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ALGEBRAIC PROOFS OF CHARACTERIZING REVERSE ORDER LAW FOR CLOSED RANGE OPERATORS IN HILBERT SPACES

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Key words: Moore-Penrose inverse, reverse order law, closed range operator.

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Abstract. We present more than 60 results, including some range inclusion results to characterize the reverse order law for the Moore-Penrose inverse of closed range Hilbert space operators. We use the basic properties of the Moore-Penrose inverse to prove the results. Some examples are also provided to illustrate failure cases of the reverse order law in an infinite-dimensional setting.

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1 Introduction

One of the fundamental research problems in the theory of generalized inverses of matrices is to establish reverse order laws for generalized inverses of matrix products. It was Erik Ivar Fredholm who seemed to have first mentioned the concept of generalized inverse in 1903. He formulated a pseudoinverse for a linear integral operator, which is not invertible in the ordinary sense. Hilbert, Schmidt, Bounitzky, Hurwitz and other mathematicians had studied the generalized inverses of integral operators and differential operators before Moore introduced the generalized inverse of matrices by algebraic methods in 1920 [17]. Bjerhammar rediscovered Moore's inverse and also noted the relationship of generalized inverses to solutions of linear systems in 1951 [5]. In 1955, Penrose [21] extended Bjerhammar's results and showed that Moore's inverse for a given matrix A is the unique matrix X satisfying the four equations:

$$AXA = A; XAX = X; (AX)^* = AX; (XA)^* = XA.$$

In honour of Moore and Penrose, this unique inverse is now commonly called the Moore-Penrose inverse and is denoted by $A^\dagger$. Meanwhile, generalized inverses were defined for operators by Tseng [24], Murray and von Neumann [19], Nashed [20] and others. Beutler discussed generalized inverses for both bounded and unbounded operators with closed and arbitrary ranges [3, 4]. Throughout the years, the Moore-Penrose inverse was extensively studied. One of the primary reasons for considering the Moore-Penrose inverse is solving systems of linear equations, which constitutes an important application in various fields.

It is well known that the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ is not true in general for various generalized inverses such as the Moore-Penrose inverse, Drazin inverse etc. Cline attempted to find a reasonable representation for the Moore-Penrose inverse of the product of matrices [9] and Greville found some necessary and sufficient conditions for the reverse order law to hold in matrix setting [13]. The reverse order law problem for bounded linear operators on Hilbert spaces was analyzed by Bouldin [6, 7] and Izumino [16]. The theory of generalized inverses on infinite-dimensional Hilbert spaces can be found in [2, 15, 25].

In this paper, we present algebraic proofs of some characterizations of reverse order law for the Moore-Penrose inverses of closed range Hilbert space operators. In the second section, we collect some definitions and lemmas which will be used in the sequel. We start the main section with some examples to show that the reverse order law does not hold for Hilbert space operators in general. In total, we present 61 results, including some range inclusion results to characterize the reverse order law in this setting. We extend the results of Arghiriade [1] and Tian [22, 23] to infinite-dimensional Hilbert spaces.

2 Preliminaries

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We abbreviate $\mathcal{B}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote by A^*, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the adjoint, the null-space and the range of A. An operator $A \in \mathcal{B}(\mathcal{H}_1)$ is said to be self-adjoint (Hermitian) if $A = A^*$. An operator $A \in \mathcal{B}(\mathcal{H}_1)$ is said to be a projection if $A^2 = A$. A projection is said to be orthogonal if $A^2 = A = A^*$. The Moore-Penrose inverse of $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the operator $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ which satisfies the Penrose equations

$$AXA = A \tag{2.1}$$
$$XAX = X \tag{2.2}$$
$$(AX)^* = AX \tag{2.3}$$
$$(XA)^* = XA. \tag{2.4}$$

A matrix X is called a $\{i, \dots, j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$ if it satisfies the $i^{th}, \dots, j^{th}$ conditions of the Penrose equations. The collection of all $\{i, \dots, j\}$-generalized inverses of A is denoted by $A\{i, \dots, j\}$. If the Moore-Penrose inverse of A exists, then it is unique and it is denoted by $A^\dagger$. It should be noted that $A^\dagger$ is bounded if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}_2$.

For the sake of clarity as well as for easier reference, we mention the following properties of the Moore-Penrose inverse without proof [25].

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a closed range operator. The following statements hold:*

(i) $(A^\dagger)^\dagger = A$.

(ii) $(A^\dagger)^* = (A^*)^\dagger$.

(iii) $A = AA^*(A^*)^\dagger = (A^*)^\dagger A^* A$.

(iv) $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*$.

(v) $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$, $(A^*A)^\dagger = A^\dagger (A^*)^\dagger$.

(vi) $A^* = A^*AA^\dagger = A^\dagger AA^*$.

(vii) $\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^\dagger)$.

(viii) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$.

(ix) $AA^\dagger = P_{\mathcal{R}(A)}$ *and* $A^\dagger A = P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^\dagger)}$.

(x) *If* $\mathcal{H}_1 = \mathcal{H}_2$, *then* $(A^n)^\dagger = (A^\dagger)^n$ *for* $n \geq 1$.

Here, $P_{\mathcal{R}(A)}$ *and* $P_{\mathcal{R}(A^*)}$ *denote the projections onto* $\mathcal{R}(A)$ *and* $\mathcal{R}(A^*)$*, respectively. We use* $A^{\dagger *}$ *instead of* $(A^\dagger)^*$ *throughout the paper.*

Lemma 2.2 ([8], Theorem 1)**.** *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a closed range operator such that $\mathcal{R}(A) = \mathcal{R}(A^*)$. Then $AA^\dagger = A^\dagger A$ and $A^n A^\dagger = A^{n-1}, n \geq 2$.*

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ be a projection. Then P is Hermitian if and only if $P = PP^*P$.*

Proof. Suppose $P = PP^*P$ and P is a projection. Let $B = P - P^*$. Then it is easy to verify that $B^3 = 0$ and $\mathcal{R}(B) = \mathcal{R}(B^*)$. By Lemma 2.2, $B^3(B^\dagger)^2 = 0$ gives $B = 0$. Thus, P is Hermitian. Converse follows directly. □

Remark 1. If P is an orthogonal projection, then P satisfies all the Penrose equations and hence $P^\dagger = P$.

Lemma 2.4 ([26], Lemma 1.3)**.** *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then*

(i) $B \in A\{1, 3\} \Leftrightarrow A^*AB = A^*$,

(ii) $B \in A\{1, 4\} \Leftrightarrow BAA^* = A^*$.

Theorem 2.1 ([12], Theorem 1)**.** *Let A and B be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) *there exists a bounded operator C on $\mathcal{H}$ so that $A = BC$.*

Theorem 2.2 ([14], Theorem 7.20)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then there exist a measure space (X, Σ, μ), a bounded measurable real-valued function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that*

$$A = U^*TU,$$

where T is the multiplication operator given by $T\psi = f\psi,\ \forall \psi \in L^2(X, \mu)$.

Definition 1. Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. The operator A is called a *positive semi-definite operator* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

Lemma 2.5. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be a positive semi-definite operator such that $A^m = A^n$ for some natural numbers $m \neq n$. Then $A^2 = A$.*

Proof. We know that a positive semi-definite operator is self adjoint. By Theorem 2.2, we can write

$$A = U^*TU,$$

where T is the multiplication operator given by $T\psi = f\psi,\ \forall \psi \in L^2(X, \mu)$. Using the positive semi-definiteness of the operator, we get $f(x) \geq 0\ \forall x \in X$.

