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COUNTABLY GENERATED EXTENSIONS OF *QTAG*-MODULES

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Communicated by V.I. Burenkov

Key words: QTAG-modules, totally projective modules, h-pure submodules, isotype submodules.

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Abstract. Let the $QTAG$-module M be the set-theoretic union of a countable collection of isotype submodules S_k of countable length. For $0 \leq k < \omega$ we prove that M is totally projective if S_k is totally projective. Certain related assertions in this direction are also presented.

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1 Introduction and terminology

Modules are the natural generalizations of abelian groups. Among many generalizations of torsion abelian groups the notion of TAG-modules and its related properties have attracted considerable attention since 1976 (see, for example, [1, 19]). Following [17], a module M_R is called a TAG-module if it satisfies the following two conditions.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfying only condition (I) is called a $QTAG$-module (see [18]). This is a very fascinating structure that has been the subject of research of many authors. They studied different notions and structures of $QTAG$-modules and developed the theory of these modules by introducing several notions, investigated some interesting properties and characterized different submodules of $QTAG$-modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. The present work translates a few of the ideas of the abelian p-groups over to the area of $QTAG$-modules and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

Throughout our discussion all the rings R here are associative with unity $(1 \neq 0)$ and modules M are unital $QTAG$-modules. A module M over a ring R is called uniserial if it has a unique decomposition series of finite length. A module M is called uniform if intersection of any two of its non-zero submodules is non-zero. An element x in M is called uniform if xR is a non-zero uniform (hence uniserial) module. For any module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element x of M, its exponent $e(x)$ is defined to be equal to the decomposition length $d(xR)$. For any $0 \neq x \in M$, $H_M(x)$ (the height of x in M) is defined by $H_M(x) = \sup\{d(yR/xR) : y \in M,\ x \in yR \text{ and } y \text{ uniform}\}$. For $k \geq 0$,

$H_k(M) = \{x \in M | \ H_M(x) \geq k\}$ denotes the submodule of M generated by the elements of height at least k and for some submodule N of M, $H^k(M) = \{x \in M | \ d(xR/(xR \cap N)) \leq k\}$ is the submodule of M generated by the elements of exponents at most k.

Let us denote by M^1, the submodule of M, containing uniform elements of infinite height. The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded [17], if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure [10] in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$.

For a $QTAG$-module M and an ordinal α, $H_\alpha(M)$ is defined as $H_\alpha(M) = \cap_{\beta<\alpha} H_\beta(M)$. For an ordinal α, a submodule N of M is said to be α-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$ and a submodule N of M is said to be isotype in M, if it is α-pure for every ordinal α [15]. For an ordinal α, a submodule $N \subseteq M$ is an α-high submodule [14] of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially.

A submodule $N \subset M$ is nice [12] in M, if $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for all ordinals α, i.e. every coset of M modulo N may be represented by an element of the same height. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

The major aim here is to extend Theorem 1 from [8] to two important classes of $QTAG$-modules the first one the class of summable modules, whereas the second one the class of α-modules, where α is a limit ordinal. The work is organized thus: in the first section, i.e. here, we have studied the basic notation as well as the terminology necessary for applicable purposes. In the second section, we proceed by proving the preliminary results, and in the third one we obtain a new simplified but more convenient for us major result, when a countable number of h-pure submodules can be a countable number of isotype submodules that seem to be interesting. In the fourth section, several applications of Theorem 3.1 in terms of total projectivity are provided which are of some importance.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [15]. Many results, stated in the present paper, are clearly generalizations from the papers [7, 8, 9]. For the better understanding of the mentioned topic here one must go through the papers [2, 3]. Most of our notations and terminology will be standard being in agreement with [4] and [5].

2 Preliminary results

We begin by defining a μ-module.

Definition 1. Let μ be a cardinal. We say that a $QTAG$-module M is a μ-module if M has cardinality μ and each submodule of M having cardinality less than μ is a direct sum of uniserial modules.

The question whether all μ-modules are direct sums of uniserial modules, has a significance in the theory of $QTAG$-modules. For every infinite cardinal μ there exists a μ-module that is a direct sum of uniserial modules with $\mu \geq \aleph_k$ and $k \geq 0$. We conjecture that the problem has a negative answer in general, but nevertheless we shall inspect in the sequel its validity for a finite cardinal μ. This follows immediately from the well-known structure of finitely generated $QTAG$-modules. However, we now have the following result.

