ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2023, Volume 14, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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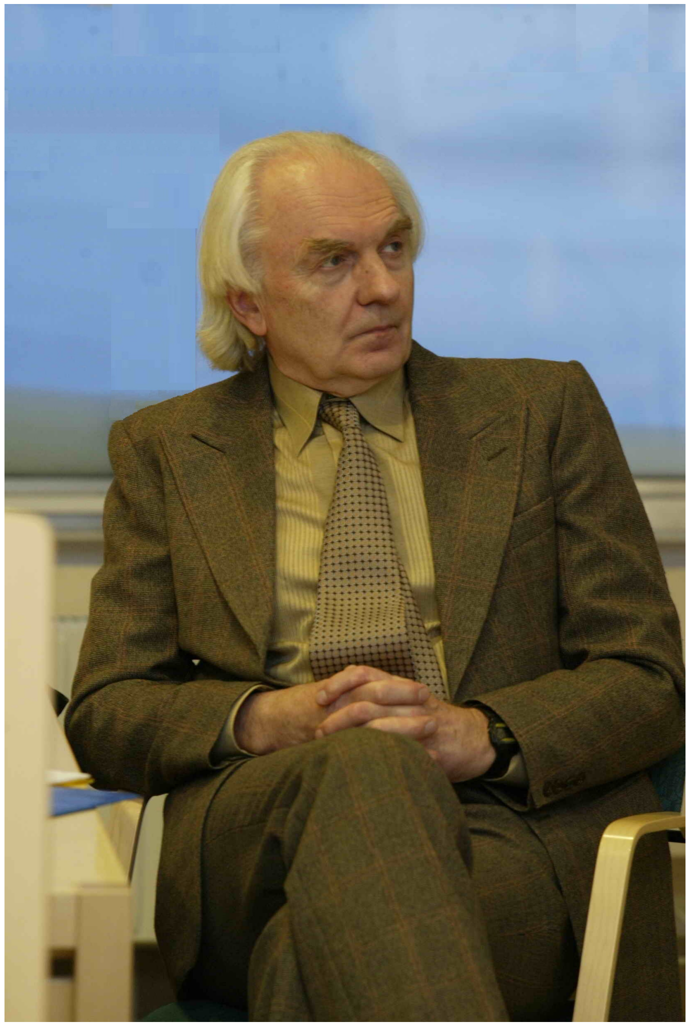
On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \leq p, \theta \leq \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 14, Number 2 (2023), 58 – 78

THREE WEIGHT HARDY INEQUALITY ON MEASURE TOPOLOGICAL SPACES

K.T. Mynbaev

Communicated by V.D. Stepanov

Key words: Hardy operator, topological space, measure space, multidimensional Hardy inequality.

AMS Mathematics Subject Classification: Primary 26D15 Secondary 47G10, 26D10.

Abstract. For the Hardy inequality to hold on a Hausdorff topological space, we obtain necessary and sufficient conditions on the weights and measures. As in the recent paper by G. Sinnamon (2022), we assume total orderedness of the family of sets that generate the Hardy operator. Sinnamon's method consists in the reduction of the problem to an equivalent one-dimensional problem. We provide a different, direct proof which develops the approach suggested by D. Prokhorov (2006) in the one-dimensional case.

DOI: https://doi.org/10.32523/2077-9879-2023-14-2-58-78

1 Introduction

Consider the inequality

$$\left[\int_{[a,b]}\left(\int_{[a,x]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_{[a,b]} f^p wd\nu\right)^{1/p} \tag{1.1}$$

for all non-negative functions f. Here $[a,b]$ is a finite or infinite segment on the extended real line, u,v and w are non-negative measurable weight functions and λ,μ,ν are Borel measures. The problem is to find a functional of the weights and measures $\Phi\left(u,v,w,\lambda,\mu,\nu\right)$ such that for the best constant C one has

$$c_1\Phi \leq C \leq c_2\Phi, \tag{1.2}$$

where the positive constants c_1, c_2 do not depend on the weights and measures. The characterizations of weights and measures for which (1.1) holds are very different for the cases $p \leq q$ and $q < p$. In particular, the proofs for the case $p \leq q$ are a lot simpler. The inequality has a long history described in several books [6], [7], [8], [10].

In [3] and [21] spherical coordinates in R^n were used to obtain the first results for the Euclidean space. Other multidimensional generalizations followed. Results for Banach function spaces and mixed L^p spaces given in [4] and [1] covered only the case $p \leq q$, when specified to usual L^p spaces. The two-dimensional result by Sawyer [19] turned out to be difficult to generalize to higher dimensions, unless under additional restrictions on the weights [23]. We believe this is caused by the fact that his domains are not totally ordered, see the definition below.

In the last several years there was a wave of new generalizations. In [16], [17] and [18] the results have been formulated in abstract settings (for homogeneous groups, hyperbolic spaces, Cartan-Hadamard manifolds, and connected Lie groups). All of them are based on the assumption of the existence of a polar decomposition, which for calculational purposes is the same as spherical coordinates. Thus, methodologically, the last three papers return to [3].

G. Sinnamon [22] made a significant contribution by providing a single framework for all Hardy inequalities, regardless of the domain dimension and covering both continuous and discrete cases. His method consists in reducing the general Hardy inequality to a special, one-dimensional one, called a normal form. In addition to being universally applicable, this approach has other advantages. The functional Φ for the normal form is relatively simple, because the weights are constant and only the upper limit of integration changes. (Note that in general there exist many functionals satisfying (1.1), see [8]). This simplicity allows Sinnamon to improve the best constants c_1, c_2 in (1.2) due to Hardy and Bliss [2].

