})-u,\ u\in[0,\eta_j^*],\ j=1,2,...,n.$$

Note that $\chi_j(0)=0,\ \chi_j(\eta_j^*)=0$, $\chi_j''(u)=-\frac{\eta_j^*}{1-e^{-\eta_j^*}}\cdot e^{-u}<0$, $j=1,2,...,n$. Therefore $\chi_j(u)\geq 0,\ u\in[0,\eta_j^*],\ j=1,2,...,n$.

Let us now verify the conditions A) - D) for the example $V_2)$. Firstly, it is obvious that $\omega_{ij}(t,0)=0,\ t\in\mathbb{R},\ i,j=1,2,...,n$. Since

$$\frac{\partial\omega_{ij}(t,u)}{\partial u}=\beta_{ij}(t)\frac{1}{(u+1)^2}>0,\ u\in[0,+\infty),\ t\in\mathbb{R},\ i,j=1,2,...,n,$$

$\omega_{ij}(t,u)\uparrow$ with respect to u on the set $[0,+\infty),\ i,j=1,2,...,n$. From the representation of $V_2)$ it follows that

$$\sup_{u\in[0,+\infty)}(\omega_{ij}(t,u))=\beta_{ij}(t),\ t\in\mathbb{R},\ i,j=1,2,...,n.$$

For the rest of examples $Q_1),\ Q_3),\ V_1)$ and $V_2)$ the verification of the corresponding conditions is made similarly.

For the sake of completeness, let us also give specific examples of $\{\mathring{K}_{ij}(x)\}_{i,j=1}^{n\times n}$, $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$ and $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$.

Examples of $\{\mathring{K}_{ij}(x)\}_{i,j=1}^{n\times n}$:

$T_1)$ $\mathring{K}_{ij}(x)=\frac{2a_{ij}}{\sqrt{\pi}}e^{-x^2},\ x\in\mathbb{R},\ i,j=1,2,...,n,$

$T_2)$ $\mathring{K}_{ij}(x)=\int\limits_a^b e^{-|x|s}d\sigma_{ij}(s),\ x\in\mathbb{R},\ i,j,=1,2,...,n,$

where $\sigma_{ij}(s),\ i,j=1,2,...,n$ are nondecreasing and continuous functions on the set $[a,b),\ 0<a<b\leq+\infty$, moreover,

$$\int\limits_a^b\frac{1}{s}d\sigma_{ij}(s)=a_{ij},\ i,j=1,2,...,n.$$

Examples of $\{\lambda_{ij}(x)\}_{i,j=1}^{n\times n}$:

$S_1)$ $\lambda_{ij}(x)=1-(1-\rho_{ij})D(x),\ x\in\mathbb{R},\ i,j=1,2,...,n,$ where $D(x):=\begin{cases}e^x, & x<0\\ 1, & x\geq 0\end{cases},$

$S_2)$ $\lambda_{ij}(x)=1-\frac{(1-\rho_{ij})}{2}\cdot(th(x)+1),\ x\in\mathbb{R},\ i,j=1,2,...,n.$

Examples of $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$:

$J_1)$ $\beta_{ij}(x) = \dfrac{\eta_j}{a_{ij}}\gamma_{ij}(x),\ x\in\mathbb{R},\ i,j=1,2,...,n,$ given that $\gamma_{ij}(x)\uparrow$ with respect to x on $\mathbb{R}$, $i,j=1,2,...,n,$

$J_2)$ $\beta_{ij}(x) = \dfrac{\eta_j\gamma_{ij}(x)}{a_{ij}-\gamma_{ij}(x)},\ x\in\mathbb{R},\ i,j=1,2,...,n,$ given that $\gamma_{ij}(x)\uparrow$ with respect to x on $\mathbb{R}$, $i,j=1,2,...,n.$

Let us take a look at example $J_2)$. First of all let us give examples of functions $\{\gamma_{ij}(x)\}_{i,j=1}^{n}$ that satisfy the condition in $J_2)$. For example in the case of $W_1)$ the functions $\gamma_{ij}(x)$ allow the following representation:

$$\gamma_{ij}(x) = \int\limits_{-\infty}^{x} \overset{\circ}{K}_{ij}(x-t)(1-\lambda_{ij}(x))dt = a_{ij}(1-\lambda_{ij}(x)),\ x\in\mathbb{R},\ i,j=1,2,...,n.$$

Note that in examples $S_1)$ and $S_2)$ the functions $(1-\lambda_{ij}(x))$, $i,j=1,2,...,n$ are increasing on $\mathbb{R}$. Therefore, if as a $\lambda_{ij}(x)$, $i,j=1,2,...,n$ we choose examples $S_1)$ and $S_2)$ we will obtain the monotonicity of the functions $\{\gamma_{ij}(x)\}_{i,j=1}^{n}$ on the set $\mathbb{R}$. But in that case the functions $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$ in examples $J_2)$ also will be nondecreasing on the set $\mathbb{R}$. For example $J_2)$ inequality (1.4) is automatically satisfied. The corresponding conditions on the functions $\{\beta_{ij}(x)\}_{i,j=1}^{n\times n}$ for example $J_1)$ are verified similarly.

It is interesting to note, that the problem of uniqueness of the solution for system (1.2) in conical segments $\{[0,\eta_j^*]\}_{j=1}^{n}$ still remains open. For system (1.1) the uniqueness of the solution (in the class of bounded on $\mathbb{R}$ vector-functions) fails, since, according to the results of Theorem 2.1, system (1.1) has a one-parameter family of nonnegative (nontrivial) and bounded (on $\mathbb{R}$) solutions.

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