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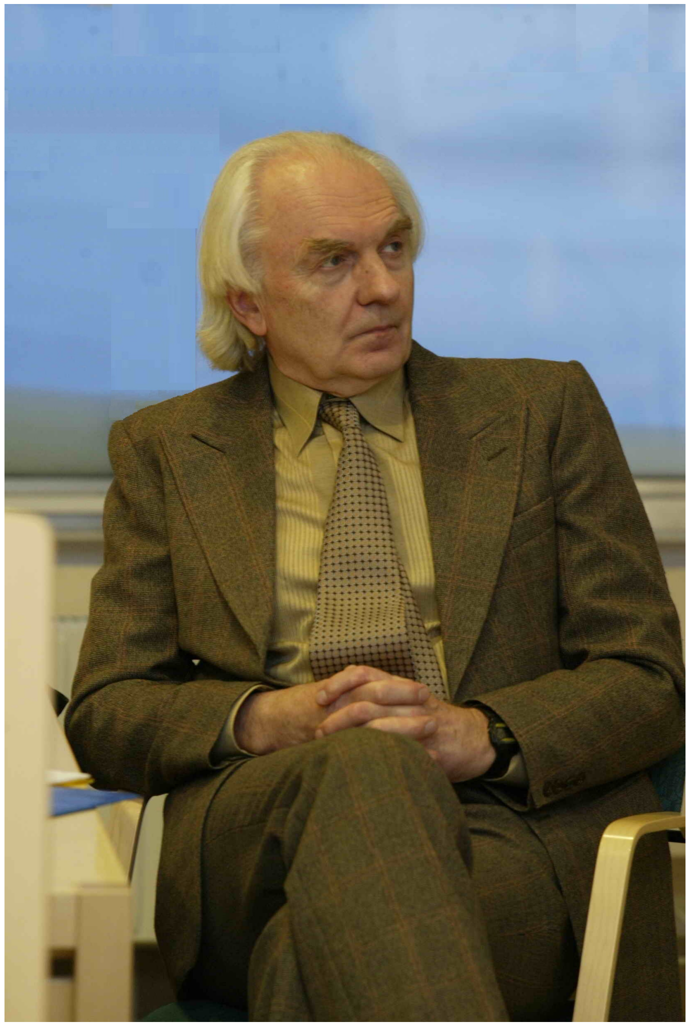
On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \leq p, \theta \leq \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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HARDY INEQUALITIES FOR p-WEAKLY MONOTONE FUNCTIONS

M. Saucedo

Communicated by V.I. Burenkov

Key words: Hardy-type inequality, generalized monotonicity.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. We prove Hardy-type inequalities

$$\left(\int_d^\infty \left|\int_d^s f(x)dx\right|^p s^\beta ds\right)^{1/p} \leq C\left(\int_d^\infty |f(s)|^q s^\alpha ds\right)^{1/q}$$

for the class of p-weakly monotone functions with q or p smaller than 1 and $d \geq 0$.

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1 Introduction

The goal of this paper is to extend the results presented in [25] and [5] by proving inequalities of the type

$$\left(\int_d^\infty \left|\int_d^s f(x)dx\right|^p s^\beta ds\right)^{1/p} \leq C\left(\int_d^\infty |f(s)|^q s^\alpha ds\right)^{1/q}$$

for p or q smaller than one and for p-weakly monotone f .

Definition 1. [31, 3] Let $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be a measurable function, then we say that f is p-weakly monotone $\big($and write $f \in WM(K, \lambda, p)$, where $K > 0, \lambda > 1, p > 0\big)$, if the inequality

$$f(x)^p \leq K \int_{x/\lambda}^{\lambda x} \frac{f(s)^p}{s} ds \tag{1.1}$$

holds for every $x > 0$. Similarly, let $f : I = [a, b] \to \mathbb{R}^+ \cup \{0\}$ be a measurable function, then we say that $f \in WM(K, \lambda, p)$ on I whenever $f\chi_I$ satisfies inequality (1.1).

Here and throughout the paper by χ_I we denote the characteristic function of I. The next concept was studied in [28] with applications to number series. It appeared in [25] as a quasi-monotonicity.

Definition 2. [28] Let $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be a function, then we say that f is weakly monotone $\big($and write $f \in WM(K)$, where $K > 0\big)$ if the inequality

$$f(x) \leq Kf(y) \tag{1.2}$$

holds for every $2y \geq x \geq y > 0$.

