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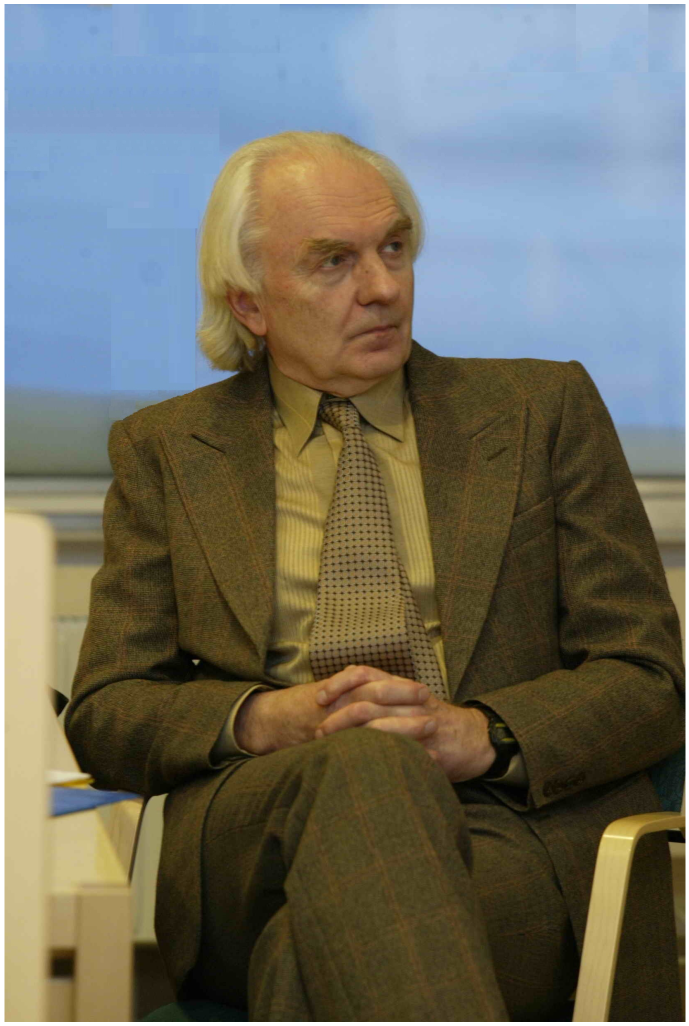
On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \leq p, \theta \leq \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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n-MULTIPLICITY AND SPECTRAL PROPERTIES FOR (M,k)-QUASI-*-CLASS Q OPERATORS

A. Nasli Bakir, S. Mecheri

Communicated by T. Bekjan

Key words: hyponormal operators, (M,k)-quasi-*-class Q operators, k-quasi-*-class $\mathbb{A}$ operators.

AMS Mathematics Subject Classification: 47A30, 47B47, 47B20.

Abstract. In the present article, we introduce a new class of operators which will be called the class of (M,k)-quasi-*-class Q operators. An operator $A \in B(H)$ is said to be (M,k)-quasi-*-class Q for certain integer k, if there exists $M > 0$ such that

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

Some properties of this class of operators are shown. It is proved that the considered class contains the class of k-quasi-*-class $\mathbb{A}$ operators. The decomposition of such operators, their restrictions on invariant subspaces, the n-multicyclicity and some spectral properties are also presented. We also show that if $\lambda \in \mathbb{C}, \lambda \neq 0$ is an isolated point of the spectrum of A, then the Riesz idempotent E for λ is self-adjoint, and verifies $EH = ker(A-\lambda) = ker(A-\lambda)^{\star}$.

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1 Introduction

Let H be an infinite dimensional complex separable Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Denote, respectively, by $ker(A)$ and $ran(A)$ the null space and the range space of an operator A in $B(H)$. As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved interesting problems for hyponormal operators (e.g., the invariant subspace problem), one of recent trends in operator theory is to study natural extensions of hyponormal operators. Below we introduce some of these non-hyponormal operators. Recall ([3, 7]) that $A \in B(H)$ is called hyponormal if $A^*A \geq AA^*$, paranormal if $\|A^2x\| \geq \|Ax\|^2$ and $*$-paranormal if $\|A^2x\| \geq \|A^*x\|^2$) for each unit vector $x \in H$. Following [7] and [12], we say that $A \in B(H)$ belongs to class $\mathbb{A}$ if $|A^2| \geq |A|^2$ where $A^*A = |A|^2$. Recently, B.P. Duggal, I.H. Jeon and I.H. Kim [6] considered the following new class of operators: an operator $A \in B(H)$ is said to belong to the $*$-class $\mathbb{A}$ if $|A^2| \geq |A^*|^2$. For brevity, we shall denote classes of hyponormal operators, paranormal operators, $*$-paranormal operators, class $\mathbb{A}$ operators, and $*$-class $\mathbb{A}$ operators by $\mathcal{H}$, $\mathcal{PN}$, $\mathcal{PN}^*$, $\mathcal{A}$ and $\mathcal{A}^*$ respectively. From [3] and [7], it is well known that