The reduction to the one-dimensional case in Sinnamon's approach requires an additional calculation to obtain the functional in terms of the original weights and measures. The present paper is different in that we give a direct proof leading to the required expressions. See Remark 1 below for a more detailed comparison.

Now we mention the contributions that directly influenced the methods employed here. Everywhere we assume that $1 < p < \infty$, $0 < q < \infty$.

For the case $q < p$ several functionals equivalent to (1.1) have been suggested. The one proposed by Maz'ya and Rozin [10] and used here has the advantage that it works both for $0 < q < 1$ and $1 \le q < p$.

D. Prokhorov [12] investigated the Hardy inequality on the real line but the merits of his measure-theoretical analysis go beyond the one-dimensional case. We follow his ideas and along the way mention some of his innovations. One of them is that he allowed the weights to be infinite on sets of positive measure and analyzed the implications.

The purpose of this paper is to obtain a criterion for the multidimensional inequality

$$\begin{aligned} &\left[\int_\Omega \left(\int_{\{y\in\Omega:\tau(y)\le\tau(x)\}} f(y)\,u(y)\,d\lambda(y)\right)^q v(x)d\mu(x)\right]^{1/q} \\ \le\ & C\left(\int_\Omega f^p w d\nu\right)^{1/p} \end{aligned} \tag{1.3}$$

(the function τ is defined in Section 2), Ω is an open set in a Hausdorff topological space X. The main restriction on the open subsets $\Omega(t)$ of Ω is that they are parameterized by real t and satisfy the monotonicity (total orderedness) condition

$$\Omega(t_1) \subset \Omega(t_2) \text{ if } t_1 < t_2. \tag{1.4}$$

Alternatively, instead of expanding, $\Omega(t)$ may be contracting but the unidirectionality is required for our method. As in Sinnamon's paper, the results can be called dimension-agnostic, because in X the dimension notion is generally not defined, and when X is a linear space, no convexity or connectedness are imposed on $\Omega(t)$ or Ω. The existing results for R^n or measure metric spaces from [1], [3], [16], [17], [18] are special cases of ours. Results of [19] (where rectangles do not satisfy the monotonicity condition) are not covered by ours. In [4] domains of integration are more general than ours and Banach function spaces are considered.

In the multidimensional case generalizations of our results in several directions are possible. For Hardy type integrals with variable kernels extensions can be obtained under the Oinarov condition [9], [11]. The generality of measures in our results may lead to their consequences for discrete problems [24] in the spirit of Sinnamon. It would be interesting to cover the Riemann–Liouville operators [13], although the lack of the derivative notion certainly makes unlikely generalizations of the results in [5].

2 Main assumptions and statements

Description of measures. The phrase "μ is a measure on Ω" means that there is a σ-algebra $\mathfrak{M}$ that contains the σ-algebra $\mathfrak{B}$ of Borel subsets of Ω and such that μ is a σ-finite and σ-additive (non-negative) function on $\mathfrak{M}$, with values in the extended real half-line $[0,+\infty]=\{0\le x\le+\infty\}$. $\mathfrak{M}_\mu$ denotes the domain of the measure μ. Everywhere λ,μ,ν are measures on Ω and λ,ν have a common domain $\mathfrak{M}_{\lambda,\nu}$.

Description of functions. The notation $f\in\{\mathfrak{M}\}^+$ means that f is defined in Ω, takes values in $[0,+\infty]$ and is $\mathfrak{M}$-measurable. The weights u,v,w satisfy $u,w\in\{\mathfrak{M}_{\lambda,\nu}\}^+$, $v\in\{\mathfrak{M}_\mu\}^+$.

Description of sets $\Omega(t)$. Ω is an open set in a Hausdorff topological space X and $\{\Omega(t):t\in[a,b]\}$, $-\infty\le a<b\le\infty$, is a one-parameter family of open subsets of Ω that satisfy monotonicity (1.4) for $a\le t_1<t_2\le b$, start at the empty set and eventually cover λ-almost all Ω:

$$\Omega(a)=\bigcap_{t>a}\Omega(t)=\varnothing,\ \lambda\left(\Omega\backslash\bigcup_{a<t<b}\Omega(t)\right)=0.$$

Let $\omega(t)=\overline{\Omega(t)}\cap\overline{(\Omega\backslash\Omega(t))}$ be the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover λ-almost all Ω:

$$\omega(t_1)\cap\omega(t_2)=\varnothing,\ t_1\ne t_2,\ t_1,t_2\in(a,b);\ \lambda\left(\Omega\backslash\bigcup_{a<t<b}\omega(t)\right)=0.$$

The last condition implies that, up to a set of λ-measure zero, for each $x\in\Omega$ there exists a unique $\tau(x)\in(a,b)$ such that $x\in\omega(\tau(x))$, which allows us to define a Hardy type operator

$$Tf(x)=\int_{\{y\in\Omega:\tau(y)\le\tau(x)\}}f\left(y\right)d\lambda\left(y\right).$$

More generally, for any $E\subset[a,b]$ such that $\Omega\left(E\right)\equiv\cup_{t\in E}\omega(t)\in\mathfrak{M}_\lambda$ we can consider the integral

$$\int_{\Omega(E)}fd\lambda.$$

In particular, we denote $\Omega\left[c,d\right]=\cup_{c\le t\le d}\omega(t)$, $\Omega[c,d)=\cup_{c\le t<d}\omega(t)$, etc. for $a\le c<d\le b$.