Let us mention that both weakly monotone and p-weakly monotone functions/sequences play an important role in various problems in analysis (see the precise references below). It is worth mentioning that the class of weakly monotone functions contains as a subclass the class of general monotone functions. Recall that for $C > 0$, the $GM(C)$ class (see [30, 27]) is defined in the following way:

$$GM(C) = \Big\{ f \in BV_{loc} : \mathrm{Var}(f)_{[x;2x]} \le C|f(x)| \quad \text{for all} \quad x \in (0,\infty) \Big\}.$$

Here assuming that f is locally absolutely continuous on $\mathbb{R}^+$, the expression $\mathrm{Var}(f)_{[x;2x]}$ can be replaced by $\int_x^{2x} |f'(t)|dt$. Similarly, any p-general monotone function is always p-weakly monotone (see [3, 27]), that is, $GM(C,\lambda,p) \subsetneq WM(K,\lambda,p)$, where K only depends on p, C and λ, and where

$$GM(C,\lambda,p) = \Big\{ f \in BV_{loc} : \mathrm{Var}(f)_{[x;2x]} \le C \left(\int_{x/\lambda}^{\lambda x} \frac{|f(t)|^p}{t} dt \right)^{1/p} \quad \text{for all } x > 0 \Big\}.$$

It is known that for $p > 1$ $GM(C) \subsetneq GM(C',\lambda,1) \subsetneq GM(C'',\lambda,p)$, where C' depends on C and λ; and C'' depends on C' and λ. For the first embedding see [27, 31], for the second one see [3]. We will see in Proposition 1.1 that the scale of weakly monotone functions has a similar structure.

Various applications of both general and weakly monotone sequences can be found in Fourier analysis and approximation theory. In particular, in the study of integrability of Fourier transforms [8, 17, 22] and trigonometric series [3, 4, 12, 14, 18], investigating various problems in approximation theory [11, 15, 20, 19, 26, 30, 31], convergence problems [7, 13, 16, 23, 27, 30], theory of number series [7, 28], and embedding theorems for smooth function spaces [3, 10, 9]. We emphasise that in many problems the consideration of either general monotone or weakly monotone sequences/functions imply completely different answers; see e.g. [3, 15, 27].

Let us present the main properties of weakly monotone and p-weakly monotone functions.

Proposition 1.1. *The following properties hold:*

1. $f \in WM(K,\lambda,p)$ *if and only if, for all* $x \in \mathbb{R}$
$$f(\exp(x))^p \le K \int_{x-\ln\lambda}^{\ln\lambda + x} f(\exp(t))^p dt;$$

2. $WM(K) \subsetneq WM(K',\lambda,p)$, *where* K' *depends only on* K, p *and* λ;

3. *Let* $q > p > 0$, *then* $WM(K,\lambda,p) \subsetneq WM(K',\lambda,q)$, *where* K' *depends only on* K, p, q *and* λ;

4. *Let* $f \in WM(K,\lambda,p)$, *then* $g(t) = f(t^{-1}) \in WM(K,\lambda,p)$. *However, if* $f \in WM(K)$, *then* g *may not be in* $WM(K')$ *for any* K';

5. *Let* $f \in WM(K,\lambda,p)$ *and* $\alpha \in \mathbb{R}$. *If* $g(t) = f(t)t^\alpha$, *then* $g \in WM(\lambda^{|\alpha|p}K,\lambda,p)$. *If* $f \in WM(K)$, *then* $g \in WM(K')$, *where* K' *depends only on* K *and* α;

6. *Let* $f \in WM(K,\lambda,p)$ *and* $\alpha \in \mathbb{R}$, *then* $g(t) = f(t)^\alpha \in WM(K,\lambda,p/\alpha)$. *If* $f \in WM(K)$, *then* $g \in WM(K')$, *where* K' *depends only on* K *and* α;

7. *Let* $f \in WM(K,\lambda,p)$ *and* $\alpha > 0$, *then* $g(t) = f(t/\alpha) \in WM(K,\lambda,p)$. *If* $f \in WM(K)$, *then* $g \in WM(K')$, *where* K' *depends only on* K *and* α.

Proof. To show 1), we use a logarithmic change of variable. Furthermore, if $f \in WM(K)$, we have that

$$\frac{x(\lambda-1)}{\lambda} f(x)^p \leq K' \int_{x/\lambda}^{x} f(y)^p dy \leq K' \int_{x/\lambda}^{\lambda x} f(y)^p dy.$$

Hence, f is p-weakly monotone and 2) follows. To see that the inclusion is proper, consider $f(x) = x^a \chi_{(0,1)\cup(1,+\infty)}(x)$. Since $f(1) = 0$, f cannot be $WM(K)$ for any K and a simple calculation shows that $f \in WM(K(\lambda, a, p), \lambda, p)$, for every $\lambda > 1, p > 0$.