It is given that $A^m = A^n$ which implies

$$f^m \psi = f^n \psi,\ \forall \psi \in L^2(X, \mu). \tag{2.5}$$

Let $x_0 \in X$ and E be a subset of X such that $x_0 \in E$ and $\mu(E) \neq 0$. Since equation (2.5) holds for the characteristic function on E, we get $f^m(x_0)(1 - f^{n-m}(x_0)) = 0$, from which we can conclude $f(x_0) = 0$ or $f(x_0) = 1$ as $f(x) \geq 0\ \forall x \in X$. As x_0 is arbitrary $f(x) = 0$ or $f(x) = 1$ for all $x \in X$.

Now, $T^2\psi(x) = T(f(x)\psi(x)) = f(x)^2\psi(x) = f(x)\psi(x) = T\psi(x)$ for all $\psi(x) \in L^2(X, \mu)$. Also, $U^*T^2U = U^*TU \Rightarrow A^2 = A$. □

Lemma 2.6. *Let A and B be orthogonal projections on a Hilbert space $\mathcal{H}$ and $m > n \geq 1$. If $(ABA)^m = (ABA)^n$, then $AB = BA$.*

Proof. $ABA = ABBA = ABB^*A^* = AB(AB)^*$. Thus ABA is Hermitian and positive semi-definite as $AB(AB)^*$ is so. Then by Lemma 2.5, $(ABA)^m = (ABA)^n$ implies $(ABA)^2 = ABA$. Consider $(ABA-AB)(ABA-AB)^* = (ABA-AB)(ABA-BA) = (ABA)^2-ABABA-ABABA+ABA = 0$. Thus $ABA = AB$. Similarly, we can verify $(ABA-BA)(ABA-BA)^* = 0$, which gives $ABA = BA$. Thus, we get $AB = BA$. □

Lemma 2.7. *Let A and B be orthogonal projections on a Hilbert space $\mathcal{H}$ and $m > n \geq 1$. If $(AB)^m = (AB)^n$, then $AB = BA$.*

Proof. Since $(AB)^2A = ABABA = ABAABA = (ABA)^2$, thus $(AB)^mA = (ABA)^m$ for all $m \geq 1$. Now it is clear that$(AB)^m = (AB)^n$ gives $(ABA)^m = (ABA)^n$. Then by Lemma 2.6, we get $AB = BA$. □

3 Main results

We start the section with some examples to show that the reverse order law does not hold good for closed range Hilbert space operators in general.

Example 1. Let $\mathcal{H} = \ell_2$ be the space of all square summable sequences. For $x = (x_1, x_2, x_3, \dots) \in \mathcal{H}$, define $Ax = (x_1 + x_2, x_2, x_3, x_4, \dots)$ and $Bx = (x_1, 0, x_3, 0, x_5, \dots)$. Then

$$AB(x) = A(x_1, 0, x_3, 0, x_5, \dots) = (x_1, 0, x_3, 0, x_5, \dots) = Bx.$$

It can be verified easily that A, B and AB are bounded and have closed ranges. We see that

$$A^*(x) = (x_1, x_1 + x_2, x_3, x_4, \dots) \text{ and } B^*(x) = (x_1, 0, x_3, 0, x_5, \dots) = Bx.$$

Using the Euler-Knopp method for finding the Moore-Penrose inverses of operators ([25], p.327) we get

$$A^\dagger(x) = (x_1 - x_2, x_2, x_3, x_4, \dots).$$

By Remark 1, we get $B^\dagger = B$ *and* $(AB)^\dagger = B^\dagger$. Hence, $B^\dagger A^\dagger(x) = B^\dagger(x_1 - x_2, x_2, x_3, x_4, \dots) = (x_1 - x_2, 0, x_3, 0, x_5, \dots) \neq (AB)^\dagger(x)$, thus $(AB)^\dagger \neq B^\dagger A^\dagger$.

Example 2. Let $\mathcal{H} = \ell_2$. For $x = (x_1, x_2, x_3, \dots) \in \mathcal{H}$, define $Ax = (0, x_2, 0, x_4, 0, \dots)$ and $Bx = (x_1 + x_2, 2x_1 + 2x_2, x_3, x_4, \dots)$. Then $AB(x) = (0, 2x_1 + 2x_2, 0, x_4, \dots)$. It is easy to verify that A, B and AB are bounded and have closed ranges. Since $A^*(x) = (0, x_2, 0, x_4, 0, x_6, \dots) = Ax$, by Remark 1 we get $A^\dagger x = Ax$. Also, $B^*(x) = (x_1 + 2x_2, x_1 + 2x_2, x_3, x_4, \dots)$ and $B^\dagger(x) = (\frac{1}{10}(x_1 + 2x_2), \frac{1}{10}(x_1 + 2x_2), x_3, x_4, \dots)$ by the Euler-Knopp method. Thus, we get

$$B^\dagger A^\dagger(x) = B^\dagger(0, x_2, 0, x_4, 0, x_6, \dots) = (\frac{x_2}{5}, \frac{x_2}{5}, 0, x_4, 0, \dots)$$

and

$$(AB)^\dagger x = (\frac{x_2}{4}, \frac{x_2}{4}, 0, x_4, 0, \dots).$$

Hence $(AB)^\dagger \neq B^\dagger A^\dagger$. One can also check that $B^\dagger A^\dagger$ satisfies the third and fourth but not the first and second Penrose equations.

Lemma 3.1 ([16], Proposition 2.1)**.** *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

The results mentioned below in Theorems 3.1 to 3.4 are proved in C^*-algebra setting [18] and for the sake of completeness, we give the proof of those in Hilbert space setting. However, our proofs are much simpler than those available for the reverse order law for closed range Hilbert space operators.

In the following result, the existence of $(A^\dagger ABB^\dagger)^\dagger$ is guaranteed by Lemma 3.1.

Theorem 3.1. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(i) $ABB^\dagger A^\dagger AB = AB$;

(ii) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;

(iii) $BB^\dagger A^\dagger A$ *is a projection* ;

(iv) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;

(v) $A^\dagger ABB^\dagger$ *is a projection* ;

(vi) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(vii) $B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$.

Proof. (i)$\Rightarrow$ (ii): If $ABB^\dagger A^\dagger AB = AB$, then

$$\begin{aligned} B^\dagger A^\dagger &= (B^*B)^\dagger B^* A^* (AA^*)^\dagger \quad \text{(by Lemma 2.1 (iv))} \\ &= (B^*B)^\dagger (AB)^* (AA^*)^\dagger \\ &= (B^*B)^\dagger (ABB^\dagger A^\dagger AB)^* (AA^*)^\dagger \quad \text{(by the assumption)} \\ &= (B^*B)^\dagger B^* A^\dagger ABB^\dagger A^* (AA^*)^\dagger \\ &= B^\dagger A^\dagger ABB^\dagger A^\dagger \quad \text{(by Lemma 2.1 (iv))}. \end{aligned}$$

(ii)$\Rightarrow$(iii): Using (ii) we see that $(BB^\dagger A^\dagger A)^2 = BB^\dagger A^\dagger ABB^\dagger A^\dagger A = BB^\dagger A^\dagger A$. Hence, it shows that $BB^\dagger A^\dagger A$ is a projection.