Theorem 2.1. *Suppose that M is a QTAG-module and ω is a first limit ordinal. If M is an $\aleph_\omega$-module, then M is a direct sum of uniserial modules.*

Proof. Let M be a $QTAG$-module of cardinality $\aleph_\omega$ such that each submodule of M having cardinality less than $\aleph_\omega$ is a direct sum of uniserial modules. Since any infinite submodule can be imbedded in an h-pure submodule of the same cardinality, it easily follows that M is the union of an ascending chain of h-pure submodules S_k of M such that $g(S) = \aleph_k$ for $0 \leq k < \omega$. For each $k < \omega$, consider $S_k = \Sigma_{i\in I_k} x_i R$ and let α denote the smallest ordinal having cardinality $\aleph_\omega$. Then there exist submodules N_β of M, for $\beta < \alpha$, such that
(1) $N_0 = 0$.
(2) N_β is h-pure in M for each $\beta < \alpha$.
(3) $N_\beta + S_k$ is h-pure in M for each $\beta < \alpha$ and $k < \omega$.
(4) $N_{\beta+1} \supseteq N_\beta$ for each β such that $\beta + 1 < \alpha$.
(5) $N_{\beta+1}/N_\beta$ is countably generated for each β such that $\beta + 1 < \alpha$.
(6) $N_\beta \cap S_k = \Sigma_{i\in I_{k,\beta}} x_i R$ for $\beta < \alpha$ and $k < \omega$, where $I_{k,\beta}$ is a subset of I_k.
(7) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ if γ is a limit ordinal less than α.
(8) $M = \cup_{\beta<\alpha} N_\beta$.

Let $\lambda < \alpha$, and suppose that a submodule N_β of M with $\beta < \lambda$ such that conditions $(1) - (7)$ hold when α is replaced by λ. After this, let us assume that a submodule N_λ of M satisfying these conditions also. Then we have two cases to consider:

Case (i). λ is a limit ordinal. In this case, let us consider $N_\lambda = \cup_{\beta<\lambda} N_\beta$. Since N_β is h-pure for each $\beta < \lambda$, so that N_λ is h-pure in M. This, in tern, implies that $N_\lambda + S_k$ is h-pure in M. Thus, we see that condition (6) from definition of N_λ is satisfied. Now, we set $I_{k,\lambda} = \cup_{\beta<\lambda} I_{k,\beta}$ for each k, then it is easy to verify that $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$. Henceforth, all the conditions $(1) - (7)$ are satisfied for $\beta < \lambda$.

Case (ii). $\lambda - 1$ exists. Consider the submodule N_λ of M such that N_λ is a countably generated extension of $N_{\lambda-1}$ and
(2^+) N_λ is h-pure in M.
(3^+) $N_\lambda + S_k$ is h-pure in M for each $k < \omega$.
(6^+) $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$ for each k, where $I_{k,\lambda}$ is a subset of I_k.

Let P be any submodule of M containing $N_{\lambda-1}$. If $P/N_{\lambda-1}$ is countably generated, there exists a submodule Q of M containing P with $g(Q/N_{\lambda-1}) \leq \aleph_0$ such that $Q/N_{\lambda-1}$ is h-pure in $M/N_{\lambda-1}$ and $[(Q/N_{\lambda-1}) + (S_k + N_{\lambda-1})/N_{\lambda-1}]/[(S_k + N_{\lambda-1})/N_{\lambda-1}]$ is h-pure in $(M/N_{\lambda-1})/[(S_k + N_{\lambda-1})/N_{\lambda-1}]$ for each $k < \omega$. From the h-purity of $N_{\lambda-1}$ and $S_k + N_{\lambda-1}$, we get that $Q + S_k = Q + S_k + N_{\lambda-1}$ is an h-pure submodule of M. Next, let J_k be a countably generated extension of the subset $I_{k,\lambda-1}$ such that $Q \cap S_k = \Sigma_{i\in J_k} x_i R$. It follows that there is an ascending chain