The embedding $WM(K, \lambda, p) \subsetneq WM(K', \lambda, q)$ follows from Hölder's inequality. To see its sharpness, for $c > 1$, consider f such that

$$f^q(\exp(x)) = g(x) = \sum_{n=1}^{\infty} c^n 4^{2^n} \chi_{[n,n+4^{-2^n-1}]}(x).$$

It is easy to see that $f \in WM(1/c, e^2, q)$ but $f \notin WM(K, \mu, q/2)$ for any K or μ. Therefore if $r = p/q < 1$ there must be some $n \geq 0$ such that

$$f^{r^n} \in WM(1/c, e^2, 1) \text{ but } f^{r^{n+1}} \notin WM(K, \mu, 1),$$

therefore $f^{r^n/q} \in WM(1/c, e^2, q)$ but $f^{r^n/q} \notin WM(K, \mu, p)$. For λ other than e^2, we can modify the previous example correspondingly.

To show the first part of 4) we use a change of variables, for the second part, consider $f(x) = x^{-1}\chi_{(0,1)}(x)$. Property 5) follows from the monotonicity of power functions while 6) is obvious. Finally, the first part of 7) follows from a change of variables and the second part is clear.

□

2 Weighted L_p spaces and Hardy inequalities

For $p > 0$ and $\alpha \in \mathbb{R}$ we denote

$$\|f\|_{p,\alpha} = \left(\int_0^{\infty} |f(s)|^p s^{\alpha} ds\right)^{1/p}, \tag{2.1}$$

and for $d > 0$ we denote

$$\|f\|_{p,\alpha}^{(d)} = \left(\int_d^{\infty} |f(s)|^p s^{\alpha} ds\right)^{1/p}. \tag{2.2}$$

Note that if $d > 0$, then $\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p} \|f\|_{p,\alpha+\varepsilon}^{(d)}$ for $\varepsilon > 0$.

First we are going to study the embeddings between weighted L_p spaces for p-weakly monotone functions.

From now on, by p and q we will denote positive numbers, by α and β, real numbers; and by C, a constant which depends only on $p, q, \alpha, \beta, K, \lambda$.

Lemma 2.1. *Let $p \geq q > 0$, $\beta \in \mathbb{R}$, and $f \in WM(K, \lambda, q)$. Let $\alpha = \frac{q}{p}(\beta+1) - 1$. Then*

$$\left(\int_0^x f^p(s) s^{\beta} ds\right)^{q/p} \leq C \int_0^{\lambda x} f(s)^q s^{\alpha} ds$$

and

$$\left(\int_x^{\infty} f^p(s) s^{\beta} ds\right)^{q/p} \leq C \int_{x/\lambda}^{\infty} f(s)^q s^{\alpha} ds.$$

Proof. Let $n \in \mathbb{Z}$ be such that $\lambda^{n-1} < x \leq \lambda^n$. For each $j \in \mathbb{Z}$, let $\lambda^j \leq s_j \leq \lambda^{j+1}$ be such that

$$\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds \leq \left(\lambda^{j+1} - \lambda^j\right) f(s_j)^p s_j^\beta.$$

Note that if for all $\lambda^j \leq t \leq \lambda^{j+1}$,

$$\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds > \left(\lambda^{j+1} - \lambda^j\right) f(t)^p t^\beta,$$

integrating both sides,

$$\left(\lambda^{j+1} - \lambda^j\right) \int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds = \int_{\lambda^j}^{\lambda^{j+1}} \left(\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds \right) dt > \left(\lambda^{j+1} - \lambda^j\right) \int_{\lambda^j}^{\lambda^{j+1}} f(t)^p t^\beta dt,$$

we arrive at a contradiction, therefore s_j must exist.

We see that

$$\left(\int_0^x f^p(s) s^\beta ds \right)^{q/p} \leq \left(\int_0^{\lambda^{n-1}} f^p(s) s^\beta ds \right)^{q/p} + \left(\int_{\lambda^{n-1}}^x f^p(s) s^\beta ds \right)^{q/p}.$$

Hence

$$\left(\int_0^{\lambda^{n-1}} f^p(s) s^\beta ds \right)^{q/p} \leq \sum_{j=-\infty}^{n-2} \left(\int_{\lambda^j}^{\lambda^{j+1}} f(s)^p s^\beta ds \right)^{q/p} \leq C \sum_{j=-\infty}^{n-2} s_j^{q\beta/p} f(s_j)^q \lambda^{qj/p}.$$

Now, since $f \in WM(K, \lambda, q)$

$$\sum_{j=-\infty}^{n-2} s_j^{q\beta/p} f(s_j)^q \lambda^{qj/p} \leq C \sum_{j=-\infty}^{n-2} s_j^{q\beta/p} \lambda^{qj/p} \int_{\lambda^{j-1}}^{\lambda^{j+2}} \frac{f(s)^q}{s} ds \leq C \int_0^{\lambda x} f(s)^q s^\alpha ds.$$