Remark 1. 1) In comparison with [22], our setup is closer to the classical one, where the domains $\Omega\left(t\right)=(0,t)$ are indexed by their boundaries $\omega\left(t\right)=t$ and can be represented as unions of boundaries $\Omega\left(t\right)=\{s:\omega\left(s\right)<\omega\left(t\right)\}$. 2) Let (S,Σ,λ) and (Y,μ) be σ-finite measure spaces and let $B:Y\to\Sigma$ be a map such that the range of B is a totally ordered subset of Σ (these are assumptions from [22]). Thus the family $\{B\left(y\right):y\in Y\}$ of subsets of Σ is indexed by $y\in Y$. Since the number $t=\lambda\left(B\left(y\right)\right)$ is unique for each $y\in Y$, we can write $\widetilde{B}(t)=B\left(y\right)$ if $t=\lambda\left(B\left(y\right)\right)$. This re-indexing is one-to-one if, as usual, we do not distinguish between two sets which differ by a set of measure zero. Hence the family $\{B\left(y\right):y\in Y\}$ can be indexed by elements from $[0,\infty)$, the total order being preserved. 3) As we mentioned in the Introduction, Sinnamon's result covers more different cases. On the other hand, our proofs are direct (they don't rely on the one-dimensional case as an intermediate step) and we have statements stemming from the assumption that weights may take values in the extended half-axis $[0,\infty]$ (see Lemmas 2.7 and 2.8); Sinnamon does not have them. Our method generalizes [12].

Conventions on improper numbers. $0+(+\infty)=a+(+\infty)=a\cdot(+\infty)=+\infty$ if $0<a\le+\infty$; $0\cdot(+\infty)=0$; $(+\infty)^\alpha=0^{-\alpha}=+\infty$, $(+\infty)^{-\alpha}=0^\alpha=0$, $\alpha\in(0,+\infty)$.

Lowercase c, with or without subscripts, denote constants that do not depend on weights and measures.

The few results in dimensions higher than 1, reviewed in the Introduction, excluding [22], employed tools of one-dimensional analysis along the radial variable, of type

$$\int_a^b \left(\int_a^x f\right)^\gamma f(x)dx = \frac{1}{\gamma+1}\int_a^b \frac{d}{dx}\left(\int_a^x f\right)^{\gamma+1} dx = \frac{1}{\gamma+1}\left(\int_a^b f\right)^{\gamma+1},$$

where dx is the Lebesgue measure. Further advancement has been held back by the lack of a truly multidimensional replacement of such tools. The significance of the following Lemmas 2.1 and 2.2 is that they are such a replacement. See [12] for the argument on the straight line.

Lemma 2.1. *Denote* $\Lambda_f(t) = \int_{\Omega[a,t]} f d\lambda$, $a \le t \le b$, $f \in \{\mathfrak{M}_\lambda\}^+$.

a) If $\gamma > 0$, *then*

$$\frac{\Lambda_f(b)^{\gamma+1}}{\max\{1,\gamma+1\}} \le \int_{\Omega[a,b]} f(x)\Lambda_f(\tau(x))^\gamma \, d\lambda(x) \le \frac{\Lambda_f(b)^{\gamma+1}}{\min\{1,\gamma+1\}}. \tag{2.1}$$

b) In the case $\gamma \in (-1,0)$, (2.1) *holds if* $\Lambda_f(b) < \infty$.

Proof. a) Let $\gamma > 0$. The second inequality in (2.1) follows from $\Lambda_f(\tau(x)) \le \Lambda_f(b)$, $x \in \Omega$. Let us prove the first inequality. Without loss of generality we assume that

$$\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma \, d\lambda(x) < \infty.$$

Then for any $t \in [a,b]$

$$\begin{aligned}\left(\int_{\omega(t)} f d\lambda\right)^{\gamma+1} &\le \int_{\omega(t)} f d\lambda \left(\int_{\Omega[a,t]} f d\lambda\right)^\gamma \le \int_{\Omega[t,b]} f(x)\Lambda_f(\tau(x))^\gamma \, d\lambda(x) \\ &\le \int_\Omega f(x)\Lambda_f(\tau(x))^\gamma \, d\lambda(x) < \infty,\end{aligned}$$

so

$$\int_{\omega(t)} f d\lambda < \infty \text{ for any } t \in [a,b]. \tag{2.2}$$

Suppose $\Lambda_f(b) = \infty$. Denote

$$E = \{t \in [a,b) : \Lambda_f(t) = \infty\}, \quad e = \begin{cases} \inf E, & \text{if } E \ne \varnothing; \\ b, & \text{if } E = \varnothing. \end{cases}$$