$$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_t \subseteq \ldots$$

of h-pure submodules of M such that Q_t is countably generated and $Q_t + S_k$ is h-pure in M for all $t, k < \omega$. Letting $Q_t \cap S_k \subseteq \Sigma_{i\in J_{k,t}} x_i R$, where $J_{k,t}$ is a countably generated extension of the subset $I_{k,\lambda-1}$ of I_k such that $Q_{t+1} \supseteq \Sigma_{i\in J_{k,t}} x_i R$ for all k. Define $N_\lambda = \cup_{t<\omega} Q_t$ and we set $I_{k,\lambda} = \cup_{t<\omega} J_{k,t}$, then $N_\lambda \cap S_k = \Sigma_{i\in I_{k,\lambda}} x_i R$. Thus, all conditions $(1) - (7)$ are satisfied for $\beta < \lambda$.

In addition, if the index set I_k is chosen to be the set of ordinals less than $\aleph_k$, then we can easily continue along condition (8) that $\beta \in I_{k,\beta}$ for all $k < \omega$ provided that $\beta \in I_k$.

In order to show that M is a direct sum of uniserial modules, it remains only to show that N_β is a direct sum of $N_{\beta+1}$ for each $\beta < \alpha$. Since N_β is h-pure and $N_{\beta+1}/N_\beta$ is countably generated, it is enough to show that $H_\omega(N_{\beta+1}/N_\beta) = 0$. Suppose that $y + N_\beta \in H_\omega(N_{\beta+1}/N_\beta) \subseteq H_\omega(M/N_\beta)$. Since $N_\beta + S_k$ is h-pure in M, then $y + N_\beta \in H_\omega((N_\beta + S_k)/N_\beta)$ where $y \in S_k$ for some k. In this connection, observe that $H_\omega((N_\beta + S_k)/N_\beta) = 0$. This completes the argument showing that

$(N_\beta + S_k)/N_\beta \cong S_k/(N_\beta \cap S_k)$ is a direct sum of uniserial modules. Setting $N_{\beta+1} = N_\beta + Q_\beta$, we obtain that $M = \Sigma_{\beta<\alpha} Q_\beta$, and the theorem is proved. □

The same idea is applicable even to $QTAG$-modules having cardinality $\aleph_\beta$, where β is cofinal with ω. So, we state without proof the following direct corollary.

Theorem 2.2. *If a QTAG-module M has cardinality $\aleph_\beta$ where β is cofinal with ω, then M is a direct sum of uniserial modules provided each submodule of M having cardinality less than $\aleph_\beta$ is a direct sum of uniserial modules.*

For freely use in the sequel, we obtain the following

Theorem 2.3. *Suppose that a QTAG-module M is a set-theoretic union of a countable number of h-pure submodules S_k for each k. If S_k is a direct sum of uniserial modules, then so is M.*

Proof. By appealing to the same reasoning as in Theorem 2.1, one may infer that the assertion follows. □

Now, we proceed by proving

Corollary 2.1. *Let S be a submodule of a QTAG-module M such that M is a direct sum of uniserial modules. Then S is a direct sum of uniserial modules.*

Proof. Suppose that M is a union of an ascending chain of h-pure submodules S_k such that S_k is bounded. Choose $P_k = S \cap S_k$ for each k. Let $Q_k \supseteq P_k$ be maximal in S with respect to $Q_k \cap H_k(S) = 0$. It is easy to see that Q_k is h-pure in S. Therefore, since Q_k is bounded, we get that Q_k is a direct sum of uniserial modules. Since $Q_k \supseteq P_k$, it follows immediately that S is a set-theoretic union of its submodules Q_k. Henceforth, according to Theorem 2.3, S is a direct sum of uniserial modules, as required. □

3 Main results

In Section 2, we have shown that if a $QTAG$-module M is the set-theoretic union of a countable number of h-pure submodules S_k, then M is a direct sum of uniserial modules if S_k is a direct sum of uniserial modules for each k. In the present section, we generalize this result by proving that if the submodules S_k are isotype then M must be totally projective provided that S_k is totally projective of countable length for each k. In particular, an ascending chain of isotype and totally projective submodules of countable length leads to a totally projective module.