Finally, by the same token

$$\left(\int_{\lambda^{n-1}}^x f^p(s) s^\beta ds \right)^{q/p} \leq C \int_{\lambda^{n-2}}^{\lambda x} f(s)^q s^\alpha ds.$$

And the result follows by adding both inequalities up. The proof of the second inequality is analogous. □

Proposition 2.1. *There is a $C > 0$ such that for all $f \in WM(K, \lambda, q)$*

$$\|f\|_{p,\beta} \leq C \|f\|_{q,\alpha} \quad \iff \quad \frac{\alpha + 1}{q} = \frac{\beta + 1}{p} \text{ and } q \leq p.$$

Proof. The "if" part is a restatement of Lemma 2.1. The proof of the "only if" part will be given in section 3. □

Remark 1. In the general case, that is, without the assumption that $f \in WM(K, \lambda, p)$, it is not possible to obtain any non-trivial embedding of the type $\|f\|_{p,\beta} \leq C \|f\|_{q,\alpha}$.

Proof. First, let f be a non-negative function which is not zero almost everywhere. For $\lambda > 0$, let $f_\lambda(t) = f(\lambda t)$. Then a change of variables shows that $\|f_\lambda\|_{p,\beta} = \lambda^{-\frac{\beta+1}{p}}\|f\|_{p,\beta}$. Therefore if such a $C > 0$ exists, we derive

$$\|f_\lambda\|_{p,\beta} = \lambda^{-\frac{\beta+1}{p}}\|f\|_{p,\beta} \leq C\|f_\lambda\|_{q,\alpha} = C\lambda^{-\frac{\alpha+1}{q}}\|f\|_{q,\alpha},$$

which implies $\frac{\alpha+1}{q} = \frac{\beta+1}{p}$.

Next, consider $f_n(t) = \chi_{(1,1+1/n)}(t)$. A simple calculation shows that

$$\lim_{n\to\infty} n^{1/p}\|f_n\|_{p,\beta} = 1.$$

Therefore, if such a $C > 0$ exists,

$$1 = \lim_{n\to\infty} \frac{n^{1/p}\|f_n\|_{p,\beta}}{n^{1/q}\|f_n\|_{q,\alpha}} \leq C \lim_{n\to\infty} n^{1/p-1/q},$$

from which it follows that $p \leq q$.

Finally, let $f(x) = x^{-(\beta+1)/p}\ln(x+1)^{-1/p}\chi_{[1,\infty)}(x)$. Then

$$\|f\|_{p,\beta}^p = \int_1^\infty \frac{1}{x\ln(x+1)}dx = \infty,$$

and

$$\|f\|_{q,\alpha}^q = \int_1^\infty \frac{1}{x^{q(\beta+1)/p-\alpha}\ln(x+1)^{q/p}}dx.$$

The last integral is finite when $\frac{\alpha+1}{q} = \frac{\beta+1}{p}$ and $q > p$. Thus the only remaining possibility is the trivial one: $p = q$ and $\alpha = \beta$. □

Proposition 2.2. *Let $d > 0$, then there is a $C > 0$ such that for all $f \in WM(K, \lambda, q)$ on $[d, \infty]$,*

$$\|f\|_{p,\beta}^{(d)} \leq C\,\|f\|_{q,\alpha}^{(d)}$$

if and only if

$$\frac{\alpha+1}{q} > \frac{\beta+1}{p} \quad \text{and} \quad q > p \quad \text{or} \quad \frac{\alpha+1}{q} \geq \frac{\beta+1}{p} \quad \text{and} \quad q \leq p.$$

Proof. For the "if" part, the $q > p$ case follows from Hölder's inequality and the $q \leq p$ case from Lemma 2.1 by considering $f\chi_{[d,\infty]}$ and the following fact:

$$\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p}\,\|f\|_{p,\alpha+\varepsilon}^{(d)} \text{ for } \varepsilon > 0 \text{ and } d > 0.$$

The proof of the "only if" part will be given in Section 3. □

We now state and prove Hardy-type inequalities for p-weakly monotone functions.

Let us recall the original Hardy inequality. Denote

$$F(x) = \int_0^x f(s)ds.$$

Theorem A. (see, e.g., [24]) *Let $p > 1$. Then*

$$\|F\|_{p,-p} \leq \frac{p}{p-1}\,\|f\|_{p,0}\,.$$

There are many generalizations of this result in various settings. Let us mention the following classical result by Bradley [5] for power weights.