If there is $\xi \in (e,b]$ such that $\int_{\Omega(\xi,b]} f d\lambda \ne 0$, then

$$\infty > \int_\Omega f(x)\Lambda_f(\tau(x))^\gamma \, d\lambda(x) \ge \Lambda_f(\xi)^\gamma \int_{\Omega(\xi,b]} f d\lambda = \infty,$$

which is impossible. Hence, for any $\xi \in (e,b]$ one has $\int_{\Omega(\xi,b]} f d\lambda = 0$. By the monotone covergence theorem $\int_{\Omega(e,b]} f d\lambda = 0$ and thus $\Lambda_f(e) = \infty$. Recalling (2.2) we see that $e > a$ and $\int_{\Omega[a,e)} f d\lambda = \infty$. By the definition of e,

$$\infty = \int_{\Omega[a,e)} f d\lambda = \int_{\Omega[a,t]} f d\lambda + \int_{\Omega(t,e)} f d\lambda$$

for $t \in [a,e)$, where $\Lambda_f(t) \in (0,\infty)$. Then, again by the definition of e,

$$\infty > \int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\, d\lambda(x) \geq \Lambda_f(t)^\gamma \int_{\Omega(t,e)} f d\lambda = \infty.$$

The contradiction arises from the assumption $\Lambda_f(b) = \infty$, so without loss of generality we can suppose that $\Lambda_f(b) < \infty$.

Changing the integration order gives

$$\begin{aligned}\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\, d\lambda(x) &= \gamma \int_\Omega f(x) \left(\int_0^{\Lambda_f(\tau(x))} s^{\gamma-1} ds \right) d\lambda(x) \\ &= \gamma \int_0^{\Lambda_f(b)} s^{\gamma-1} \left(\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\, d\lambda(x) \right) ds.\end{aligned}$$

For $s \geq 0$ put $E_s = \{t \in [a,b] : \Lambda_f(t) < s\}$. If $E_s = \varnothing$, then $\Lambda_f(\tau(x)) \geq s$ for any $x \in \Omega$ and

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\, d\lambda(x) = \Lambda_f(b) \geq \Lambda_f(b) - s.$$

Suppose $E_s \neq \varnothing$ and let $e_s = \sup E_s$. Take a sequence $\left\{t_n^{(s)}\right\} \subset E_s$ such that $t_n^{(s)} \uparrow e_s$ as $n \to \infty$. Then in the case $e_s \in E_s$ we have $\Lambda_f(\tau(x)) < s$ for $\tau(x) \leq e_s$, $\Lambda_f(\tau(x)) \geq s$ for $\tau(x) > e_s$ and

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\, d\lambda(x) = \int_{\Omega(e_s,b]} f d\lambda = \Lambda_f(b) - \Lambda_f(e_s) \geq \Lambda_f(b) - s,$$

while in the case $e_s \notin E_s$

$$\int_\Omega f(x)\chi_{[0,\Lambda_f(\tau(x))]}(s)\, d\lambda(x) = \int_{\Omega[e_s,b]} f d\lambda = \Lambda_f(b) - \lim_{n\to\infty} \Lambda_f\left(t_n^{(s)}\right) \geq \Lambda_f(b) - s.$$

So, summarizing,

$$\begin{aligned}\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\, d\lambda(x) &\geq \gamma \int_0^{\Lambda_f(b)} s^{\gamma-1} (\Lambda_f(b) - s)\, ds \\ &= \Lambda_f(b)^{\gamma+1} - \frac{\gamma}{\gamma+1}\Lambda_f(b)^{\gamma+1} = \frac{\Lambda_f(b)^{\gamma+1}}{\gamma+1},\end{aligned}$$

which completes the argument for $\gamma > 0$.

b) Now let $\gamma \in (-1,0)$ and $\Lambda_f(b) < \infty$. Then the first inequality in (2.1) follows from $\Lambda_f(b)^\gamma \leq \Lambda_f(\tau(x))^\gamma$, $x \in \Omega$. Let us prove the second inequality. Start with

$$\begin{aligned}\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\, d\lambda(x) &= -\gamma \int_\Omega f(x) \left(\int_{\Lambda_f(\tau(x))}^\infty s^{\gamma-1} ds \right) d\lambda(x) \\ = -\gamma \int_\Omega f(x) &\left(\int_0^{\Lambda_f(b)} s^{\gamma-1}\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\, ds + \int_{\Lambda_f(b)}^\infty s^{\gamma-1} ds \right) d\lambda(x) \\ = -\gamma \int_0^{\Lambda_f(b)} s^{\gamma-1} &\left(\int_\Omega f(x)\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\, d\lambda(x) \right) ds + \Lambda_f(b)^{\gamma+1}.\end{aligned}$$

For $s \geq 0$ define $E_s = \{t \in [a,b] : \Lambda_f(t) \leq s\}$. In case $E_s = \varnothing$ we have $\Lambda_f(\tau(x)) > s$ for all $x \in \Omega$ and

$$\int_\Omega f(x)\chi_{[\Lambda_f(\tau(x)),\Lambda_f(b)]}(s)\, d\lambda(x) = 0 \leq s.$$

Suppose $E_s \neq \varnothing$, let $e_s = \sup E_s$ and take a sequence $\left\{t_n^{(s)}\right\} \subset E_s$ such that $t_n^{(s)} \uparrow e_s$ as $n \to \infty$. Then in case $e_s \in E_s$ we have $e_s \leq b$, $\Lambda_f(\tau(x)) \leq \Lambda_f(e_s) \leq s$ for $\tau(x) \leq e_s$ and $\Lambda_f(\tau(x)) > s$ for $\tau(x) > e_s$, so that

$$\int_\Omega f(x)\chi_{\left[\Lambda_f(\tau(x)),\Lambda_f(b)\right]}(s)\, d\lambda(x) = \int_{\Omega[a,e_s]} f d\lambda = \Lambda_f(e_s) \leq s.$$

On the other hand, in the case $e_s \notin E_s$

$$\int_\Omega f(x)\chi_{\left[\Lambda_f(\tau(x)),\Lambda_f(b)\right]}(s)\, d\lambda(x) = \int_{\Omega[a,e_s)} f d\lambda = \lim_{n\to\infty} \Lambda_f\left(t_n^{(s)}\right) \leq s.$$

As a result,

$$\int_\Omega f(x)\Lambda_f(\tau(x))^\gamma\, d\lambda(x) \leq -\gamma \int_0^{\Lambda_f(b)} s^\gamma ds + \Lambda_f(b)^{\gamma+1} = \frac{\Lambda_f(b)^{\gamma+1}}{\gamma+1}.$$

□

A similar statement holds for the integral with a variable lower limit of integration.