$$\mathcal{H} \subset \mathcal{A} \subset \mathcal{PN}$$

and

$$\mathcal{H} \subset \mathcal{A}^* \subset \mathcal{PN}^*.$$

Recently, the authors of [23] have extended $*$-class $\mathbb{A}$ operators to quasi-$*$-class $\mathbb{A}$ operators. An operator $A \in B(H)$ is said to be quasi-$*$-class $\mathbb{A}$ if $A^*|A^2|A \geq A^*|A^*|^2A$, and quasi-$*$-paranormal if

$$\|A^*Ax\|^2 \leq \|A^3x\|\|Ax\|$$

for all $x \in H$. In [19], many results on quasi-$*$-paranormal operators were proved. In particular, quasi-$*$-paranormal operators have Bishop's property (β) [19]. If we denote the class of quasi-$*$-class $\mathbb{A}$ operators by $\mathcal{QA}^*$ and of quasi-$*$-paranormal operators by $\mathcal{QPN}^*$, we have

$$\mathcal{H} \subset \mathcal{A}^* \subset \mathcal{QA}^* \subset \mathcal{QPN}^*.$$

As a further generalization, S.Mecheri in [16, 14] introduced the class of k-quasi-$*$-class $\mathbb{A}$ operators and the class of k-quasi-$*$-paranormal operators [20]. An operator T is said to be a k-quasi-$*$-class $\mathbb{A}$ operator if

$$A^k(|A^2| - |A^*|^2)A^k \geq 0$$

where k is a natural number and k-quasi-$*$-paranormal if

$$\|A^*A^kx\|^2 \leq \|A^{k+2}x\|\|A^kx\|$$

for all unit vector $x \in H$ where k is a natural number. 1-quasi-$*$-class $\mathbb{A}$ is quasi-$*$-class $\mathbb{A}$ and 1-quasi-$*$-paranormal is quasi-$*$-paranormal. It is shown that a k-quasi-$*$-class $\mathbb{A}$ operator is a k-quasi-$*$-paranormal operator [20].

An operator A in $B(H)$ is said to be an M-*-class Q operator [5], if there exists $M > 0$ such that

$$MA^{*2}A^2 - 2AA^* + I \geq 0.$$

$A \in B(H)$ is said to be (M,k)-quasi-*-class Q operator [5], if

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

For $k = 1$, A is an M-quasi-*-class Q operator. It is clear that

$$M\text{-*-class } Q \subset M\text{-quasi-*-class } Q \subset (M,k)\text{-quasi-*-class } Q$$

and that

$$(M,k)\text{-quasi-*-class } Q \subset (M,k+1)\text{-quasi-*-class } Q.$$

Example Consider on the Hilbert space l_2, equipped with its standard orthonormal basis $(e_n)_n$, the weighted right shift defined by $Se_n = \lambda_n e_{n+1}$, where $(\lambda_n)_n$ is a decreasing complex sequence. Then, S is an (M,k)-quasi-*-class Q operator if and only if

$$M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 + 1 \geq 2|\lambda_{n+k-1}|^2$$

for all n. Indeed, we have

$$\begin{aligned}
&\langle S^{*k}(MS^{*2}S^2 - 2SS^* + I)S^ke_n, e_n\rangle \geq 0 \\
&\Leftrightarrow (M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1)\lambda_n\lambda_{n+1}...\lambda_{n+k-1}\overline{\lambda_{n+k-1}\lambda_{n+k-2}...\lambda_n} \geq 0 \\
&\Leftrightarrow (M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1)|\lambda_n|^2|\lambda_{n+1}|^2...|\lambda_{n+k-1}|^2 \geq 0 \\
&\Leftrightarrow M|\lambda_{n+k}|^2|\lambda_{n+k+1}|^2 - 2|\lambda_{n+k-1}|^2 + 1 \geq 0.
\end{aligned}$$

In this paper, we are interested in the study of (M,k)-quasi-*-class Q operators. Some properties of this class of operators are shown. It is proved that this class of operators contains the class of k-quasi-*-class $\mathbb{A}$ operators. The decomposition of such operators, their restrictions on invariant subspaces and other related results are also presented.

2 Main results

We will start by the following useful theorem.

Theorem 2.1. *An operator $A \in B(H)$ is an (M,k)-quasi-*-class Q operator if and only if*

$$2\left\|A^*A^kx\right\|^2 \le M\left\|A^{k+2}x\right\|^2 + \left\|A^kx\right\|^2$$

for all x in H.

Proof. There exists $M > 0$ such that

$$\left\langle A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^kx, x\right\rangle \ge 0$$

for all $x \in H$. Hence,

$$M\left\langle A^{*k+2}A^{k+2}x, x\right\rangle + \left\langle A^{*k}A^kx, x\right\rangle \ge 2\left\langle A^*A^kx, A^*A^kx\right\rangle.$$

Thus,

$$2\left\|A^*A^kx\right\|^2 \le M\left\|A^{k+2}x\right\|^2 + \left\|A^kx\right\|^2.$$

The converse can be proved in a similar way. □

Remark 1. It is clear that the class of (M,k)-quasi-*-class Q operators is nested with respect to M, i.e.,

$$(M_1,k)\text{-quasi-*-class } Q \subset (M_2,k)\text{-quasi-*-class } Q$$

whenever $M_1 \le M_2$.