Theorem B. [5] *Let $1 < q \le p$. Then there is a $C > 0$ such that*

$$\|F\|_{p,\beta} \le C\,\|f\|_{q,\alpha} \qquad \Longleftrightarrow \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1 \quad \text{and} \quad \beta < -1.$$

For $q < 1$ it is necessary to restrict ourselves to a narrower class of functions, as the following example shows.

Example 1. Let $1 > \varepsilon > q$. Consider the following function

$$f(x) = \sum_{n=1}^{\infty} 4^n \chi_{[n,n+4^{-\varepsilon n}]}(x)$$

An easy calculation shows that

$$\|f\|_{q,\alpha} \le C \left(\sum_{n=1}^{\infty} 4^{(q-\varepsilon)n} n^{\alpha} \right)^{1/q} < \infty$$

and, if $2 \le n \le x < n+1$,

$$\int_0^x f(s)ds \ge \sum_{j=1}^{n-1} 4^{j(1-\varepsilon)} \ge C 4^{n(1-\varepsilon)}.$$

Hence,

$$\|F\|_{p,\beta} \ge C \left(\sum_{n=2}^{\infty} n^{\beta} 4^{(1-\varepsilon)pn} \right)^{1/p} = \infty.$$

We mention that the Hardy inequalities $\|F\|_{p,\alpha-p} \le C\,\|f\|_{p,\alpha}$ for $0 < p < 1$ and $-1 < \alpha < p-1$ under some monotone-type condition of f have been recently studied in [6, 1, 2]. This topic has been originated by Konuyshkov [21], who considered quasi-monotone functions, and Leindler [25], who restricted himself to consideration of functions from the $WM(K)$ class.

In this paper we investigate the (p,q) case and weakly monotone functions.

Theorem 2.1. *Let $p \ge q \le 1$, and $\beta < -1$. Let $f \in WM(K,\lambda,q)$. Then,*

$$\|F\|_{p,\beta} \le C\,\|f\|_{q,\alpha} \qquad \Longleftrightarrow \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1.$$

Furthermore, if $0 < p < q < \infty$ there is no such C.

Proof. Note that F is monotonically increasing and thus $F \in WM(K,\lambda,p)$ for any λ and p. Hence, applying Proposition 2.1, we obtain

$$\|F\|_{p,\beta} \le C\,\|F\|_{q,\frac{q(\beta+1)}{p}-1}.$$

Let $\gamma = \alpha - q = \frac{q(\beta+1)}{p} - 1 < -1$. Then, by Lemma 2.1 with $p=1$,

$$\|F\|_{q,\gamma}^q = \int_0^\infty x^\gamma \left(\int_0^x f(s)ds \right)^q dx \le C \int_0^\infty x^\gamma \int_0^{\lambda x} \frac{f(s)^q}{s^{1-q}} ds dx = C \int_0^\infty \frac{f(s)^q}{s^{1-q}} \int_{s/\lambda}^\infty x^\gamma dx ds.$$

Since $\gamma < -1$, we continue as follows

$$C\left(\int_0^\infty \frac{f(s)^q}{s^{1-q}} s^{1+\gamma} ds\right)^{1/q} = C\left(\int_0^\infty f(s)^q s^{q+\gamma}\right)^{1/q} = C\left\|f\right\|_{q,q+\gamma} = C\left\|f\right\|_{q,\alpha}.$$

The "only if" part as well as the $q > p$ case will be proved in Section 3. $\square$

Remark 2. Note that Theorem 2.1 is optimal with respect to q, that is, for every $1 > q \leq p$ and $q' > q$ there exists $f \in WM(K, \mu, q')$ such that the inequality $\left\|F\right\|_{p,\beta} \leq C\left\|f\right\|_{q,\alpha}$ does not hold.

Proof. Let $q' > q < 1$ and $\lambda > 1$ such that $q' > \lambda^{-1} > q$. Consider the following function:

$$g = \sum_{n=1}^\infty 4^{\lambda^n} \chi_{[n,n+4^{-\lambda^{n-1}}]}$$

and let $f(e^x) = g(x)$. Note that if $1 \leq n \leq x < n+1$, one has

$$g(x)^{q'} \leq 4^{q'\lambda^n} \leq 4^{(\lambda^n)(q'-\lambda^{-1})\lambda^m} \leq \int_{n+m}^{n+m+1} g(s)^{q'} ds \leq \int_{x-m-1}^{x+m+1} g(s)^{q'} ds$$

for $m \in \mathbb{N}$ such that $(q' - \lambda^{-1})\lambda^m > q'$. Thus, from Proposition 1.1 we conclude that $f \in WM(1, e^{m+1}, q')$.