Recall from [11] that an h-reduced $QTAG$-module M is said to be totally projective if it possesses a collection $\mathcal{N}$ consisting of nice submodules of M such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\Sigma_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated. Call a collection $\mathcal{N}$ of nice submodules of M which satisfies conditions (i), (ii) and (iii) a nice system for M. It is well-known that any countably generated h-reduced $QTAG$-module is totally projective by induction on the length of M. Thus the direct sum of any number of countably generated h-reduced $QTAG$-modules is totally projective.

Before presenting our main attainment, we prove the following working lemma.

Lemma 3.1. *Let α be an arbitrary ordinal and M a QTAG-module of countable length. Suppose that*

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\beta \subseteq \ldots, \ \beta < \alpha$$

is a chain of nice submodules of M satisfying the following conditions:
(a) $N_{\beta+1}/N_\beta$ is countably generated.
(b) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ where γ is a limit.
(c) $M = \cup_{\beta<\alpha} N_\beta$.
Then M is totally projective.

Proof. Let μ be the first uncountable ordinal such that $\alpha < \mu$. Therefore, M satisfies a nice system of countability. In fact, for an arbitrary α, it is not evident that conditions $(a)-(c)$ imply a nice system of countability.

By hypothesis, M embeds as an isotype submodule of a totally projective module, for a height-preserving monomorphism from N_β to $N_{\beta+1}$. Since the length of M is countable, M itself is totally projective and hence the result follows. □

The main result is now the following.

Theorem 3.1. *Let a QTAG-module M be a set-theoretic union of a countable number of isotype submodules S_k. If S_k is totally projective of countable length for each k, then M is totally projective.*

Proof. First we note that M has countable length. Let us assume that length of $M = \eta$ and let the submodules S_k be indexed by the nonnegative integers. For $k < \omega$, let $S_k = \Sigma_{i \in I_k} T_i$ where T_i is countably generated for each i. Suppose that

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\beta \subseteq \ldots, \ \beta < \lambda$$

is a chain of submodules of M satisfying the following conditions:
(a) $N_{\beta+1}/N_\beta$ is countably generated.
(b) $N_\gamma = \cup_{\beta<\gamma} N_\beta$ where γ is a limit.
(c) $N_\beta \cap S_k = \Sigma_{i \in I_{k,\beta}} T_i$ for each k and β, where $I_{k,\beta}$ is a subset of I_k.
(d) $\langle H_\lambda(M), N_\beta\rangle \cap \langle S_k, N_\beta\rangle = \langle H_\lambda(S_k), N_\beta\rangle$ for each k and β and for each $\lambda \leq \eta$.

We consider two possibilities. Firstly, if λ is a limit ordinal, we define $N_\lambda = \cup_{\beta<\lambda} N_\beta$ and see that conditions $(a)-(d)$ are satisfied for the chain of submodules N_β. Secondly, if $\lambda - 1$ exists. For an arbitrary countably generated extension T of $N_{\lambda-1}$ in M, there exists a countably generated extension P of T such that $P \cap S_k = \Sigma_{i \in J_k} T_i$, for each k, where J_k is a countable generated extension of $I_{k,\lambda-1}$.

Next, with this in hand, we ascertain the same argument that there exists a countably generated extension Q of $P \supseteq T \supseteq N_{\lambda-1}$ in M such that

$$\langle H_\lambda(M), Q\rangle \cap \langle S_k, Q\rangle = \langle H_\lambda(S_k), Q\rangle,$$

for all $\lambda \leq \eta$ and all $k < \omega$. Let $\{x_i\}_{i<\omega}$ be a set of representatives for the cosets of $P/N_{\lambda-1}$. For each triple (i,k,λ) with $i,k < \omega$ and $\lambda \leq \eta$ such that $y + z_k \in x_i + N_{\lambda-1}$ where $y \in H_\lambda(M)$ and $z_k \in S_k$. If we choose a representative $y = y_{i,k,\lambda}$ for the triple (i,k,λ), then clearly there are only a countable number of such representatives. Setting $Q_1 = \langle P, y_{i,k,\lambda}\rangle$, one may see that

$$\langle H_\lambda(M), P\rangle \cap \langle S_k, P\rangle \subseteq \langle H_\lambda(S_k), Q_1\rangle.$$

If we replace Q_1 by P, then Q_{j+1} is replaced by Q_j such that $Q_{j+1} = \langle Q_j, y_{i,k,\lambda}\rangle$. Hence, the desired properties follows if $Q = \cup_{j<\omega} Q_j$.