Remark 2. The class of (M,k)-quasi-*-class Q operators is not convex. For example, the operators $A=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are 4-quasi-*-class Q. However, the operator $C = \frac{1}{3}A + \frac{2}{3}B$ is not a 4-quasi-*-class Q operator since

$$2\|C^*C(0,-1)\|^2 = \frac{20}{81} > 4\|C^3(0,-1)\|^2 + \|C(0,-1)\|^2 = \frac{85}{729}.$$

Remark 3. Also, the operator $A - I$ is not a $(4,k)$-quasi-*-class Q operator. This shows that the above class is not translation invariant.

Theorem 2.2. *If $A \in B(H)$ is an (M,k)-quasi-*-class Q operator with dense range, then A is an M-*-class Q operator.*

Proof. Let $x \in H$. Since A has dense range, there exists a sequence $(x_n)_n$ in H for which $\lim_{n\to\infty} Ax_n = x$. By the continuity of A, $\lim_{n\to\infty} A^kx_n = A^{k-1}x$. Hence, and by the continuity of the inner product,

$$\begin{aligned}
\left\|A^*A^{k-1}x\right\|^2 &= \left\|\lim_{n\to\infty} A^*A^kx_n\right\|^2 = \lim_{n\to\infty}\left\|A^*A^kx_n\right\|^2 \\
&\le \lim_{n\to\infty}\frac{1}{2}(M\left\|A^{k+1}x_n\right\|^2 + \left\|A^kx_n\right\|^2) \\
&= \frac{1}{2}(M\left\|\lim_{n\to\infty} A^{k+1}x_n\right\|^2 + \left\|\lim_{n\to\infty} A^kx_n\right\|^2) \\
&= \frac{1}{2}(M\left\|A^{k+1}x\right\|^2 + \left\|A^{k-1}x\right\|^2).
\end{aligned}$$

Thus, A is an $(M, k-1)$-quasi-*-class Q operator. Since $ran(A)$ is dense in H, A is an $(M, k-2)$-quasi-*-class Q operator. By iteration, A is an M-*-class Q operator. □

Corollary 2.1. *Let A be a nonzero (M, k)-quasi-*-class Q operator, but not an M-*-class Q operator. Then A admits a non trivial closed invariant subspace.*

Proof. Suppose that A has no non trivial closed invariant subspaces. Since $A \neq 0$, $ker(A) \neq H$ and $\overline{ran(A)} \neq \{0\}$ are closed invariant subspaces for A. Thus, necessarily, $ker(A) = \{0\}$ and $\overline{ran(A)} = H$. By Theorem 2.2, A is an M-*-class Q operator. This contradicts the hypothesis. □

Theorem 2.3. *Let $A \in B(H)$ be an (M, k)-quasi-*-class Q operator. If $N \subset H$ is a closed A-invariant subspace, then the restriction $A|N$ is an (M, k)-quasi-*-class Q operator.*

Proof. Let
$$A = \begin{pmatrix} T & S \\ 0 & R \end{pmatrix} \text{ on } H = N \oplus N^{\perp}.$$

Then, for all integer m, $m \geq 2$, we get

$$A^m = \begin{pmatrix} T^m & \sum_{p=0}^{m-1} T^{m-1-p} S R^p \\ 0 & R^m \end{pmatrix}.$$

Since A is (M, k)-quasi-*-class Q, there exists $M > 0$ such that

$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k \geq 0.$$

Then,
$$A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^k = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$
where, $X = T^{*k}(MT^{*2}T^2 - 2TT^* - 2SS^* + I)T^k$, Y is a bounded operator from N to $N^{\perp}$ and Z is a bounded operator on $N^{\perp}$. According to [4, Theorem 6], $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0, Z \geq 0$ and $Y = X^{\frac{1}{2}} C Z^{\frac{1}{2}}$ for some contraction C on H. Therefore,
$$X = T^{*k}(MT^{*2}T^2 - 2TT^* - 2SS^* + I)T^k \geq 0.$$
Since $SS^* \geq 0$,
$$T^{*k}(MT^{*2}T^2 - 2TT^* + I)T^k \geq 0$$
which completes the proof. □

Theorem 2.4. *If $B \in B(H)$ is unitarily equivalent to an (M, k)-quasi-*-class Q operator A on H, then B is also an (M, k)-quasi-*-class Q operator.*

Proof. There exists a unitary operator V on H satisfying $B = VAV^*$. Since A is an (M, k)-quasi-*-class Q operator,

$$\begin{aligned}
&B^{*k}(MB^{*2}B^2 - 2BB^* + I)B^k \\
&= (VAV^*)^{*k}\left[M(VAV^*)^{*2}(VAV^*)^2 - 2VAV^*(VAV^*)^* + I\right](VAV^*)^k \\
&= VA^{*k}V^*\left[MVA^{*2}V^*VA^2V^* - 2VA^2V^* + I\right]VA^kV^* \\
&= VA^{*k}(MA^{*2}A^2 - 2AA^* + I)A^kV^* \geq 0.
\end{aligned}$$

Thus, B is an (M, k)-quasi-*-class Q operator. □

Remark 4. Theorem 2.4 is in general false if the operator U is invertible and not unitary. Indeed, the bilateral weighted shift S defined on the Hilbert space $\ell_2(\mathbb{Z})$ by

$$Se_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \sqrt{2}e_3 & n = 2 \end{cases}$$

is in particuliar a $(3,k)$-quasi-*-class Q, and the operator

$$Ue_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \frac{1}{3}e_3 & n = 2 \end{cases}$$

is invertible and not unitary. Nonetheless, the operator $U^{-1}SU$ is not a $(3,k)$-quasi-*-class Q operator.