(iii)$\Rightarrow$(iv): We have

$$\begin{aligned} BB^\dagger A^\dagger A(BB^\dagger A^\dagger A)^* BB^\dagger A^\dagger A &= BB^\dagger A^\dagger A(A^\dagger A)^*(BB^\dagger)^* BB^\dagger A^\dagger A \\ &= BB^\dagger A^\dagger A(A^\dagger A)(BB^\dagger)BB^\dagger A^\dagger A \\ &= BB^\dagger A^\dagger ABB^\dagger A^\dagger A = BB^\dagger A^\dagger A. \end{aligned}$$

Then by Lemma 2.3, we get $(BB^\dagger A^\dagger A)^* = BB^\dagger A^\dagger A$, since $BB^\dagger A^\dagger A$ is a projection. Thus $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$.

(iv)$\Rightarrow$(v): It is given that $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. We have

$$\begin{aligned} (A^\dagger ABB^\dagger)^2 &= A^\dagger ABB^\dagger A^\dagger ABB^\dagger = BB^\dagger A^\dagger AA^\dagger ABB^\dagger \\ &= BB^\dagger A^\dagger ABB^\dagger = A^\dagger ABB^\dagger BB^\dagger = A^\dagger ABB^\dagger. \end{aligned}$$

(v)$\Rightarrow$(vi): Using the fact that A is a projection if and only if A^* is a projection, it is easy to verify all Penrose equations.

(vi)$\Rightarrow$(vii): Pre- and post-multiplying by $B^\dagger$ and $A^\dagger$ respectively in (vi), we get the desired result.

(vii)$\Rightarrow$(i): It is given that $B^\dagger A^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$. We have

$$\begin{aligned} ABB^\dagger A^\dagger AB &= ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger AB \\ &= AA^\dagger ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger ABB^\dagger B \\ &= AA^\dagger ABB^\dagger B = AB, \end{aligned}$$

where all the equalities follow using the first Penrose equation. □

Next, we give ten equivalent conditions for $B^\dagger A^\dagger$ to be a $\{1,2,3\}$-generalized inverse of AB in Hilbert space setting. The existence of $(ABB^\dagger)^\dagger$ follows as the ranges of $ABB^\dagger$ and AB are equal.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $AB(AB)^\dagger = ABB^\dagger A^\dagger$;

(ii) $B^\dagger A^\dagger \in AB\{1,2,3\}$;

(iii) $BB^\dagger A^* AB = A^* AB$;

(iv) $(AB)(AB)^\dagger A = ABB^\dagger$;

(v) $A^* ABB^\dagger = BB^\dagger A^* A$;

(vi) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(vii) $B^\dagger (ABB^\dagger)^\dagger = B^\dagger A^\dagger$;

(viii) $B\{1,3\}A\{1,3\} \subseteq AB\{1,3\}$;

(ix) $B^\dagger A^\dagger \in AB\{1,3\}$;

(x) $(BB^*)^\dagger A^\dagger \in ABB^*\{1,2,3\}$.

Proof. We prove the equivalence of all the statements in the following order of implications:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(ix)} \Leftrightarrow \text{(viii)}, \text{(ix)} \Rightarrow \text{(i)}.$$

(i)⇒(ii): Since $ABB^\dagger A^\dagger = AB(AB)^\dagger$, post-multiplying by AB we get,

$$ABB^\dagger A^\dagger AB = AB(AB)^\dagger AB = AB.$$

Hence, $B^\dagger A^\dagger \in AB\{1\}$. By Theorem 3.1, $B^\dagger A^\dagger \in AB\{2\}$. Now using the assumption we get, $(ABB^\dagger A^\dagger)^* = (AB(AB)^\dagger)^* = AB(AB)^\dagger = ABB^\dagger A^\dagger$. Thus $B^\dagger A^\dagger \in AB\{1,2,3\}$.

(ii)⇒(iii): Suppose $B^\dagger A^\dagger \in AB\{1,2,3\}$. Then by Theorem 3.1, $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. Thus, we get

$$\begin{aligned} A^*AB &= A^*AA^\dagger ABB^\dagger B = A^*ABB^\dagger A^\dagger AB \\ &= A^*(ABB^\dagger A^\dagger)^* AB \quad (\text{since } B^\dagger A^\dagger \in AB\{3\} \\ &= A^* A^{\dagger^*} BB^\dagger A^* AB = (A^\dagger A)^* BB^\dagger A^* AB \\ &= A^\dagger ABB^\dagger A^* AB = BB^\dagger A^\dagger AA^* AB \\ &= BB^\dagger A^* AB \quad (\text{by Lemma 2.1 (vi)}). \end{aligned}$$

(iii)⇒(iv): We have

$$\begin{aligned} (AB)(AB)^\dagger A &= ((AB)(AB)^\dagger)^* A^{**} = (A^*AB(AB)^\dagger)^* \\ &= (BB^\dagger A^* AB(AB)^\dagger)^* \quad (\text{by the assumption}) \\ &= ((AB)(AB)^\dagger)^*(BB^\dagger A^*)^* = AB(AB)^\dagger ABB^\dagger \\ &= ABB^\dagger. \end{aligned}$$

(iv)⇒(v): Pre-multiplying the given condition by A^*, we get $A^*ABB^\dagger = A^*AB(AB)^\dagger A$. As the RHS of the previous equality is Hermitian, $A^*ABB^\dagger$ is also Hermitian and

$$A^*ABB^\dagger = (A^*ABB^\dagger)^* = BB^\dagger A^* A.$$

(v)⇒(vi): We show this by verifying all Penrose equations. Given that $A^*ABB^\dagger = BB^\dagger A^*A$. Pre-multiplying by $A^{\dagger *}$, we get $ABB^\dagger = AA^\dagger ABB^\dagger = (AA^\dagger)^*ABB^\dagger = A^{\dagger *}A^*ABB^\dagger = A^{\dagger *}BB^\dagger A^*A$. Hence

$$\begin{aligned}(ABB^\dagger)(BB^\dagger A^\dagger)(ABB^\dagger) &= ABB^\dagger A^\dagger ABB^\dagger = A^{\dagger *}BB^\dagger A^*AA^\dagger ABB^\dagger \\ &= A^{\dagger *}A^*ABB^\dagger BB^\dagger = (AA^\dagger)^*ABB^\dagger = ABB^\dagger.\end{aligned}$$

This shows that $BB^\dagger A^\dagger \in ABB^\dagger\{1\}$. Now,

$$\begin{aligned}BB^\dagger A^\dagger &= BB^\dagger A^*(AA^*)^\dagger \quad \text{(by Lemma 2.1 (iv))} \\ &= (ABB^\dagger)^*(AA^*)^\dagger = (ABB^\dagger BB^\dagger A^\dagger ABB^\dagger)^*(AA^*)^\dagger \\ &= (ABB^\dagger A^\dagger ABB^\dagger)^*(AA^*)^\dagger = BB^\dagger A^\dagger ABB^\dagger A^*(AA^*)^\dagger \\ &= BB^\dagger A^\dagger ABB^\dagger BB^\dagger A^\dagger \quad \text{(by Lemma 2.1 (iv)).}\end{aligned}$$

Thus, $BB^\dagger A^\dagger \in ABB^\dagger\{1,2\}$. Also,

$$(ABB^\dagger)(BB^\dagger A^\dagger) = (A^{\dagger *}A^*ABB^\dagger)(BB^\dagger A^\dagger) = A^{\dagger *}BB^\dagger A^*ABB^\dagger A^\dagger.$$