Lemma 2.2. *Let $\bar\Lambda_f(t) = \int_{\Omega[t,b]} f d\lambda$. a) If $\gamma > 0$, then*

$$\frac{\bar\Lambda_f(a)^{\gamma+1}}{\max\{1,\gamma+1\}} \leq \int_{\Omega[a,b]} f(x)\bar\Lambda_f(\tau(x))^\gamma\, d\lambda(x) \leq \frac{\bar\Lambda_f(a)^{\gamma+1}}{\min\{1,\gamma+1\}}. \tag{2.3}$$

b) For $\gamma \in (-1,0)$ (2.3) holds if $\bar\Lambda_f(a) < \infty$.

The proof of the next lemma can be found in [15] (it is dimensionless).

Lemma 2.3. *Let $1 < p < \infty$, $u \in \{\mathfrak{M}_\lambda\}^+$, $E \in \mathfrak{M}_\lambda$. If $\int_E u^{p'} d\lambda = \infty$, then there exists $f \in \{\mathfrak{M}_\lambda\}^+$ such that $\int_E f^p d\lambda < \infty$ and $\int_E f u d\lambda = \infty$.*

Lemma 2.4. *a) Let $E \subset [a,b]$ be such that $\Omega(E) \in \mathfrak{M}_\lambda$. Define $E_t = E \cap [a,t]$, $\bar E_t = E \cap [t,b]$, $t \in E$. If $\lambda(\Omega(E_t)) = 0$ for any $t \in E$ or $\lambda\left(\Omega\left(\bar E_t\right)\right) = 0$ for any $t \in E$, then $\lambda(\Omega(E)) = 0$.*

b) Alternative formulation. Take a set $E \subset \Omega$ that belongs to $\mathfrak{M}_\lambda$ and define $E_y = E \cap \Omega[a,\tau(y)]$, $\bar E_y = E \cap \Omega[\tau(y),b]$. If $\lambda(E_y) = 0$ for any $y \in E$ or $\lambda\left(\bar E_y\right) = 0$ for any $y \in E$, then $\lambda(E) = 0$.

Proof. If E is empty, the statement is obvious. Let $E \neq \varnothing$, put $s = \sup E$ and take a sequence $\{s_n\}$ such that $s_n \uparrow s$, $s_n \in E$ for all n. If $s \in E$, then $E = E \cap [a,s] = E_s$ and $\lambda(\Omega(E)) = \lambda(\Omega(E_s)) = 0$. If $s \notin E$, then $E = \cup_n (E \cap [a,s_n]) = \cup_n E_{s_n}$ and $\lambda(\Omega(E)) = \lim \lambda(\Omega(E_{s_n})) = 0$. This proves a). The proof of b) is similar. □

In the next lemma we look at the special case of (1.3) with $\nu = \lambda$ and $w \equiv 1$. The lemma is a long way of saying that replacing $g = fh$ is all it takes to pass from (2.4) to (2.5).

Lemma 2.5. *Consider weights $u, h \in \{\mathfrak{M}_\lambda\}^+$ and $v \in \{\mathfrak{M}_\mu\}^+$. The inequalities*

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} u f h d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f \in \{\mathfrak{M}_\lambda\}^+, \tag{2.4}$$

and

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} u g d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega g^p h^{-p} d\lambda\right)^{1/p}, \tag{2.5}$$
$$g \in \{\mathfrak{M}_\lambda\}^+,$$

are equivalent.

Proof. Fix $f \in \{\mathfrak{M}_\lambda\}^+$ and let (2.5) be true. Plugging $g = fh$ in (2.5) we get

$$\begin{aligned}\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} ufhd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} &\leq C\left(\int_\Omega (fh)^p h^{-p} d\lambda\right)^{1/p} \\ &\leq C\left(\int_\Omega f^p d\lambda\right)^{1/p}\end{aligned}$$

because $h^p h^{-p} \leq 1$, where in case $h = \infty$ or $h = 0$ we have $(\infty)^p (\infty)^{-p} = \infty \cdot 0 = 0 < 1$.