Theorem 2.5. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator. If A commutes with an isometric operator $S \in B(H)$, then AS is an (M,k)-quasi-*-class Q operator.*

Proof. We have $AS = SA$ and $S^*S = I$. Since A is an (M,k)-quasi-*-class Q operator,

$$\begin{aligned}
&(AS)^{*k}(M(AS)^{*2}(AS)^2 - 2AS(AS)^* + I)(AS)^k \\
&= S^{*k}A^{*k}\left[MS^*A^*S^*A^*ASAS - 2ASS^*A^* + I\right]S^kA^k \\
&= A^{*k}S^{*k}\left[MA^{*2}A^2 - 2ASS^*A^* + I\right]S^kA^k \\
&= A^{*k}S^{*k-1}\left[MS^*A^{*2}A^2S - 2S^*ASS^*A^*S + S^*S\right]S^{k-1}A^k \\
&= A^{*k}S^{*k-1}\left[MA^{*2}A^2 - 2AA^* + I\right]S^{k-1}A^k \\
&= S^{*k-1}A^{*k}\left[MA^{*2}A^2 - 2AA^* + I\right]A^kS^{k-1} \geq 0.
\end{aligned}$$

Thus, AS is an (M,k)-quasi-*-class Q operator.

□

Theorem 2.6. *Let $A \in B(H)$ be an (M,k) -quasi-*-class Q operator. Assume that $\overline{A^kH} \neq H$, and that*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with respect to the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{*k})$. Then, A_1 is an M --class Q operator and $A_3^k = 0$. Moreover, $\sigma(A) = \sigma(A_1) \cup \{0\}$, where $\sigma(A)$ denotes the spectrum of A.*

Proof. Since A is an (M,k)-quasi-*-class Q operator,

$$\left\langle A^{*k}(MA^{*2}A^2 - 2AA^* + I)A^ky, y\right\rangle \geq 0$$

for all $y \in H$. Hence,

$$\left\langle (MA^{*2}A^2 - 2AA^* + I)A^ky, A^ky\right\rangle \geq 0.$$

Thus, for all $x \in \overline{ran(A^k)}$,

$$\left\langle (MA^{*2}A^2 - 2AA^* + I)x, x\right\rangle = \left\langle (MA_1^{*2}A_1^2 - 2A_1A_1^* + I)x, x\right\rangle \geq 0.$$

Consequently, A_1 is an M-*-class Q operator. Let now, P be the orthogonal projection on $\overline{ran(A^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$, we have

$$\left\langle A_3^kx_2, y_2\right\rangle = \left\langle A^k(I-P)x, (I-P)y\right\rangle = \left\langle (I-P)x, A^{*k}(I-P)y\right\rangle = 0.$$

Thus, $A_3^k = 0$. Furthermore,

$$\sigma(A_1) \cup \sigma(A_3) = \sigma(A) \cup \Omega$$

where Ω is the union of the holes in $\sigma(A)$ which happen to be subsets of $\sigma(A_1) \cap \sigma(A_3)$ using [9, Corollary 7], and $\sigma(A_1) \cap \sigma(A_3)$ has no interior points and A_3 is nilpotent. Thus, $\sigma(A) = \sigma(A_1) \cup \{0\}$. □

It is shown in [25] that for $A, B, Q \in B(H)$, the equation $BX - XA = Q$ admits a unique solution whenever $\sigma(A)$ and $\sigma(B)$ are disjoint. For more details, reader can see [13, 18, 24] and [26].

Corollary 2.2. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator. If the restriction $A_1 = A\big|_{\overline{ran(A^k)}}$ is invertible, then A is similar to the sum of an M-*-class Q operator and a nilpotent operator.*

Proof. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{on } H = \overline{ran(A^k)} \oplus ker(A^{*k}).$$

Then, A_1 is an M-*-class Q operator by the above Theorem. Since A_1 is invertible, $0 \notin \sigma(A)$. Hence, $\sigma(A_1) \cap \sigma(A_3) = \emptyset$. By Rosenblum's result [18, 25, 27], there exists $C \in B(H)$ for which $A_1C - CA_3 = A_2$. Thus,

$$\begin{aligned} A &= \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}. \end{aligned}$$

□

Let $A \in B(H)$. Denote by $\mathcal{R}(\sigma(A))$ the set of all rational analytic functions on $\sigma(A)$. The operator A is said to be n-multicyclic [11], if there exist n (generating) vectors $x_1, x_2, .., x_n$ in H such that

$$\bigvee \{g(A)x_i \ , \ 1 \leq i \leq n \ , \ g \in \mathcal{R}(\sigma(A))\} = H$$

where $\bigvee$ denotes the linear span, that is, the set of all finite linear combinations.