As the RHS of the last equality is Hermitian, $(ABB^\dagger)(BB^\dagger A^\dagger)$ is so. Similarly, we can prove $(BB^\dagger A^\dagger)(ABB^\dagger)$ is Hermitian. It ensures that $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$.
(vi)⇒(vii): Pre-multiplying the given condition by $B^\dagger$, we get (vii).
(vii)⇒(ix): It is clear that $ABB^\dagger(ABB^\dagger)^\dagger ABB^\dagger = ABB^\dagger$. Then by (vii), we have $ABB^\dagger A^\dagger ABB^\dagger = ABB^\dagger$. Post-multiplying by B we get $ABB^\dagger A^\dagger AB = AB$. Thus $B^\dagger A^\dagger \in AB\{1\}$. Also, $ABB^\dagger A^\dagger$ is Hermitian since $ABB^\dagger A^\dagger = ABB^\dagger(ABB^\dagger)^\dagger$. Thus $B^\dagger A^\dagger \in AB\{1,3\}$.
(ix)⇒(viii): Let $CD \in B\{1,3\}A\{1,3\}$ where $C \in B\{1,3\}$ and $D \in A\{1,3\}$. By Lemma 2.4, C and D satisfy $B^*BC = B^*$ and $A^*AD = A^*$. Also, we note that $B^\dagger BC = (B^*B)^\dagger B^*BC = (B^*B)^\dagger B^* = B^\dagger$ and, similarly we can prove $A^\dagger AD = A^\dagger$. By using $B^\dagger A^\dagger \in AB\{1,3\}$, we get

$$\begin{aligned}(AB)^*(AB)CD &= (ABB^\dagger A^\dagger AB)^*ABCD = (AB)^*ABB^\dagger A^\dagger ABCD \\ &= (AB)^*AA^\dagger ABB^\dagger BCD \quad \text{(by Theorem 3.1)} \\ &= (AB)^*AA^\dagger ABB^\dagger D = (AB)^*ABB^\dagger A^\dagger AD \\ &= (AB)^*ABB^\dagger A^\dagger = (ABB^\dagger A^\dagger AB)^* = (AB)^*.\end{aligned}$$

(viii)⇒(ix): Obvious.
(ix)⇒(i): By the assumption, we have $ABB^\dagger A^\dagger AB = AB$ and $(ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger$. Post-multiplying by $(AB)^\dagger$ in the first equation and taking adjoint on both sides, we get $AB(AB)^\dagger = ABB^\dagger A^\dagger$.
(ix)⇒(x): The first and third Penrose conditions follow easily from (ix). The second Penrose condition can be verified with the help of Theorem 3.1 (iv).
(x)⇒(ix): Since $(BB^*)^\dagger A^\dagger \in ABB^*\{1\}$, we get $ABB^*(BB^*)^\dagger A^\dagger ABB^* = ABB^*$ i.e., $ABB^\dagger A^\dagger ABB^* = ABB^*$ by Lemma 2.1 (iv) and Theorem 3.1. Post-multiplying by $B^{\dagger *}$ and using Lemma 2.1 (iii), we get

$$ABB^\dagger A^\dagger AB = AB.$$

Also, $ABB^*(BB^*)^\dagger A^\dagger = ABB^\dagger A^\dagger$ is Hermitian. It shows that $B^\dagger A^\dagger \in AB\{1,3\}$. □

The following result is similar to Theorem 3.2. It gives ten equivalent conditions for $B^\dagger A^\dagger$ to be a $\{1,2,4\}$-generalized inverse of AB in Hilbert space setting. Here, the existence of $(A^\dagger AB)^\dagger$ is guaranteed as the ranges of $(A^\dagger AB)^*$ and $(AB)^*$ are the same.

Theorem 3.3. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^{\dagger}AB = B^{\dagger}A^{\dagger}AB$;

(ii) $B^{\dagger}A^{\dagger} \in AB\{1,2,4\}$;

(iii) $ABB^{*} = ABB^{*}A^{\dagger}A$;

(vi) $B(AB)^{\dagger}AB = A^{\dagger}AB$;

(v) $A^{\dagger}ABB^{*} = BB^{*}A^{\dagger}A$;

(vi) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A$;

(vii) $(A^{\dagger}AB)^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$;

(viii) $B\{1,4\}A\{1,4\} \subseteq AB\{1,4\}$;

(ix) $B^{\dagger}A^{\dagger} \in AB\{1,4\}$;

(x) $B^{\dagger}(A^{*}A)^{\dagger} \in A^{*}AB\{1,2,4\}$.

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 3.4. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$;

(ii) $(AB)(AB)^{\dagger} = ABB^{\dagger}A^{\dagger}$ *and* $(AB)^{\dagger}AB = B^{\dagger}A^{\dagger}AB$;

(iii) $A^{*}AB = BB^{\dagger}A^{*}AB$ *and* $ABB^{*} = ABB^{*}A^{\dagger}A$;

(iv) $AB(AB)^{\dagger}A = ABB^{\dagger}$ *and* $B(AB)^{\dagger}AB = A^{\dagger}AB$;

(v) $A^{*}ABB^{\dagger} = BB^{\dagger}A^{*}A$ *and* $BB^{*}A^{\dagger}A = A^{\dagger}ABB^{*}$;

(vi) $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}$*and* $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A$;

(vii) $B^{\dagger}(ABB^{\dagger})^{\dagger} = B^{\dagger}A^{\dagger}$*and* $(A^{\dagger}AB)^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$;

(viii) $B\{1,3\}A\{1,3\} \subseteq AB\{1,3\}$ *and* $B\{1,4\}A\{1,4\} \subseteq AB\{1,4\}$;

(ix) $B^{\dagger}A^{\dagger} \in AB\{1,3,4\}$;

(x) $(BB^{*})^{\dagger}A^{\dagger} \in ABB^{*}\{1,2,3\}$ *and* $B^{\dagger}(A^{*}A)^{\dagger} \in A^{*}AB\{1,2,4\}$.

Proof. Follows from Theorems 3.2 and 3.3. □

Remark 2. Consider the operators A and B on $\mathcal{H}$ defined in Example 1. Then for all $x \in \mathcal{H}$, $A^{*}ABx = (x_1, x_1, x_3, 0, x_5, \ldots)$ and $BB^{\dagger}A^{*}ABx = (x_1, 0, x_3, 0, x_5, \ldots)$. Hence

$$A^{*}AB \neq BB^{\dagger}A^{*}AB \text{ and } ABB^{*}x = (x_1, 0, x_3, 0, x_5, \ldots) = ABB^{*}A^{\dagger}Ax.$$

Note that the conditions in (iii) of Theorem 3.4 are not satisfied and $(AB)^{\dagger} \neq B^{\dagger}A^{\dagger}$ which was shown in Example 1.

Theorem 3.5. *Let the conditions of Theorem* 3.1 *hold. Then the following statements hold:*

(i) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^*AB)$.

(ii) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*)$.