Conversely, let (2.4) hold. Put

$$F_t = \{x \in \Omega[a,t] : h(x) = \infty,\ u(x) \neq 0\}, \quad E = \{t \in [a,b] : \lambda(F_t) > 0\}.$$

We want to show that

$$\int_{\Omega(E)} v d\mu = 0. \tag{2.6}$$

If $t_1 < t_2$, then $F_{t_1} \subset F_{t_2}$ by monotonicity of $\{\Omega(t)\}$ and $\lambda(F_{t_1}) \leq \lambda(F_{t_2})$. Hence, $\Omega(E)$ is Borel measurable. If E is empty, (2.6) is obvious. Let $E \neq \varnothing$ and fix $t \in E$. By Lemma 6.9 from [15] there is a function $f \in \mathfrak{M}_\lambda$ such that $\int_\Omega f^p d\lambda < \infty$ and $0 < f(x) < 1$ on Ω. Then

$$\int_{F_t} ufhd\lambda = \infty,$$

because $u(x)f(x) > 0$, $h(x) = \infty$ on F_t and $\lambda(F_t) > 0$. Plugging $f\chi_{F_t}$ in (2.4) we obtain $F_t \subset \Omega[a,t] \subset \Omega[a,\tau(x)]$ for $\tau(x) \geq t$ and

$$\begin{aligned}\left(\int_{\Omega[t,b]} v d\mu\right)^{1/q} \int_{F_t} ufhd\lambda &\leq \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} uf\chi_{F_t} hd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \\ &\leq C\left(\int_\Omega f^p d\lambda\right)^{1/p} < \infty.\end{aligned}$$

This shows that

$$\int_{E\cap\Omega[t,b]} v d\mu \leq \int_{\Omega[t,b]} v d\mu = 0 \text{ for all } t \in E$$

and by Lemma 2.4 (2.6) follows. Hence, to prove (2.5) it suffices to prove that

$$\left[\int_{\Omega([a,b]\setminus E)} \left(\int_{\Omega[a,\tau(x)]} ugd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega g^p h^{-p} d\lambda\right)^{1/p}, \quad g \in \{\mathfrak{M}_\lambda\}^+. \tag{2.7}$$

Note that

$$\lambda\left(F_{\tau(x)}\right) = 0 \text{ for any } x \in \Omega([a,b]\setminus E) \tag{2.8}$$

by the definition of E.

Now take any $g \in \{\mathfrak{M}_\lambda\}^+$. If $\int_\Omega g^p h^{-p} d\lambda = \infty$, then (2.7) is trivial. Suppose $\int_\Omega g^p h^{-p} d\lambda < \infty$. Then $g^p h^{-p}$ is finite λ-almost everywhere. In particular, for the set $E_1 = \{x \in \Omega : g(x) \neq 0,\ h(x) = 0\}$, where $g^p h^{-p} = \infty$, we have

$$\lambda(E_1) = 0. \tag{2.9}$$

Using (2.6) and $f(y) = g(y)\, h(y)^{-1}$, $y \in \Omega$, in (2.4) we get

$$\left[\int_{\Omega([a,b]\setminus E)} \left(\int_{\Omega[a,\tau(x)]} ugh^{-1}hd\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \tag{2.10}$$
$$\leq C\left(\int_\Omega \left(gh^{-1}\right)^p d\lambda\right)^{1/p}.$$

If we show that

$$\lambda(\{y \in \Omega[a,\tau(x)] : u(y)\, g(y) \neq u(y)\, g(y)\, h(y)^{-1} h(y)\}) = 0 \quad \text{for any } x \in \Omega([a,b]\setminus E), \tag{2.11}$$

then (2.10) will imply (2.7). Using the definitions of $F_{\tau(x)}$ and E_1 we see that

$$\begin{aligned} & \{y \in \Omega[a,\tau(x)] : u(y)\, g(y) \neq u(y)\, g(y)\, h(y)^{-1} h(y)\} \\ \subset\ & \{y \in \Omega[a,\tau(x)] : u(y)\, g(y) \neq 0,\ h(y)^{-1} h(y) \neq 1\} \\ =\ & \{y \in \Omega[a,\tau(x)] : u(y)\, g(y) \neq 0,\ h(y) = 0\} \\ & \cup \{y \in \Omega[a,\tau(x)] : u(y)\, g(y) \neq 0,\ h(y) = \infty\} \\ \subset\ & \{y \in \Omega[a,\tau(x)] : g(y) \neq 0,\ h(y) = 0\} \\ & \cup \{y \in \Omega[a,\tau(x)] : u(y) \neq 0,\ h(y) = \infty\} \subset E_1 \cup F_{\tau(x)}. \end{aligned}$$

We can use (2.8) and (2.9). This implies (2.11) and finishes the proof. □

We give the proof of the next well-known fact [10] just because we consider a more general situation.

Lemma 2.6. *Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to λ, that is, ν_a is absolutely continuous with respect to λ and ν_s is singular with respect to λ. Then the inequalities*

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fd\lambda\right)^q d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p w d\nu\right)^{1/p},\ \ f \in \{\mathfrak{M}_\lambda\}^+, \tag{2.12}$$

and

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fd\lambda\right)^q d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p w d\nu_a\right)^{1/p},\ \ f \in \{\mathfrak{M}_\lambda\}^+, \tag{2.13}$$

are equivalent.