We have then

Theorem 2.7. *If A is an n-multicyclic (M,k)-quasi-*-class Q operator, then its restriction on $\overline{ran(A^k)}$ is also n-multicyclic.*

Proof. Put

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

on the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{*k})$. Since $\sigma(A_1) \subset \sigma(A)$ by Theorem 2.6, $\mathcal{R}(\sigma(A_1)) \subset \mathcal{R}(\sigma(A))$. The operator A is n-multicyclic. Then, there exist n generating vectors $x_1, x_2, .., x_n \in H$ for which

$$\bigvee \{g(A)x_i, \ 1 \leq i \leq n, \ g \in \mathcal{R}(\sigma(A))\} = H.$$

Put $y_i = A^k x_i$, $1 \leq i \leq n$. Hence,

$$\begin{aligned}\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\} &= \bigvee\left\{g(A_1)A^kx_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \bigvee\left\{g(A)A^kx_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \bigvee\left\{A^kg(A)x_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\right\}\\ &= \overline{ran(A^k)}.\end{aligned}$$

But

$$\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A))\}\subset\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ g\in\mathcal{R}(\sigma(A_1))\}.$$

Thus,

$$\overline{ran(A^k)}\subset\bigvee\{g(A_1)y_i,\ 1\le i\le n,\ \ g\in\mathcal{R}(\sigma(A_1))\}.$$

Therefore, $\{y_i\}_{i=1}^n$ are n-generating vectors of A_1, and A_1 is n-multicyclic. □

Recall that an operator $A\in B(H)$ is said to be a class $\mathbb{A}$ operator [8, 20, 23] if $|A^2|-|A|^2\ge 0$. This class was introduced by Furuta-Ito-Yamazaki [8], and it is shown that it contains both p-hyponormal operators and log-hyponormal operators. It is also proved in [8, 28] that the class $\mathbb{A}$ is a subclass of paranormal operators. It is known that p-hyponormal operators are normaloid, i.e., $\|A\|=r(A)$ where $r(A)$ denotes the spectral radius of A. However, a quasi-class $\mathbb{A}$ operator is not normaloid [23], [28]. $A\in B(H)$ is said to be in the -*-class $\mathbb{A}$ if $|A|^2-|A^*|^2\ge 0$, and in the k-quasi-*-class $\mathbb{A}$ if $A^{*k}(|A|^2-|A^*|^2)A^k\ge 0$ for a positive integer k. A 1-quasi-*-class $\mathbb{A}$ operator is quasi-*-class $\mathbb{A}$.

In the sequel, we will show that the (M,k)-quasi-*-class Q operators contains the k-quasi-*-class $\mathbb{A}$. We need first the following result

Lemma 2.1. *If $A\in B(H)$ is a k-quais-*-class $\mathbb{A}$ operator, then*

$$\||A|^2A^kx\|\le\|A^{k+2}x\|$$

for all $x\in H$.

Proof. Let x be any vector in H. Since A is a k-quasi-*-class $\mathbb{A}$, we have

$$\begin{aligned}\||A|^2A^kx\|^2=\|A^*AA^kx\|^2 &= \langle A^*A^{k+1}x,A^*A^{k+1}x\rangle\\ &= \langle x,A^*(A^{*k}AA^*A^k)Ax\rangle\\ &= \langle Ax,(A^{*k}AA^*A^k)Ax\rangle\\ &= \langle(A^{*k}AA^*A^k)Ax,Ax\rangle\\ &\le \langle A^{*k+1}A^{k+1}Ax,Ax\rangle\\ &= \|A^{k+2}x\|^2.\end{aligned}$$

□

Theorem 2.8. *An operator belonging to the k-quasi-*-class $\mathbb{A}$ is an (M,k)-quasi-*-class Q operator.*

Proof. Let A be a k-quasi-*-class $\mathbb{A}$ operator. Then,

$$A^{*k}(|A|^2-|A^*|^2)A^k\ge 0.$$

Hence, for $M\ge 1$ we have

$$A^{*k}(\sqrt{M}\,|A|^2-|A^*|^2)A^k\ge 0.$$

Thus, for all $x \in H$,

$$\begin{aligned} \langle A^{*k} |A^*|^2 A^k x, x\rangle &= \langle A^{*k} A A^* A^k x, x\rangle \\ &= \left\|A^* A^k x\right\|^2 \\ &\leq \left\langle \sqrt{M} A^{*k} |A|^2 A^k x, x\right\rangle \\ &= \left\langle \sqrt{M} |A|^2 A^k x, A^k x\right\rangle. \end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemma 2.1,

$$\begin{aligned} \left\|A^* A^k x\right\|^2 &\leq \sqrt{M} \left\||A|^2 A^k x\right\| \left\|A^k x\right\| \\ &\leq \sqrt{M} \left\|A^{k+2} x\right\| \left\|A^k x\right\| \\ &\leq \frac{1}{2}(M \left\|A^{k+2} x\right\|^2 + \left\|A^k x\right\|^2). \end{aligned}$$

This shows that A is an (M, k)-quasi-*-class Q operator.