Furthermore, suppose that the conditions (a) and (d) holds. Then there exists a countable generated extension N_λ of $N_{\lambda-1}$ containing T that satisfies both conditions (c) and (d). Hence, a chain of submodules satisfying conditions $(a)-(d)$ is applicable to deduce that M is totally projective.

To complete the proof of the theorem, it remains only to show that N_β is nice in M for each β. It suffices to show that

$$H_\lambda(M/N_\beta) = \langle H_\lambda(M), N_\beta\rangle/N_\beta \tag{3.1}$$

for all $\lambda \leq \eta$. The proof is by induction on λ in conjunction with

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = \langle H_\lambda(S_k), N_\beta\rangle/N_\beta = H_\lambda(\langle S_k, N_\beta\rangle/N_\beta) \tag{3.2}$$

Clearly, for a given λ the second equality in condition (3.2) is a consequence of the first equality. However, the second equality is valid, because of condition (c). We claim that condition (3.2) hold good for $\lambda = \sigma$, where σ is a limit. Then it is suffices to show that condition (3.2) holds for all $\lambda < \sigma$. By the choice of σ, condition (3.1) holds for all $\lambda < \sigma$. Hence, if $\lambda < \sigma$, we observe that

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = (\langle H_\lambda(M), N_\beta\rangle \cap \langle S_k, N_\beta\rangle)/N_\beta.$$

Thus, by condition (d), we write

$$H_\lambda(M/N_\beta) \cap \langle S_k, N_\beta\rangle/N_\beta = \langle H_\lambda(S_k), N_\beta\rangle/N_\beta,$$

and so condition (3.2) holds for $\lambda = \sigma$. This gives that

$$H_\sigma(M/N_\beta) \subseteq \cup_{k<\omega}\langle H_\sigma(S_k), N_\beta\rangle/N_\beta \subseteq \langle H_\sigma(M), N_\beta\rangle/N_\omega,$$

which allows us to infer that N_β is nice in M for each β. □

4 Applications

The purpose of the present section is to explore some structural corollaries of Theorem 3.1. Several such applications are now presented.

4.1 Summability

Singh [17] proved that a $QTAG$-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. Apparently, M is a direct sum of uniserial modules if and only if $Soc(M) = \oplus_{k\in\omega}S_k$ and $H_M(x) = k$ for every $x \in S_k$. This led to the notion of summable modules, see, [16]. Let us recall the definition: an h-reduced $QTAG$-module M is summable if $Soc(M) = \oplus_{\beta<\alpha}N_\beta$, where N_β is the set of all elements of $H_\beta(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M. It is self-evident that a $QTAG$-module of length ω is a direct sum of uniserial modules if and only if the $QTAG$-module is summable. However, for the sake of completeness, the following corollaries are immediate.
(i) Countably generated h-reduced $QTAG$-modules are summable.
(ii) Direct sums of countably generated h-reduced $QTAG$-modules are summable.
(iii) Isotype submodules of summable modules of countable length are summable.

We start here with the following easy observation.

Theorem 4.1. *Let M be a summable $QTAG$-module of countable length α. If $M/H_\beta(M)$ is totally projective for each limit ordinal $\beta < \alpha$, then M is totally projective.*

Proof. The proof is by induction on α. If there is a limit ordinal β such that both $H_\beta(M)$ and $M/H_\beta(M)$ are totally projective, then M is itself totally projective. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \dots$ be an increasing sequence of ordinals whose limit is α. We choose $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq \dots$ be an ascending chain of submodules, so that S_k is α_k-high in M. Observe that $Soc(S) = Soc(M)$, where $S = \cup_{k<\omega} S_k$. Therefore, since S is h-pure in M, we get that $S = M$. Thus in view of Theorem 3.1, it suffices to show that S_k is totally projective for each k. However, we know that S_k is isomorphic to an isotype submodule of $M/H_{\alpha_k}(M)$ under the natural map. Henceforth, a simple technical argument applies to get that S_k is totally projective which gives the desired total projectivity of M. □

The following statement generalizes Theorem 2.2.