Proof. (i) Suppose that $B^\dagger = (AB)^\dagger A$. Pre-multiplying by AB, we get $ABB^\dagger = (AB)(AB)^\dagger A$, which is equivalent to $BB^\dagger A^*AB = A^*AB$ by Theorem 3.2. This implies $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ as $BB^\dagger$ is the projection onto $\mathcal{R}(B)$. Now by Lemma 2.1 (iv),

$$B^\dagger = (AB)^\dagger A = [(AB)^*(AB)]^\dagger (AB)^* A$$

implies that $B^{\dagger^*} = A^*AB[(AB)^*(AB)]^\dagger$, as $AB)^*(AB)$ is Hermitian. Thus

$$\mathcal{R}(B) = \mathcal{R}(B^{\dagger^*}) = \mathcal{R}(A^*AB[(AB)^*(AB)]^\dagger) \subseteq \mathcal{R}(A^*AB).$$

Conversely, $\mathcal{R}(B) = \mathcal{R}(A^*AB) \implies BB^\dagger A^*AB = A^*AB$. By Theorem 3.2, $B^\dagger A^\dagger \in AB\{1,2,3\}$, $\mathcal{R}(B^{\dagger^*}) = \mathcal{R}(B) = \mathcal{R}(A^*AB) \subseteq \mathcal{R}(A^*)$ and $A^\dagger A$ is the projection onto $\mathcal{R}(A^*)$ gives $A^\dagger AB = B$ and $A^\dagger AB^{\dagger^*} = B^{\dagger^*} \implies B^\dagger A^\dagger A = B^\dagger$. It shows that $(A^\dagger AB)^\dagger = B^\dagger = B^\dagger A^\dagger A$. By Theorem 3.3, $B^\dagger A^\dagger \in AB\{1,2,4\}$ and hence $(AB)^\dagger = B^\dagger A^\dagger$. Therefore $B^\dagger = B^\dagger A^\dagger A = (AB)^\dagger A$.
(ii) Proof is similar to (i). □

Theorem 3.6. *Let the conditions of Theorem* 3.1 *hold. Then the following statements hold:*

(i) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB)$.

(ii) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger \Leftrightarrow \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

Proof. (i) If we replace A by $A^\dagger$ and B by AB in Theorem 3.5 (i), we get $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AB) = \mathcal{R}(A^{\dagger^*}A^\dagger AB) = \mathcal{R}(A^{\dagger^*}B)$. Now by Theorem 2.1, there exists a bounded operator C such that $AB = A^{\dagger^*}BC$. Pre-multiplying by AA^* we get $AA^*AB = AA^*A^{\dagger^*}BC = A(A^\dagger A)^*BC = ABC$. Thus we get $\mathcal{R}(AA^*AB) \subseteq \mathcal{R}(AB)$. Similarly, we can prove $\mathcal{R}(AB) \subseteq \mathcal{R}(AA^*AB)$.
(ii) Replace A by AB and B by $B^\dagger$ in Theorem 3.5 (ii) and use a similar argument as above. □

Theorem 3.7. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(i) $(AB)^\dagger = B^\dagger A^\dagger$;

(ii) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$;

(iii) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$;

(iv) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}[(ABB^*B)^*] = \mathcal{R}[(AB)^*]$;

(v) $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$.

Proof. First we note that the condition $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ is equivalent to Theorem 3.2 (iii) and the condition $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ is equivalent to Theorem 3.3 (iii).
(i)⇔(ii): Follows from Theorem 3.4 (iii).
(i)⇒(iii): By Theorem 3.4 (vii), $(AB)^\dagger = B^\dagger A^\dagger = (A^\dagger AB)^\dagger A^\dagger$. Then (iii) follows from Theorem 3.6 (i).
(iii)⇒(v): By Theorem 2.1, there exists an operator T such that $AA^*AB = ABT$. Pre-multiplying by $A^\dagger$ we get $A^*AB = A^\dagger ABT = A^\dagger ABB^*B^{\dagger^*}T = BB^*A^\dagger AB^{\dagger^*}T$ by Theorem 3.3 (v). Also, $A^*ABB^* = BB^*A^\dagger AB^{\dagger^*}TB^*$. Thus $\mathcal{R}(A^*ABB^*) \subseteq \mathcal{R}(BB^*A^*A)$. Similarly, $AB = AA^*ABS$, for some operator S. Pre- and post-multiplying by $A^\dagger$ and B^*A^*A respectively, we get $A^\dagger ABB^*A^*A = A^*ABSB^*A^*A$.

Then by Theorem 3.3 (v), $BB^*A^\dagger AA^*A = A^*ABSB^*A^*A$. Thus, $BB^*A^*A = A^*ABSB^*A^*A$. It shows that

$$\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A).$$

(v)⇒(ii): By Theorem 2.1, there exists an operator T such that $A^*ABB^* = BB^*A^*AT$. Pre-multiplying by $BB^\dagger$ and post-multiplying by $B^{\dagger^*}$ we get,

$$BB^\dagger A^*ABB^*B^{\dagger^*} = BB^*A^*ATB^{\dagger^*}.$$

Hence, $BB^\dagger A^*AB = BB^*A^*ATB^{\dagger^*} = A^*ABB^*B^{\dagger^*} = A^*AB$, thus $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$. Similarly, we can prove $A^\dagger ABB^*A^* = BB^*A^*$ and hence $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.
We can give the proof of (i)⇒(iv) and (iv)⇒(v) in a similar fashion. □

Remark 3. Let A and B be as defined in Example 1. Then we get $A^*ABx = (x_1, x_1, x_3, 0, x_5, \ldots)$. Thus $\mathcal{R}(A^*AB) \nsubseteq \mathcal{R}(B)$, $A^*(x) = (x_1, x_1 + x_2, x_3, x_4, \ldots)$ and $BB^*A^*x = (x_1, 0, x_3, 0, x_5, \ldots)$ implies $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$. This shows that $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ is indispensable for the reverse order law to hold.

Lemma 3.2. *Let the conditions of Theorem* 3.1 *hold. If* $P = (AA^*)^mA$ *and* $Q = B(B^*B)^n$, *then* P, Q *and* PQ *have closed ranges.*

Proof. By Lemma 3.1, AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range. Take A as $(AA^*)^m$ and B as A to apply Lemma 3.1. Then, we have $(AA^*)^{m\dagger}(AA^*)^mAA^\dagger = ((AA^*)^\dagger(AA^*))^mAA^\dagger = (AA^\dagger)^mAA^\dagger = AA^\dagger AA^\dagger = AA^\dagger$. Now, $\mathcal{R}(AA^\dagger) = \mathcal{R}(A)$ is closed implies $\mathcal{R}(P)$ is closed. Similar argument works for Q also.

Now again by Lemma 3.1, PQ has a closed range if and only if $P^\dagger PQQ^\dagger$ has a closed range. For,

$$\begin{aligned} P^\dagger PQQ^\dagger &= [(AA^*)^mA]^\dagger(AA^*)^mAB(B^*B)^n[B(B^*B)^n]^\dagger \\ &= A^\dagger[(AA^*)^m]^\dagger(AA^*)^mAB(B^*B)^n[(B^*B)^n]^\dagger B^\dagger \\ &= A^\dagger AA^\dagger ABB^\dagger BB^\dagger = A^\dagger ABB^\dagger. \end{aligned}$$

Here, the reverse order law is applied for $[(AA^*)^mA]^\dagger$ and $[B(B^*B)^n]^\dagger$ as they satisfy condition (ii) in Theorem 3.7. □

The next result is an extension of Theorem 11.1 of [23] to infinite e-dimensional setting. Djordjević and Dinčić [10, 11] have extended the results of Tian [22, 23] using the operator matrix method to different settings. By Lemma 3.1, $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(A^\dagger ABB^\dagger) = \mathcal{R}(A^*A^{\dagger*}BB^\dagger) = \mathcal{R}(A^\dagger AB^{\dagger*}B^*)$ is closed. This happens if and only if $\mathcal{R}(A^{\dagger*}B)$ and $\mathcal{R}(AB^{\dagger*})$ are closed. Thus $(A^{\dagger*}B)^\dagger$ and $(AB^{\dagger*})^\dagger$ exist. Also, $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed implies $\mathcal{R}(BB^\dagger A^\dagger A)$ is closed and hence $\mathcal{R}(B^\dagger A^\dagger)$ is closed. For natural numbers m and n, the existence of the Moore-Penrose inverse of $(AA^*)^m$ and $(B^*B)^n$ is guaranteed as they are powers of Hermitian operators with closed ranges, according to the spectral mapping theorem. The existence of the Moore-Penrose inverse of all other operators discussed below can be guaranteed with the closedness of the ranges of $AB, A^{\dagger*}B, AB^{\dagger*}$ and $B^\dagger A^\dagger$.