First, we show that

$$\int_0^\infty f(x)^q x^\alpha dx = \int_{-\infty}^\infty g(s)^q e^{s(\alpha+1)} ds \leq C' \sum_{n=1}^\infty 4^{(q-\lambda^{-1})\lambda^n} e^{n(\alpha+1)} < \infty.$$

Now,

$$\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^\beta dx = \int_{-\infty}^\infty \left(\int_{-\infty}^x g(y)e^y dy\right)^p e^{s(\beta+1)} dx$$

and, for $n \in \mathbb{N}$,

$$\int_{n+1}^{n+2} \left(\int_{-\infty}^x g(y)e^y dy\right)^p e^{s(\beta+1)} dx \geq C' e^{(n+1)(\beta+1)} \left(\int_{-\infty}^{n+1} g(y)e^y dy\right)^p$$
$$\geq C' e^{(n+1)(\beta+1)} 4^{\lambda^n p(1-\lambda^{-1})} e^{pn}.$$

Therefore,

$$\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^\beta dx \geq C' \sum_{n=1}^\infty e^{(n+1)(\beta+1)} 4^{\lambda^n p(1-\lambda^{-1})} e^{pn} = \infty$$

and consequently, the inequality $\left\|F\right\|_{p,\beta} \leq C\left\|f\right\|_{q,\alpha}$ is not valid. $\square$

Now, similarly to F, we define an average of f with a lower limit of the integral being non zero and we will see that in this case the set of admissible parameters α, β becomes wider. For $d > 0$, we denote

$$F_d(x) = \int_d^x f(s)ds.$$

Theorem 2.2. *Let $d > 0$. Let $p \geq q \leq 1$, and $\beta < -1$. Let $f \in WM(K, \lambda, q)$ on $[d, \infty]$. Then,*

$$\left\|F_d\right\|_{p,\beta}^{(d)} \leq C\left\|f\right\|_{q,\alpha}^{(d)} \quad \Longleftrightarrow \quad \frac{\alpha+1}{q} \geq \frac{\beta+1}{p} + 1.$$

Proof. Applying Theorem 2.1 to $f\chi_{[d,\infty]}$ we obtain the result in the case $\frac{\alpha+1}{q} = \frac{\beta+1}{p}+1$. The remaining cases follow from the following fact:

$$\|f\|_{p,\alpha}^{(d)} \leq d^{-\varepsilon/p} \|f\|_{p,\alpha+\varepsilon}^{(d)} \text{ for } \varepsilon > 0 \text{ and } d > 0.$$

The proof of the "only if" part will be given in Section 3. □

Theorem 2.3. *Let $d > 0$. Let $q > p \leq 1$, and $\beta < -1$. Let $f \in WM(K, \lambda, p)$ on $[d, \infty]$ Then,*

$$\|F_d\|_{p,\beta}^{(d)} \leq C \|f\|_{q,\alpha}^{(d)} \qquad \text{if and only if} \qquad \frac{\alpha+1}{q} > \frac{\beta+1}{p} + 1.$$

Proof. Applying Theorem 2.1 to $f\chi_{[d,\infty]}$ we obtain

$$\|F_d\|_{p,\beta}^{(d)} \leq C \|f\|_{p,\beta+p}^{(d)}.$$

Finally, since $q > p$ we can use Proposition 2.2 to obtain

$$\|f\|_{p,\beta+p}^{(d)} \leq C \|f\|_{q,\alpha}^{(d)}$$

for $\frac{\alpha+1}{q} > \frac{\beta+1+p}{p}$. The proof of the "only if" part will be given in Section 3. □

Note that since F is non-decreasing, we have $\|F\|_{\infty,\beta} = \sup_{x\in[0,\infty]} F(x) = \int_0^\infty f(s)ds$. Thus,

Theorem 2.4 (Case $p = \infty$). *Let $q \leq 1$ and $f \in WM(K, \lambda, q)$, then*

$$\int_0^\infty f(s)ds \leq C \left(\int_0^\infty f^q(s) s^\alpha ds \right)^{1/q}$$

if and only if $\alpha = q - 1$.

Proof. The "if" part is a restatement of Lemma 2.1. The proof of the "only if" part will be given in Section 3. □

As an immediate corollary, we obtain

Theorem 2.5. *Let $d > 0$. Let $q \leq 1$ and $f \in WM(K, \lambda, q)$ on $[d, \infty]$, then*

$$\int_d^\infty f(s)ds \leq C \left(\int_d^\infty f^q(s) s^\alpha ds \right)^{1/q}$$

if and only if $\alpha \geq q - 1$.

For $0 < D \leq \infty$, denote

$$G^*(x) = \int_x^D g(s)ds$$

and

$$\|g\|_{p,\alpha}^{*,(D)} = \left(\int_0^D |g(s)|^p s^\alpha ds \right)^{1/p}.$$

The following result is well known, see for example, [24].