Theorem 4.2. *Let M be a QTAG-module of cardinality $\aleph_\beta$ where β is cofinal with ω. If each submodule of M having cardinality less than $\aleph_\beta$ is contained in a totally projective submodule of M having countable length, then M is totally projective.*

Proof. Assume that S_k is a submodule of M having cardinality $\aleph_{\beta_k}$, where $\beta_1 < \beta_2 < \cdots < \beta_k < \dots$ be an increasing sequence of ordinals whose limit is β. Note that if S_k is contained in an isotype submodule P_k of M having the same cardinality $\aleph_{\beta_k}$ as that of S_k, then P_k is contained in a totally projective submodule Q_k of M having countable length. Since P_k is isotype in Q_k, P_k is totally projective. It follows that M is the union of a countable ascending chain $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_k \subseteq \dots$ of isotype and totally projective submodules P_k. One seeing readily in view of Theorem 3.1 that M is totally projective, as wanted. □

This brings us to another technical observation.

Theorem 4.3. *Let M_1 and M_2 be QTAG-modules of countable type λ and suppose that M_1 is totally projective. If, for each ordinal $\beta \leq \lambda$, there exists a height-preserving isomorphism between $Soc(M_1/H_{\beta\omega}(M_1))$ and $Soc(M_2/H_{\beta\omega}(M_2))$, then M_2 is totally projective and $M_1 \cong M_2$.*

Proof. By hypothesis, there exists a height-preserving isomorphism between $Soc(M_1)$ and $Soc(M_2)$. It is easy to see that M_1 and M_2 have the same *Ulm* invariants (and are therefore isomorphic) if M_2 is totally projective.

We induct on λ to show that M_2 is totally projective. Since M_1 is summable, M_2 is also summable. If $\lambda = 1$, then M_2 is a direct sum of uniserial modules. Thus, assuming that $\lambda > 1$ and that $\lambda - 1$ exists. Observe that

$$M_1/H_{\omega(\lambda-1)}(M_1)/H_{\omega\beta}(M_1/H_{\omega(\lambda-1)}(M_1)) \cong M_1/H_{\omega\beta}(M_1),$$

for all $\beta \leq \lambda - 1$, and similarly for M_2. Hence in virtue of inductive hypothesis, $M_2/H_{\omega(\lambda-1)}(M_2)$ is totally projective. Since both $Soc(H_{\omega(\lambda-1)}(M_1))$ and $Soc(H_{\omega(\lambda-1)}(M_2))$ have a height-preserving isomorphism, we deduce that $H_{\omega(\lambda-1)}(M_2)$ is a direct sum of uniserial modules. This guarantees that M_2 is totally projective.

In the remaining case when λ is a limit ordinal, we assume that $Soc(M_1) = \Sigma U_\beta$ and $Soc(M_2) = \Sigma V_\beta$ be decompositions of $Soc(M_1)$ and $Soc(M_2)$, respectively, such that $H_{U_\beta}(x) = H_{V_\beta}(y) = \beta$, for some $x \in U$, $y \in V$. Let $1 \leq \lambda_1 \leq \lambda_2 < \cdots < \lambda_k < \dots$ be an increasing sequence of ordinals whose limit is λ, we choose $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \dots$ and $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k \subseteq \dots$ be the ascending chain of submodules such that $Soc(N_k) = \Sigma_{\beta<\omega\lambda_k} U_\beta$ and $Soc(L_k) = \Sigma_{\beta<\omega\lambda_k} V_\beta$. Note that $M_1 = \cup_{k<\omega} N_k$ and $M_2 = \cup_{k<\omega} L_k$ since $\cup_{k<\omega} N_k$ and $\cup_{k<\omega} L_k$ are h-pure submodules of M_1 and M_2, respectively, containing $Soc(M_1)$ and $Soc(M_2)$.