Theorem 3.8. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $B(AB)^\dagger A = BB^\dagger A^\dagger A$;

(3) $AA^*(B^*A^*)^\dagger B^*B = AB$;

(4) $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$;

(5) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(6) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(7) $(AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(8) $B^\dagger A^\dagger \in AB\{1,3,4\}$;

(9) $(AB)(AB)^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB = B^* A^\dagger A (B^\dagger)^*$;

(10) $(A^{\dagger *} B)^\dagger = B^\dagger A^*$;

(11) $A^\dagger (B^* A^\dagger)^\dagger B^* = A^\dagger ABB^\dagger$;

(12) $AA^\dagger (B^* A^\dagger)^\dagger B^* B = AB$;

(13) $(A^{\dagger *} B)^\dagger = B^\dagger A^\dagger ABB^\dagger A^*$;

(14) $(A^{\dagger *} B)^\dagger = (A^\dagger AB)^\dagger A^*$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(15) $(A^{\dagger *} B)^\dagger = B^\dagger (A^{\dagger *} BB^\dagger)^\dagger$ *and* $(A^{\dagger *} BB^\dagger)^\dagger = BB^\dagger A^*$;

(16) $(A^{\dagger *} B)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(17) $B^\dagger A^* \in A^{\dagger *} B\{1,3,4\}$;

(18) $(B^* A^\dagger)^\dagger B^* A^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $B^* A^\dagger (B^* A^\dagger)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$;

(19) $(AB^{\dagger *})^\dagger = B^* A^\dagger$;

(20) $B^{\dagger *} (AB^{\dagger *})^\dagger A = BB^\dagger A^\dagger A$;

(21) $AA^* (B^\dagger A^*)^\dagger B^\dagger B = AB$;

(22) $(AB^{\dagger *})^\dagger = B^* A^\dagger ABB^\dagger A^\dagger$;

(23) $(AB^{\dagger *})^\dagger = (A^\dagger AB^{\dagger *})^\dagger A^\dagger$ *and* $(A^\dagger AB^{\dagger *})^\dagger = B^* A^\dagger A$;

(24) $(AB^{\dagger *})^\dagger = B^* (ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(25) $(AB^{\dagger *})^\dagger = B^* (A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(26) $B^* A^\dagger \in AB^{\dagger *}\{1,3,4\}$;

(27) $(B^\dagger A^*)^\dagger B^\dagger A^* = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$ *and* $B^\dagger A^* (B^\dagger A^*)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$;

(28) $(B^\dagger A^\dagger)^\dagger = AB$;

(29) $A^\dagger (B^\dagger A^\dagger)^\dagger B^\dagger = A^\dagger ABB^\dagger$;

(30) $(AA^*)^\dagger (B^\dagger A^\dagger)^\dagger (B^* B)^\dagger = A^{\dagger *} A^{\dagger *}$;

(31) $(B^\dagger A^\dagger)^\dagger = ABB^\dagger A^\dagger AB$;

(32) $(B^\dagger A^\dagger)^\dagger = A(B^\dagger A^\dagger A)^\dagger$ *and* $(B^\dagger A^\dagger A)^\dagger = A^\dagger AB$;

(33) $(B^\dagger A^\dagger)^\dagger = (BB^\dagger A^\dagger)^\dagger B$ *and* $(BB^\dagger A^\dagger)^\dagger = ABB^\dagger$;

(34) $(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B$ *and* $(BB^\dagger A^\dagger A)^\dagger = A^\dagger ABB^\dagger$;

(35) $AB \in B^\dagger A^\dagger\{1,3,4\}$;

(36) $B^\dagger A^\dagger (B^\dagger A^\dagger)^\dagger = B^\dagger A^\dagger AB = B^* A^\dagger AB^{\dagger *}$ *and* $(B^\dagger A^\dagger)^\dagger B^\dagger A^\dagger = ABB^\dagger A^\dagger = A^{\dagger *} BB^\dagger A^*$;

(37) $(AB)^\dagger = (A^*AB)^\dagger A^*$ *and* $(A^*AB)^\dagger = B^\dagger (A^*A)^\dagger$;

(38) $(AB)^\dagger = B^*(ABB^*)^\dagger$ *and* $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$;

(39) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$ *and* $(A^*ABB^*)^\dagger = (BB^*)^\dagger (A^*A)^\dagger$;

(40) $(AB)^\dagger = (B^*B)^n((AA^*)^m AB(B^*B)^n)^\dagger (AA^*)^m$ *and*

$$((AA^*)^m AB(B^*B)^n)^\dagger = (B(B^*B)^n)^\dagger ((AA^*)^m A)^\dagger;$$

(41) $(AB)^\dagger = B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger (A^*A)^m A^*$ *and*

$$((A^*A)^{m+1}(BB^*)^{n+1})^\dagger = ((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}.$$

Proof. (1) $\Rightarrow$ (2): Straightforward.
(2) $\Rightarrow$ (3): Pre- and post-multiplying the given condition by B^* and A^*, respectively, we get $B^*B(AB)^\dagger AA^* = B^*A^*$, equivalently $AA^*(B^*A^*)^\dagger B^*B = AB$.
(3) $\Rightarrow$ (1): We have $B^*B(AB)^\dagger AA^* = B^*A^*$. Pre- and post-multiplying by $(B^*B)^\dagger$ and $(AA^*)^\dagger$ respectively, we get $B^\dagger B(AB)^\dagger AA^\dagger = B^\dagger A^\dagger$. It is clear that $\mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*) \subseteq R(B^*)$ and $R((AB)^{\dagger *}) = R(AB) \subseteq R(A)$. Thus $B^\dagger B(AB)^\dagger = (AB)^\dagger$ and $(AB)^\dagger AA^\dagger = (AB)^\dagger$. Hence, $B^\dagger B(AB)^\dagger AA^\dagger = (AB)^\dagger = B^\dagger A^\dagger$.
(1) $\Rightarrow$ (4): It is easy to see from the assumption that