Theorem C. (see, e.g., [24]). *Let $1 < q \leq p$, then there exists C such that*

$$\|G^*\|_{p,\beta}^{*,(\infty)} \leq C \|g\|_{q,\alpha}^{*,(\infty)} \qquad \Longleftrightarrow \qquad \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1 \text{ and } \beta > -1.$$

We obtain the following counterparts of Theorems $2.5, 2.7, 2.8, 2.9$ and 2.10.

Theorem 2.6. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D \leq \infty$. Let also $\beta > -1$.*

1. *Let $g \in WM(K,\lambda,q)$. If $1 \geq q \leq p \leq \infty$, then $\|G^*\|_{p,\beta}^{*,(\infty)} \leq C \|g\|_{q,\alpha}^{*,(\infty)} \iff \frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1$. Furthermore, if $\infty > q > p > 0$ there is no such C.*

2. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D < \infty$. If $1 \geq q \leq p \leq \infty$, then $\|G^*\|_{p,\beta}^{*,(D)} \leq C \|g\|_{q,\alpha}^{*,(D)} \iff \frac{\alpha+1}{q} \leq \frac{\beta+1}{p} + 1$.*

3. *Let $g \in WM(K,\lambda,q)$ on $[0,D]$ for $0 < D < \infty$. If $q > p \leq 1$, then $\|G^*\|_{p,\beta}^{*,(D)} \leq C \|g\|_{q,\alpha}^{*,(D)} \iff \frac{\alpha+1}{q} < \frac{\beta+1}{p} + 1$.*

Proof. Let $d = 1/D$. Denote
$$g(t) = f(t^{-1})t^{-2}.$$
Note that $g(t^{-1})t^{-2} = f(t)$. Using the properties of p-weakly monotone functions, we know that $g \in WM(K',\lambda,p)$ on $[0,1/d]$ if and only if $f \in WM(K,\lambda,p)$ on $[d,\infty]$.
We have
$$G^*(x) = \int_x^{1/d} g(s)ds.$$
Since
$$\int_x^{1/d} g(s)ds = \int_d^{1/x} g(t^{-1})t^{-2}dt = \int_d^{1/x} f(t)dt,$$
we have
$$G^*(x) = F(x^{-1}).$$
Similarly, we derive that
$$\|G^*\|_{p,-\beta-2}^{*,(1/d)} = \|F\|_{p,\beta}^{(d)} \quad \text{and} \quad \|g\|_{q,2q-2-\alpha}^{*,(1/d)} = \|f\|_{q,\alpha}^{(d)}.$$
Thus,
$$\|G^*\|_{p,-\beta-2}^{*,(1/d)} \leq C \|g\|_{q,2q-2-\alpha}^{*,(1/d)} \quad \text{if and only if} \quad \|F\|_{p,\beta}^{(d)} \leq C \|f\|_{q,\alpha}^{(d)}.$$
Finally, using Theorems $2.5, 2.7, 2.8, 2.9, 2.10$, the result follows. □

3 Optimality

Note that if we prove the sharpness of Theorems $2.1, 2.2, 2.3, 2.4$, then the sharpness of Propositions 2.1 and 2.2 follows. Remark also that for $\gamma > -1$, $g(x) = x^\gamma \chi_{[0,1]}$ is $WM(K,\lambda,p)$ for every p and λ since
$$x^{p\gamma} \leq K \int_{x/\lambda}^x s^{p\gamma-1}ds \leq K \int_{x/\lambda}^{\lambda x} \frac{g(s)^p}{s}ds.$$
Denote $\int_0^x g(s)ds = G(x)$.

Now, $\|g\|_{q,\alpha} < \infty \iff \gamma > \frac{-1-\alpha}{q}$, and, for $\beta < -1$, $\|G\|_{p,\beta} = \infty$ if and only if either $\gamma \leq -1$ or $\gamma \leq \frac{-1-\beta}{p} - 1$. So if $\frac{-1-\alpha}{q} < \gamma < \frac{-1-\beta}{p} - 1$, $\|G\|_{p,\beta} = \infty$ and $\|g\|_{q,\alpha} < \infty$. Thus, the inequality in Theorem 2.1 cannot possibly hold for $\frac{1+\alpha}{q} > \frac{1+\beta}{p} + 1$.

For the $p = \infty$ case (Theorem 2.4), the same considerations for $x^\gamma \chi_{[0,1]}$ suffice to obtain the condition $\frac{1+\alpha}{q} \leq 1$.