Proof. Since $\nu = \nu_a + \nu_s$, (2.13) obviously implies (2.12). Suppose (2.12) is true. Since ν_s and λ are mutually singular, there exists a set $A_s \in \mathfrak{M}_\lambda$ such that $\lambda(A_s) = 0$ and ν_s is concentrated on A_s, implying

$$\nu_s(\Omega\setminus A_s) = 0,\ \ \lambda(\Omega[a,\tau(x)] \cap A_s) = 0. \tag{2.14}$$

By absolute continuity of ν_a with respect to λ

$$\nu_a(A_s) = 0. \tag{2.15}$$

Defining $\tilde{f} = f\chi_{\Omega\setminus A_s}$ we have by (2.14)

$$\int_{\Omega[a,\tau(x)]} fd\lambda = \int_{\Omega[a,\tau(x)]\cap A_s} fd\lambda + \int_{\Omega[a,\tau(x)]\setminus A_s} fd\lambda = \int_{\Omega[a,\tau(x)]} \tilde{f}d\lambda$$

and by (2.14), (2.15)

$$\begin{aligned}\int_\Omega \tilde{f}^p w d\nu &= \int_{\Omega\setminus A_s} f^p w d\nu_a + \int_{\Omega\setminus A_s} f^p w d\nu_s \\ &= \int_{\Omega\setminus A_s} f^p w d\nu_a + \int_{A_s} f^p w d\nu_a = \int_\Omega f^p w d\nu_a.\end{aligned}$$

(2.12) and the last two equations give the desired result:

$$\begin{aligned}\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f d\lambda\right)^q d\mu(x)\right]^{\frac{1}{q}} &= \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} \tilde{f} d\lambda\right)^q d\mu(x)\right]^{\frac{1}{q}} \\ &\le C\left(\int_\Omega \tilde{f}^p w d\nu\right)^{\frac{1}{p}} = C\left(\int_\Omega f^p w d\nu_a\right)^{\frac{1}{p}}.\end{aligned}$$

□

Denote

$$I_0 = \left\{x\in\Omega : \int_{\Omega[a,\tau(x)]} u d\lambda = 0\right\},\ I_\infty = \left\{x\in\Omega : \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = \infty\right\}.$$

By monotonicity I_0 is adjacent to point a and I_∞ is adjacent to point b. See Lemma 2.8 for more information on the structure of these sets. Consider a version of inequality (1.3) with $\nu=\lambda$ and $w\equiv 1$:

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fu d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \le C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+. \tag{2.16}$$

The next lemma tells us that I_0, I_∞ do not influence the validity of (2.16). The values of f on I_0 should not matter because, as it will be shown,

$$\int_{I_0} u d\lambda = 0. \tag{2.17}$$

By Hölder's inequality for $x\in I_\infty$

$$\int_{\Omega[a,\tau(x)]} fu d\lambda \le \left(\int_{\Omega[a,\tau(x)]} f^p d\lambda\right)^{1/p}\left(\int_{\Omega[a,\tau(x)]} u^{p'} d\lambda\right)^{1/p'}$$

where the last integral on the right is infinite. Hence, by Lemma 2.3 the integral on the left may be infinite. For (2.16) to hold, such values must be suppressed and for this it should be true that

$$\int_{I_\infty} v d\mu = 0. \tag{2.18}$$

That is why the values of the integral $\int_{\Omega[a,\tau(x)]} fu d\lambda$ on I_∞ should not matter. (2.17) and (2.18) arise from allowing weights and measures with improper values and have been discovered in [12].

Lemma 2.7. *a)* (2.17) *is true. b) Put* $I=\Omega\setminus[I_0\cup I_\infty]$. (2.16) *holds with* $C<\infty$ *if and only if* (2.18) *holds and*

$$\left[\int_I \left(\int_{\Omega[a,\tau(x)]} fu d\lambda\right)^q v(x) d\mu(x)\right]^{1/q} \le C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+. \tag{2.19}$$

Proof. a) First note that $I_0, I_\infty, I \in \mathfrak{M}_\lambda$. Define the measure $d\lambda_u = ud\lambda$ and note that for any $x \in I_0$ the set $E_x = I_0 \cap \Omega\,[a, \tau\,(x)] \subset \Omega\,[a, \tau\,(x)]$ satisfies

$$\lambda_u\,(E_x) = \int_{E_x} d\lambda_u \leq \int_{\Omega[a,\tau(x)]} ud\lambda = 0.$$

Hence, (2.17) follows by Lemma 2.4.

b) It is obvious that (2.16) implies (2.19) and that (2.18) holds in case of an empty I_∞. Let us derive (2.18) in case $I_\infty \neq \varnothing$. Take any $x \in I_\infty$. By Lemma 2.3 there is a function $f \in \{\mathfrak{M}_\lambda\}^+$ such that

$$\int_{\Omega[a,\tau(x)]} f^p d\lambda < \infty, \ \int_{\Omega[a,\tau(x)]} fud\lambda = \infty.$$

Plugging $f\chi_{\Omega(\tau(x))}$ in (2.16) we get

$$\begin{aligned} &\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{1/q} \int_{\Omega[a,\tau(x)]} fud\lambda \\ \leq\ & \left[\int_\Omega \left(\int_{\Omega[a,\tau(y)]} f\chi_{\Omega[a,\tau(x)]}ud\lambda\right)^q v(y)d\mu(y)\right]^{1/q} \\ \leq\ & C\left(\int_{\Omega[a,\tau(x)]} f^p d\lambda\right)^{1/p} < \infty. \end{aligned}$$

Thus, we should have $\int_{\Omega[\tau(x),b]} vd\mu = 0$ for any $x \in I_\infty$ and by Lemma 2.4 (2.18) follows.