$$(AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger.$$

(4) $\Rightarrow$ (5): Pre-multiplying the given condition by B and post-multiplying by A, we get $B(AB)^\dagger A = BB^\dagger A^\dagger ABB^\dagger A^\dagger A = (BB^\dagger A^\dagger A)^2$. $(BB^\dagger A^\dagger A)^4 = (BB^\dagger A^\dagger A)^2(BB^\dagger A^\dagger A)^2 = B(AB)^\dagger AB(AB)^\dagger A = B(AB)^\dagger A = (BB^\dagger A^\dagger A)^2$. Since $BB^\dagger$ and $A^\dagger A$ are orthogonal projections by Lemma 2.7, $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$. The statements $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ and $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$ can be proved by verifying all Penrose equations using $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$.
(5) $\Rightarrow$ (6): Note that by substituting the second condition in the first condition of (5), we get $(AB)^\dagger = B^\dagger A^\dagger$. Thus, we have $AB = AB(AB)^\dagger AB = ABB^\dagger A^\dagger AB$, $(AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$, $ABB^\dagger A^\dagger$ is a projection and Hermitian. Now $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ is easily verifiable, using Theorem 3.1 (iii). Moreover, $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger (AB)(AB)^\dagger = B^\dagger BB^\dagger A^\dagger AB(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger (AB)(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$.
(6) $\Rightarrow$ (7): Suppose $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$. Then

$$(AB)(AB)^\dagger = ABB^\dagger (ABB^\dagger)^\dagger = ABB^\dagger BB^\dagger A^\dagger = ABB^\dagger A^\dagger.$$

Thus, $AB = ABB^\dagger A^\dagger AB$. Now by Theorem 3.1 (vi), we get $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$. Since $BB^\dagger A^\dagger = (ABB^\dagger)^\dagger$, $(A^\dagger ABB^\dagger)^\dagger = (ABB^\dagger)^\dagger A$. Therefore $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger (ABB^\dagger)^\dagger AA^\dagger = (AB)^\dagger AA^\dagger =$

$(AB)^\dagger$.
(7) $\Rightarrow$ (8): We have

$$\begin{aligned}AB = AB(AB)^\dagger AB &= ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger AB\\ &= ABB^\dagger BB^\dagger A^\dagger AA^\dagger AB = ABB^\dagger A^\dagger AB\end{aligned}$$

and $(AB)(AB)^\dagger = ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = ABB^\dagger BB^\dagger A^\dagger AA^\dagger = ABB^\dagger A^\dagger$. Similarly, $(AB)^\dagger AB = B^\dagger A^\dagger AB$. Thus $B^\dagger A^\dagger \in AB\{1,3,4\}$.
(8) $\Rightarrow$ (9): Since $B^\dagger A^\dagger \in AB\{1,3,4\}$, $AB = ABB^\dagger A^\dagger AB$ and $ABB^\dagger A^\dagger = (ABB^\dagger A^\dagger)^*$. Now,

$$\begin{aligned}(AB)(AB)^\dagger = ABB^\dagger A^\dagger AB(AB)^\dagger &= (ABB^\dagger A^\dagger)^* AB(AB)^\dagger\\ &= (B^\dagger A^\dagger)^*(AB)^* AB(AB)^\dagger\\ &= (B^\dagger A^\dagger)^*(AB)^*(\text{ by Lemma 2.1 (vi)})\\ &= (ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger\\ &= A^{\dagger *}BB^\dagger A^*.\end{aligned}$$

Similarly, we can prove the other relation.
(9) $\Rightarrow$ (10): Since $AB(AB)^\dagger = ABB^\dagger A^\dagger$, $AB(AB)^\dagger AB = ABB^\dagger A^\dagger AB$. Then by Theorem 3.1 (iv), $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. It is clear from the assumption that $B^\dagger A^* \in A^{\dagger *}B\{3,4\}$. Also, it is easy to verify that $B^\dagger A^* \in A^{\dagger *}B\{1,2\}$.
(10) $\Rightarrow$ (1): Applying Theorem 3.1 for $A^{\dagger *}$ and B, we get $A^*A^{\dagger *}BB^\dagger = BB^\dagger A^*A^{\dagger *}$ i.e., $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$. Using the third and fourth Penrose conditions for (10), we get

$$\begin{aligned}ABB^\dagger A^\dagger AB &= AA^\dagger ABB^\dagger B = AB,\\ B^\dagger A^\dagger ABB^\dagger A^\dagger &= B^\dagger BB^\dagger A^\dagger AA^\dagger = B^\dagger A^\dagger,\\ (ABB^\dagger A^\dagger)^* &= A^{\dagger *}BB^\dagger A^* = ABB^\dagger A^\dagger,\\ (B^\dagger A^\dagger AB)^* &= B^*A^\dagger AB^{\dagger *} = B^\dagger A^\dagger AB.\end{aligned}$$

The equivalences of (10)-(18) can be established by replacing A by $A^{\dagger *}$ in (1)-(9). Similarly, the equivalences of (19)-(27) can be established by replacing B by $B^{\dagger *}$ in (1)-(9) and the equivalences of (28)-(36) can be established by replacing A by $B^\dagger$ and B by $A^\dagger$ in (1)-(9). The equivalence of (1) and (19) is similar to that of (1) and (10). The equivalence of (1) and (28) follows by applying the Moore-Penrose inverse on both sides of (1) and (28).
(1) $\Rightarrow$ (37): We use Theorem 3.1 to prove $(A^*AB)^\dagger = B^\dagger(A^*A)^\dagger$. We get the first Penrose equation verified as below. By using Lemma 2.1 (iv) and (vi), we get

$$A^*ABB^\dagger(A^*A)^\dagger A^*AB = A^*ABB^\dagger A^\dagger AB = A^*AA^\dagger ABB^\dagger B = A^*AB.$$

By Lemma 2.1, Theorem 3.1 and Theorem 3.2 (v), we get

$$[A^*ABB^\dagger(A^*A)^\dagger]^* = [BB^\dagger A^*A(A^*A)^\dagger]^* = (BB^\dagger A^\dagger A)^* = A^\dagger ABB^\dagger.$$

The right-hand side is Hermitian, so is the left-hand side. Similarly, we can prove the second and fourth Penrose equations. Also, we get

$$(A^*AB)^\dagger A^* = B^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger = (AB)^\dagger.$$

(1) $\Rightarrow$ (38): Similar to (1) $\Rightarrow$ (37).
(1) $\Rightarrow$ (39): By using Lemma 2.1 and Theorem 3.1, we get

$$\begin{aligned} A^*ABB^*(BB^*)^\dagger(A^*A)^\dagger A^*ABB^* &= A^*ABB^\dagger A^\dagger ABB^* \\ &= A^*AA^\dagger ABB^\dagger BB^* \\ &= A^*ABB^*. \end{aligned}$$

Now, using Lemma 2.1 and Theorem 3.2 (v), we have

$$\begin{aligned} [A^*ABB^*(BB^*)^\dagger(A^*A)^\dagger]^* &= [A^*ABB^\dagger(A^*A)^\dagger]^* = [BB^\dagger A^*A(A^*A)^\dagger]^* \\ &= (BB^\dagger A^\dagger A)^*, \end{aligned}$$

which is Hermitian by Theorem 3.1. Hence the first and third conditions of the Penrose equations are satisfied. The second and fourth conditions follow similarly.
(1) $\Rightarrow$ (40): Let $P = (AA^*)^m A$ and $Q = B(B^*B)^n$. Then $\mathcal{R}(P)$, $\mathcal{R}(Q)$ and $\mathcal{R}(PQ)$ have closed ranges by Lemma 3.2. We prove $[(AA^*)^m AB(B^*B)^n]^\dagger = [B(B^*B)^n]^\dagger[(AA^*)^m A]^\dagger$ i.e., $(PQ)^\dagger = Q^\dagger P^\dagger$ by verifying the Penrose equations. By Lemma 2.1 and Theorem 3.1, we get

$$\begin{aligned} PQQ^\dagger P^\dagger PQ &= (AA^*)^m AB(B^*B)^n[(B^*B)^n]^\dagger B^\dagger A^\dagger[(AA^*)^m]^\dagger(AA^*)^m AB(B^*B)^n \\ &= (AA^*)^m ABB^\dagger BB^\dagger A^\dagger AA^\dagger AB(B^*B)^n \\ &= (AA^*)^m ABB^\dagger A^\dagger AB(B^*B)^n = (AA^*)^m AA^\dagger ABB^\dagger B(B^*B)^n \\ &= (AA^*)^m AB(B^*B)^n = PQ. \end{aligned}$$