□

Theorem 2.9. *If $A \in B(H)$ with $\|A\| \leq \frac{1}{\sqrt{2}}$, then A is an (M, k)-quasi-*-class Q operator.*

Proof. Let $x \in H$. We have $\|A^* x\| \leq \frac{1}{\sqrt{2}} \|x\|$. Hence,

$$\begin{aligned} \langle (M A^{*2} A^2 - 2AA^* + I)x, x\rangle &= M \left\|A^2 x\right\|^2 - 2\left\|A^* x\right\|^2 + \|x\|^2 \\ &\geq M \left\|A^2 x\right\|^2 - \|x\|^2 + \|x\|^2 \geq M \left\|A^2 x\right\|^2 \\ &\geq 0. \end{aligned}$$

Thus,

$$\langle A^{*k}(M A^{*2} A^2 - 2AA^* + I)A^k x, x\rangle \geq 0.$$

□

Recall that an operator A in $B(H)$ is said to have the *Single Valued Extension Property*, briefly SVEP, at a complex number α, if for each open neighborhood V of α, the unique analytic function $f\colon V \to H$ that satisfies

$$\forall \lambda \in V : (A - \lambda) f(\lambda) = 0$$

is the function $f \equiv 0$. If furthermore, A has SVEP at every $\alpha \in \mathbb{C}$, we say that A has SVEP. For more details see ([2, 17, 15, 21]).

Also, the *local resolvent set* of A at a vector $x \in H$, denoted by $\rho_A(x)$, is defined to consist of all complex elements z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, for which $(A - z)f(z) = x$. [2]

The set $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$ is called the *local spectrum* of A at x. We've then the following important result.

Theorem 2.10. *Let*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

be an (M, k)-quasi--class Q operator with respect to the decomposition $H = \overline{ran(A^k)} \oplus ker(A^{\star k})$. Then, for all $x = x_1 + x_2 \in H$:*

i. $\sigma_{A_3}(x_2) \subset \sigma_A(x_1 + x_2)$.

ii. $\sigma_{A_1}(x_1) = \sigma_A(x_1 + 0)$.

Proof. i. Let $z_0 \in \rho_A(x_1 + x_2)$. By the definition of the local resolvent set of A at x, there exists a neighborhood U of z_0 and an analytic function $f(z)$ defined on U, with values in H, for which

$$(A - z)f(z) = x, \; z \in U. \tag{2.1}$$

Let $f = f_1 + f_2$ where

$$f_1 : U \to \overline{ran(A^k)}), \quad f_2 : U \to ker(A^{*k})$$

are in the Frechet spaces $O(U, \overline{ran(A^k)}), O(U, ker(A^{*k}))$ respectively, consisting of analytic functions on U with values in H, and equipped with the topology of uniform convergence, [2]. Equality (2.1) can then be written

$$\begin{pmatrix} A_1 - z & A_2 \\ 0 & A_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then, for all $z \in U$,

$$(A_3 - z)f_2(z) = x_2.$$

Hence,

$$z_0 \in \rho_{A_3}(x_2).$$

Thus, (i) holds by passing to the complement.

ii. For $z_1 \in \rho_A(x_1 + 0)$, there exists a neighborhood V_1 of z_1 and an analytic function g defined on V_1 with values in H verifying

$$(A - z)f(z) = x_1 + 0, \; z \in V_1. \tag{2.2}$$

Let $g = g_1 + g_2$, where

$$g_1 \in O(V_1, \overline{ran(A^k)}), \; g_2 \in O(V_1, ker(A^{*k}))$$

be as in (i). From equation (2.2), we obtain

$$(A_1 - z)g_1(z) + A_2 g_2(z) = x_1$$

and

$$(A_3 - z)g_2(z) = 0, \; z \in V_1$$

Since A_3 is nilpotent by Theorem 2.6, A_3 has SVEP by [2]. Thus,

$$g_2(z) = 0$$

Consequently,

$$(A_1 - z)g_1(z) = x_1$$

Therefore, $z_1 \in \rho_{A_1}(x_1)$, and then

$$\rho_T(x_1 + 0) \subset \rho_{A_1}(x_1)$$

Thus,

$$\sigma_{A_1}(x_1) \subset \sigma_A(x_1 + 0)$$

Now, if $z_2 \in \rho_{A_1}(x_1)$, then, there exists a neighborhood V_2 of z_2 and an analytic function h from V_2 onto H, such that

$$(A_1 - z)h(z) = x_1, \; z \in V_2$$

Thus,

$$(A-z)(h(z)+0)=(A_1-z)h(z)=x_1=x_1+0$$

Hence,

$$z_2 \in \rho_A(x_1+0)$$

□

Definition 1. An operator $A \in B(H)$ is said to be (M,k)-quasi-*-paranormal if there exists M and a positive integer k such that

$$A^{*k}(MA^{*2}A^2-2\lambda AA^*+\lambda^2)A^k \geq 0$$

for all $\lambda > 0$.