Now for $d \geq 0$. Let $p \neq \infty$, define

$$g(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p} \left(-\frac{\beta+1}{p} - \frac{1}{p} \frac{x}{(x+b)\ln(x+b)} \right).$$

Note that since $\beta < -1$, for large enough b, $f(x) > 0$ for $x > 0$, and

$$Dx^{-(\beta+1+p)/p} \ln(x+b)^{-1/p} > g(x) > Cx^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}$$

for some $C, D > 0$. It is easy to see that

$$G(x) = \int_0^x g(s)ds = x^{-(\beta+1)/p} \ln(x+b)^{-1/p}.$$

Set

$$f(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}.$$

It is clear that $f(x)$ and $g(x)$ have the same behaviour at infinity and so do $F(x) = \int_0^x f(s)ds$ and $G(x)$.

Now assume that there is a locally integrable function h and $M > 0$ such that

1. $h(x) = 0$ for $x < d+1$;
2. $h(x) = f(x)$ for $x > M$;
3. $\int_d^x h(s)ds = H(x) = F(x) > D^{-1}G(x)$ for $x > M$;
4. $h \in WM(K, \lambda, r)$ on $[d, \infty]$ for any r.

Then

$$\left(\|H\|_{p,\beta}^{(d)} \right)^p > D^{-1} \int_M^\infty \frac{1}{x\ln(x+b)} dx = \infty$$

and

$$\left(\|h\|_{q,\alpha}^{(d)} \right)^q = \int_{d+1}^M h^q(x) x^\alpha dx + \int_M^\infty \frac{1}{x^{q(\beta+1+p)/p-\alpha} \ln(x+b)^{q/p}} dx,$$

which is finite provided either $\frac{q(\beta+1+p)}{p} - \alpha > 1$ or $\frac{q(\beta+1+p)}{p} - \alpha = 1$ and $q > p$.

So if $\frac{q(\beta+1+p)}{p} - \alpha > 1$ (or, equivalently, $\frac{\alpha+1}{q} < \frac{\beta+1}{p} + 1$) or if $\frac{q(\beta+1+p)}{p} - \alpha = 1$ (or, equivalently, $\frac{\alpha+1}{q} = \frac{\beta+1}{p} + 1$) and $q > p$, the inequalities in Theorems 2.1, 2.2 and 2.3 cannot hold.

All that remains is to build h satisfying the former properties. Since $f(x) = x^{-(\beta+1+p)/p} \ln(x+b)^{-1/p}$, if $(\beta+1+p)/p < 0$, f will be eventually monotonically increasing, say for $x > N$. Now, let $n \in \mathbb{N}$ be such that $(d+1)\lambda^n \geq N$ and $n \geq 4$. For $m \in \mathbb{R}^+$, let

$$h(x,m) = \begin{cases} 0, & x < d+1 \\ m, & d+1 \leq x \leq (d+1)\lambda^2, \\ 0, & (d+1)\lambda^2 < x < \lambda^n(d+1), \\ f(x), & x \geq (d+1)\lambda^n. \end{cases} \tag{3.1}$$

Note that for any $x > 0$, h is monotonic on $[x/\lambda, \lambda x]$, thus $h \in WM(K, \lambda, r)$ for any $r > 0$ and for some K. Furthermore, by construction $h(x) = 0$, for $x < d+1$ and $h(x) = f(x)$ for $x \geq M = (d+1)\lambda^n$.

Finally, since

$$\int_0^M h(x,0)dx \leq \int_0^M f(s)ds < \int_0^M h(x,\infty)dx = \infty,$$

by continuity there must be some m^* such that

$$\int_0^M h(x,m^*)dx = \int_0^M f(s)ds.$$

So if $h(x) = h(x,m^*)$, one has

$$\int_0^x h(s,m^*)ds = \int_0^x f(s)ds$$

for $x > M$.

Obviously, if $(\beta+1+p)/p \geq 0$, $f(x)$ will be always decreasing. For $m \in \mathbb{R}^+$, let

$$h(x,m) = \begin{cases} 0, & x < d+1, \\ m, & d+1 \leq x \leq (d+1)\lambda^2, \\ f(x), & x \geq (d+1)\lambda^2. \end{cases} \tag{3.2}$$

Note that if $m \geq f((d+1)\lambda^2)$, then for any $x > 0$, h is monotonic on $[x/\lambda, \lambda x]$, thus $h \in WM(K,\lambda,r)$ for any $r > 0$ and for some K.

Let $m = \frac{F(\lambda^2(d+1))}{\lambda^2(d+1)-(d+1)} \geq \frac{F(\lambda^2(d+1))}{\lambda^2(d+1)} \geq f(\lambda^2(d+1))$, where the last inequality holds because f is decreasing. Then $h(x,m)$ is the desired counterexample.

The only case that remains is when $p = \infty$. To deal with it, it suffices to use the previously described idea to build a locally monotonic function which agrees with x^γ for large enough x.

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