Similarly, we can prove the second Penrose equation. For proving the third one we use the following facts that for all $m \geq 1$, $A^\dagger(AA^*)^m = A^*(AA^*)^{m-1}$, $[(AA^*)^m]^\dagger A = [(AA^*)^{m-1}]^\dagger A^{\dagger *} = A^{\dagger *}$ and $(ABB^\dagger A^\dagger)^* = ABB^\dagger A^\dagger$. We have

$$\begin{aligned} (PQQ^\dagger P^\dagger)^* &= ((AA^*)^m ABB^\dagger A^\dagger[(AA^*)^m]^\dagger)^* \\ &= [(AA^*)^m]^\dagger ABB^\dagger A^\dagger(AA^*)^m \\ &= A^{\dagger *}BB^\dagger A^* = (ABB^\dagger A^\dagger)^*. \end{aligned}$$

The right-hand side is Hermitian so is the left-hand side. Similarly, the fourth Penrose equation can be proved. Also, we have

$$\begin{aligned} (B^*B)^n[(AA^*)^m AB(B^*B)^n]^\dagger(AA^*)^m &= (B^*B)^n[B(B^*B)^n]^\dagger[(AA^*)^m A]^\dagger(AA^*)^m \\ &= (B^*B)^n[(B^*B)^n]^\dagger B^\dagger A^\dagger[(AA^*)^m]^\dagger(AA^*)^m \\ &= B^\dagger BB^\dagger A^\dagger AA^\dagger = B^\dagger A^\dagger = (AB)^\dagger. \end{aligned}$$

(1) $\Rightarrow$ (41): Let $P = (AA^*)^{m+1}$ and $Q = (B^*B)^{n+1}$. We can prove the existence of $(PQ)^\dagger$ by a similar argument in Lemma 3.2.

$$\begin{aligned} PQQ^\dagger P^\dagger PQ &= (AA^*)^{m+1}B^\dagger BAA^\dagger(B^*B)^{n+1} \\ &= (AA^*)^{m+1}AA^\dagger B^\dagger B(B^*B)^{n+1} \\ &= (AA^*)^{m+1}(B^*B)^{n+1} = PQ. \end{aligned}$$

Using the fact $AA^*B^\dagger B = B^\dagger BAA^*$, we have

$$\begin{aligned} (PQQ^\dagger P^\dagger)^* &= (AA^*)^{m+1}B^\dagger B[(AA^*)^{m+1}]^\dagger \\ &= B^\dagger B(AA^*)^{m+1}[(AA^*)^{m+1}]^\dagger \\ &= B^\dagger BAA^\dagger. \end{aligned}$$

The right-hand side is Hermitian so is the left-hand side. Similarly, the second and fourth Penrose equations can be proved. Now we have

$$\begin{aligned} B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger(A^*A)^m A^* &= B^*(BB^*)^n[(BB^*)^{n+1}]^\dagger \\ &\quad [(A^*A)^{m+1}]^\dagger(A^*A)^m A^* \\ &= B^*BB^\dagger(BB^*)^\dagger(A^*A)^\dagger A^\dagger AA^* \\ &= B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger = (AB)^\dagger. \end{aligned}$$

<u>(37) – (41) ⇒ (1)</u>: Using the given conditions we get

$$(AB)^\dagger = (A^*AB)^\dagger A^* = B^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger.$$

Similarly, other implications also follow by substituting the second set of equations into the first ones and using the properties of the Moore-Penrose inverse given in Lemma 2.1. □

Proposition 3.1. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are true.*

(i) $A^*ABB^\dagger A^\dagger A$ *is Hermitian if and only if* $B^\dagger A^\dagger \in AB\{3\}$.

(ii) $BB^\dagger A^\dagger ABB^*$ *is Hermitian if and only if* $B^\dagger A^\dagger \in AB\{4\}$.

Proof. Since $ABB^\dagger A^\dagger = A^{\dagger *}A^*ABB^\dagger A^\dagger AA^\dagger$ we get $B^\dagger A^\dagger \in AB\{3\}$ if and only if $A^*ABB^\dagger A^\dagger A$ is Hermitian. Similarly, we can prove (ii). □

The next result is a continuation of Theorem 3.8.

Theorem 3.9. *Let the conditions of Theorem* 3.1 *hold. Then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(42) $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$ *and* $BB^\dagger A^\dagger ABB^*$ *is Hermitian;*

(43) $(A^*AB)^\dagger = B^\dagger(A^*A)^\dagger$ *and* $A^*ABB^\dagger A^\dagger A$ *is Hermitian;*

(44) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $BB^\dagger A^\dagger ABB^*$ *is Hermitian;*

(45) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$ *and* $A^*ABB^\dagger A^\dagger A$ *is Hermitian;*

(46) $(A^*ABB^\dagger)^\dagger = (BB^*)^\dagger(A^*A)^\dagger$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(47) $(A^\dagger ABB^*)^\dagger = (BB^*)^\dagger A^\dagger A$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(48) $(A^*ABB^\dagger)^\dagger = BB^\dagger(A^*A)^\dagger$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian;*

(49) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$; $A^*ABB^\dagger A^\dagger A$ *and* $BB^\dagger A^\dagger ABB^*$ *are Hermitian.*

Proof. In all the implications of the proof, Proposition 3.1 is also used.
<u>(1) ⇒ (42)</u>: Follows from Theorem 3.8 (38).
<u>(42) ⇒ (1)</u>: Follows from Theorem 3.2 (x).
<u>(1) ⇒ (43)</u>: Follows from Theorem 3.8 (37).
<u>(43) ⇒ (1)</u>: Follows from Theorem 3.3 (x).
<u>(1) ⇒ (44)</u>: Follows from Theorem 3.8 (6).
<u>(44) ⇒ (1)</u>: Follows from Theorem 3.2 (vi).

(1) $\Rightarrow$ (45): Follows from Theorem 3.8 (5).
(45) $\Rightarrow$ (1): Follows from Theorem 3.3 (vi).
(1) $\Rightarrow$ (46): Follows from Theorem 3.8 (39).
(46) $\Rightarrow$ (1): It is easy to verify that $B^\dagger A^\dagger \in AB\{1,2\}$.
(46) $\Leftrightarrow$ (47): Replacing B by BB^* in the equivalence (43) $\Leftrightarrow$ (45).
(46) $\Leftrightarrow$ (48): Replacing A by A^*A in the equivalence (42) $\Leftrightarrow$ (44).
(47) $\Leftrightarrow$ (49): Replacing A by $A^\dagger A$ in the equivalence (42) $\Leftrightarrow$ (44). □

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