This definition is equivalent to

$$\|A^\star A^k x\|^2 \leq \sqrt{M}\|A^{k+2}x\|\|A^k x\|$$

for all $x \in H$.

Theorem 2.11. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator such that A^2 is an isometry on H. Then A is (M,k)-quasi-*-paranormal.*

Proof. Since A^2 is an isometry, $A^{\star 2}A^2 = I$, and then $\|A^2x\| = \|x\|,\ x \in H$. By iteration, $\|A^{k+2}x\| = \|A^k x\|,\ k \geq 1$. Since A is an (M,k)-quasi-*-class Q operator,

$$\begin{aligned} 2\|A^\star A^k x\|^2 &\leq M\|A^{k+2}x\|^2+\|A^k x\|^2 \\ &\leq \left(\sqrt{M}\|A^{k+2}x\|-\|A^k x\|\right)^2+2\sqrt{M}\|A^{k+2}x\|\|A^k x\| \\ &\leq 2\sqrt{M}\|A^{k+2}x\|\|A^k x\| \end{aligned}$$

□

Definition 2. An operator $A \in B(H)$ is said to be isoloid, if every isolated point of its spectrum is an eigenvalue of A.

We have then the following result.

Theorem 2.12. *Each (M,k)-quasi-*-class Q operator is isoloid.*

Proof. Let A be an (M,k)-quasi-*-class Q operator. Suppose that A has a representation given in Theorem 2.6. Let z be an isolated point in $\sigma(A)$. Since $\sigma(A) = \sigma(A_1) \cup \{0\}$, z is an isolated point in $\sigma(A_1)$ or $z = 0$.

If z is an isolated point in $\sigma(A_1)$, then $z \in \sigma_p(A_1)$. Assume that $z = 0$ and $z \notin \sigma(A_1)$. Then, for $x \in \ker A_3$, we get $(-A_1^{-1}A_2x \oplus x) \in \ker A$. □

Theorem 2.13. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator, and let $N \subseteq H$ be a closed A-invariant subspace for which the restriction $A|N$ is an injective and normal operator. Then N reduces A, that is, N is invariant for A and A^*.*

Proof. Suppose that P is an orthogonal projection of H onto $\overline{ranA^k}$. Since A is an (M,k)-quasi-*-class Q operator, we have

$$P(MA^{*2}A^2 - AA^*)P \geq 0.$$

By assumption, $A|_N$ is an injective normal operator. Then, $E \leq P$ for the orthogonal projection E of H onto N, and $\overline{ranA^k|_N} = N$ because $A|_N$ has a dense range. Therefore, $N \subseteq \overline{ranA^k}$ and hence

$$E(MA^{*2}A^2 - AA^*)E \geq 0.$$

Let

$$A = \begin{pmatrix} A\,|N & A_2 \\ 0 & A_3 \end{pmatrix},$$

on $N \oplus N^{\perp}$. Then,

$$AA^* = \begin{pmatrix} A\,|N\ A^*\,|N + A_2A_2^* & A_2A_3^* \\ A_3A_2^* & A_3A_3^* \end{pmatrix}$$

and

$$MA^{\star 2}A^2 = \begin{pmatrix} MA^{*2}\,|N\ A^2\,|N & S \\ T & R \end{pmatrix}$$

for some bounded linear operators S, T and R. Thus,

$$\begin{pmatrix} A\,|N\ A^*\,|N + A_2A_2^* & 0 \\ 0 & 0 \end{pmatrix} = E(AA^*)E = E|A^*|^2E \leq E(A^{*2}A^2)^{\frac{1}{2}}E$$
$$\leq (E(A^{*2}A^2E))^{\frac{1}{2}}$$
$$= \begin{pmatrix} A^{*2}\,|N\ A^2\,|N & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}}$$

This implies that

$$A|_NA^*|_N + A_2A_2^* \leq A|_NA^*|_N.$$

Since $A|N$ is normal and $A_1A_1^*$ is positive, it follows that $A_2 = 0$. Hence N reduces A. □

Remark 5. The previous result is in general false if the restriction $A\,|N$ is not injective. In fact, if A is a nilpotent operator of order k, such that $A^{k-1} \neq 0$, then $A\left|\overline{ranA^{k-1}}\right. = 0$ is a normal operator. Assume that $\overline{ranA^{k-1}}$ reduces A. Then, $A^{\star}A^{k-1}H \subset \overline{ranA^{k-1}}$. Thus, $A^{*k-1}A^{k-1}H \subset \overline{ranA^{k-1}}$ and $kerA^{*k-1} \subset kerA^{*k-1}A^{k-1} = kerA^{k-1}$. Since $A^{*k} = A^{*k-1}A^* = 0$, $A^{k-1}A^* = 0$. Hence, $A^{k-1}A^{*k-1} = 0$. Therefore, $A^{k-1} = 0$. This contradicts the hypotheses on A.