What remains to show is that L_k is totally projective. Our future aim, which we pursue, is to check the existence of a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k))$ and $Soc(L_k/H_{\omega\beta}(L_k))$

for each $\beta \leq \lambda_k$. To that goal, we have two cases to consider. First, if $\beta = \lambda_k$, then there is a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k)) = Soc(N_k)$ and $Soc(L_k/H_{\omega\beta}(L_k)) = Soc(L_k)$ by the choice of N_k and L_k. For the second case where $\beta < \lambda_k$, it is easily observed that $M_1 = \langle N_k, H_{\omega\beta}(M_1)\rangle$ since N_k is $\omega\lambda_k$-high in M_1. Similarly, $M_2 = \langle L_k, H_{\omega\beta}(M_2)\rangle$. Therefore, $M_1/H_{\omega\beta}(M_1) \cong N_k/H_{\omega\beta}(N_k)$ and $M_2/H_{\omega\beta}(M_2) \cong L_k/H_{\omega\beta}(L_k)$. Then there exists a height-preserving isomorphism between $Soc(N_k/H_{\omega\beta}(N_k))$ and $Soc(L_k/H_{\omega\beta}(L_k))$ for each $\beta \leq \lambda_k$. It follows by induction hypothesis that L_k is totally projective.

In addition, since L_k is isotype in M_2 and $M_2 = \cup_{k<\omega} L_k$, so referring to Theorem 3.1, we can conclude that M_2 is totally projective, as promised. □

4.2 α-modules

For the definition of an α-module, the reader can see [13] or [6] where it is given in all details. However, for a convenience of the reader, we shall include it in the text. A *QTAG*-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$.

It is well-known that every totally projective module is an α-module. Besides, it is simple to checked that an α-module of length α is a direct sum of countably generated modules if and only if it is summable.

Now, we are ready to formulate the following

Theorem 4.4. *Let M be a QTAG-module of length α such that M is an α-module. If M is a set-theoretic union of countable number of submodules S_k where the heights of the nonzero uniform elements of S_k in M are bounded by some ordinal $\alpha_k < \alpha$, then M is totally projective.*

Proof. Suppose that $M = \cup_{k<\omega} S_k$ where S_k is isotype of length $\alpha_k < \alpha$ for each k. Since M has countable length, then S_k is totally projective. Because $S_k \cong \langle S_k, H_{\alpha_k}(M)\rangle / H_{\alpha_k}(M)$ is isomorphic to an isotype submodule of a totally projective module $M/H_{\alpha_k}(M)$ having countable length α_k, it easily follows that M is totally projective by the usage of Theorem 3.1 if α is countable.

Similarly, if the interval (γ, α) of ordinals is countable for some γ less than α, then $H_\gamma(M)$ is totally projective. Indeed, since $H_\gamma(M)$ has countable length and $H_\gamma(M) = \cup_{k<\omega} H_\gamma(S_k)$ such that $H_\gamma(S_k)$ is isotype in $H_\gamma(M)$. Therefore, $H_\gamma(M)$ is totally projective if (γ, α) is countable for some $\gamma < \alpha$. Moreover, since M is an α-module, then $M/H_\gamma(M)$ is totally projective. This means that M is totally projective, and the result follows for (γ, α) is countable with $\gamma < \alpha$.

We next assume that (γ, α) is uncountable for every ordinal γ less than the length α of M. In particular, if $\gamma < \alpha$, then $\gamma + \omega < \alpha$. Without loss of generality, we may assume that S_k is maximal with $S_k \cap H_{\alpha_k}(M) = 0$. Then S_k is an α_k-high submodule of M.

Finally, consider the case $\alpha_k = \sigma_k + \omega$ for some ordinal σ_k. Suppose now the ordinal α_k has the form $\alpha_k + \omega$ for some $\alpha_k < \alpha$. Since $\alpha_k = \sigma_k + \omega$ and S_k is an α_k-high in M, we have $M = \langle S_k, H_{\sigma_k}(M)\rangle$ and $M/H_{\sigma_k}(M) \cong S_k/H_{\sigma_k}(S_k)$ for each k. Consequently, $S_k/H_{\sigma_k}(S_k)$ is totally projective, and we get $H_{\sigma_k}(S_k)$ is totally projective since it is isomorphic to an isotype submodule of the totally projective module $H_{\sigma_k}(M)/H_{\alpha_k}(M)$. Therefore, we conclude that S_k is totally projective, and again the application of Theorem 3.1 leads to M being totally projective, as expected. □

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