Theorem 2.14. *Let A be an (M,k)-quasi-*-class Q operator. Equation $(A-\lambda)x = 0$ implies $(A-\lambda)^{\star}x = 0$ for all non-zero complex scalar λ.*

Proof. Assume that $x \neq 0$. Let $N = \text{span}\{x\}$ and

$$A = \begin{pmatrix} \lambda & T \\ 0 & S \end{pmatrix} \quad \text{on } H = N \oplus N^{\perp}.$$

Let $P : H \to N$ be the orthogonal projection. Then, $A\,|N = \lambda$ is an injective normal operator. Hence, N reduces A by Theorem 2.11. Thus, $T = 0$. □

Theorem 2.15. *Let $A \in B(H)$ be an (M,k)-quasi-*-class Q operator, and let $\lambda \in \mathbb{C}, \lambda \neq 0$ be an isolated point of the spectrum of A. Then, the Riesz idempotent E for λ is self-adjoint, and satisfies the following equality*

$$EH = ker(A-\lambda) = ker(A-\lambda)^{\star}.$$

Proof. By Theorem 2.12, λ is an eigenvalue of A, and $EH = ker(A-\lambda)$. According to Theorem 2.14, it sufficies to show that $ker(A-\lambda)^\star \subset ker(A-\lambda)$. The subspace $ker(A-\lambda)$ reduces A by Theorem 2.14, and the restriction of A on its reducing subspace is an (M,k)-quasi-*-class Q operator by Theorem 2.3. It follows that

$$A = \lambda \oplus B \text{ on } H = ker(A-\lambda) \oplus (ker(A-\lambda))^\perp$$

where B is (M,k)-quasi-*-class Q and $ker(B-\lambda) = \{0\}$. We've

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(B)$$

and λ is isolated. Then, either $\lambda \notin \sigma(B)$, or λ is an isolated point of $\sigma(B)$, which contradicts the fact that $ker(A-\lambda) = \{0\}$. Since B is invertible on $(ker(A-\lambda))^\perp$,

$$ker(A-\lambda) = ker(A-\lambda)^\star.$$

Furthermore, since $EH = ker(A-\lambda) = ker(A-\lambda)^\star$,

$$((z-A)^\star)^{-1}E = \overline{(z-\lambda)^{-1}}E.$$

Thus,

$$\begin{aligned} E^\star = -\frac{1}{2\pi i}\int_{\partial D}((z-A)^\star)^{-1}E\, d\overline{z} &= -\frac{1}{2\pi i}\int_{\partial D}\overline{(z-\lambda)^{-1}}E\, d\overline{z} \\ &= \overline{\frac{1}{2\pi i}\int_{\partial D}(z-\lambda)^{-1}\, dz}E = E. \end{aligned}$$

So, E is self-adjoint. □

3 Weyl's Theorem

An operator $A \in B(H)$ is called Fredholm if $R(A)$ is closed, $\alpha(A) = \dim N(A) < \infty$ and $\beta(A) - \dim H \setminus R(A) < \infty$. Moreover if $i(A) = \alpha(A) - \beta(A) = 0$, then A is called Weyl. The Weyl spectrum $w(A)$ of A is defined by

$$w(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}.$$

According to [10], we say that Weyl's theorem holds for A if

$$\sigma(A) \setminus w(A) = \pi_{00}(A),$$

where

$$\pi_{00}(A) = \{\lambda \in iso\sigma(A) : 0 < \dim N(A - \lambda I < \infty\}.$$

In [22], Patel showed that Weyl's theorem holds for 2-isometric operators, i.e., operators satisfying

$$A^{*2}A^2 - 2A^*A + I = 0$$

[1], which has been extended to many non normal operators [16, 19]. In this section, we prove that Weyl's theorem holds for (M,k)-quasi-$*$-class Q operators without any additional conditions.

Theorem 3.1. *Weyl's theorem holds for any (M,k)-quasi-*-class Q operator.*

Proof. Suppose that A is a (M,k)-quasi-*-class Q operator. Then A has SVEP at zero. Either $\sigma(A_1) \subseteq \partial\mathcal{D}$ or $\sigma(A_1) = \overline{\mathcal{D}}$, where $\mathcal{D}$ denotes the open unit disc, and $\partial\mathcal{D}$ is its boundary. If $\sigma(A_1) \subseteq \partial\mathcal{D}$, then A has SVEP everywhere: else $\sigma(A_1) = \overline{\mathcal{D}}$. The operator A has SVEP on $\sigma(A) \setminus w(A)$, then $< 0 \dim(A-\lambda) < \infty$. We have $\lambda \in \sigma_p(A) \subseteq \partial\mathcal{D} \cup \{0\}$, An operator such that its point spectrum has empty interior has SVEP [2, Remark 2.4(d)]. Hence A has SVEP. Also, if $\sigma(A_1) = \sigma(A) = \overline{\mathcal{D}}$, then $iso\sigma(A) = \emptyset$. If $\sigma(A_1) \subset \partial\mathcal{D}$, then A_1 is polaroid, that is, the isolated points of the spectrum of A_1 are poles of the resolvent. Hence, A is polaroid. This proves Weyl's theorem for A. □

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