Conversely, if (2.18) and (2.19) are true, then, taking into account also (2.17), we see that (2.19) implies (2.16). □

Denote

$$\begin{aligned} U\,(x) &= \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda, \ V\,(x) = \int_{\Omega[\tau(x),b]} vd\mu, \\ A\,(x) &= V\,(x)^{1/q}\,U\,(x)^{1/p'}, \ x \in \Omega, \ A = \sup_{x\in\Omega} A(x), \\ A' &= \sup_{x\in I} A\,(x) \ \text{ if } I \neq \varnothing, \ A' = 0 \text{ if } I = \varnothing. \end{aligned}$$

Lemma 2.8. *a) Define* $s = \sup\{\tau\,(x) : x \in I_0\}$, $i = \inf\{\tau\,(x) : x \in I_\infty\}$. *Then* $I_0 = \Omega\,[a, s]$, $I_\infty = \Omega\,[i, b]$

b) The inequality $A < \infty$ *is equivalent to the combination of* (2.18) *and*

$$A' < \infty. \tag{2.20}$$

Besides, $A = A'$.

Proof. a) If $x \in I_0$, then the monotonicity $\tau\,(y) < \tau\,(x)$ implies

$$\int_{\Omega[a,\tau(y)]} ud\lambda \leq \int_{\Omega[a,\tau(x)]} ud\lambda = 0.$$

Hence, with any $x \in I_0$, I_0 contains $\Omega\,[a, \tau\,(x)]$ and $\Omega\,[a, \tau\,(x)] \cap I_0 = \Omega\,[a, \tau\,(x)]$. Choose $\{s_n\} \subset \{\tau\,(x) : x \in I_0\}$ so that $s_n \uparrow s$. Then

$$\int_{\Omega[a,s]} ud\lambda = \lim \int_{\Omega[a,s_n]} ud\lambda = 0,$$

so $I_0 = \Omega\left[a, s\right]$. Similarly, if $\tau\left(y\right) > \tau\left(x\right)$ and $x \in I_\infty$ then by (2.21)

$$\int_{\Omega[\tau(y),b]} v d\mu \leq \int_{\Omega[\tau(x),b]} v d\mu = 0.$$

Hence, $\Omega\left[\tau\left(x\right), b\right] \cap I_\infty = \Omega\left[\tau\left(x\right), b\right]$. Choosing $\{i_n\} \subset \{\tau\left(x\right) : x \in I_\infty\}$ so that $i_n \downarrow i$ and using the above equation we see that $I_\infty = \Omega\left[i, b\right]$.

a) Let $A < \infty$. For any $x \in I_\infty$ we have $U\left(x\right) = \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = \infty$, so for $A < \infty$ it is necessary that

$$V\left(x\right) = \int_{\Omega[\tau(x),b]} v d\mu = 0, \ x \in I_\infty. \tag{2.21}$$

In Lemma 2.4 put $E = I_\infty$, $d\mu_v = v d\mu$. Then $\bar{E}_x = I_\infty \cap \Omega\left[\tau\left(x\right), b\right] \subset \Omega\left[\tau\left(x\right), b\right]$ and

$$\mu_v\left(\bar{E}_x\right) \leq \int_{\Omega[\tau(x),b]} v d\mu = 0, \ x \in I_\infty.$$

By Lemma 2.4 (2.18) follows. Besides, from the definition of I_0 we see that

$$U\left(x\right) = \int_{\Omega[a,\tau(x)]} u^{p'} d\lambda = 0, \ x \in I_0. \tag{2.22}$$

By (2.21) and (2.22) $A\left(x\right) = 0$ on $I_0 \cup I_\infty$, so $A' = A$.

Conversely, let (2.18) and (2.20) hold. By (2.22) $A\left(x\right) = 0$ on I_0. Besides, part a) and (2.18) imply

$$V\left(x\right) = \int_{\Omega[\tau(x),b]} v d\mu \leq \int_{I_\infty} v d\mu = 0, \ x \in I_\infty.$$

Thus, $A\left(x\right) = 0$ on I_∞ and $A = A' < \infty$. □

Theorem 2.1. *Let $1 < p \leq q < \infty$. Inequality (2.16) holds if and only if $A < \infty$, with the equivalence $c_1 A \leq C \leq c_2 A$.*

Proof. We want to show that $A = C = 0$ in case $I = \varnothing$. By monotinicity $V\left(x\right) = 0$ on I_∞ and by definition $\int_{\Omega[a,\tau(x)]} u d\lambda = 0$ on I_0. Hence $A = A' = 0$. On the other hand, $\Omega = I_0 \cup I_\infty$ implies

$$\int_\Omega \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q v(x) d\mu(x) = \int_{I_\infty} \left(\int_{\Omega[a,\tau(x)]\backslash I_0} f u d\lambda\right)^q v(x) d\mu(x) = 0.$$

Thus, $C = 0$. In what follows we can safely assume that $I \neq \varnothing$.

Sufficiency. Let $x \in I$. Since $\Omega\left[a, \tau\left(x\right)\right] \subseteq \Omega \backslash I_\infty$ and u is λ-everywhere zero on I_0, by Hölder's inequality

$$\begin{aligned} \int_{\Omega[a,\tau(x)]} f u d\lambda &= \int_{\Omega[a,\tau(x)] \cap I} f u d\lambda \\ &\leq \left(\int_{\Omega[a,\tau(x)] \cap I} f^p U^{1/p'} d\lambda\right)^{1/p} \left(\int_{\Omega[a,\tau(x)] \cap I} u^{p'} U^{-1/p} d\lambda\right)^{1/p'}. \end{aligned}$$

By Lemma 2.1 with $\gamma = -1/p$ and $I \cap \Omega\left[a, \tau\left(x\right)\right]$ in place of Ω and using the definition of I_0

$$\begin{aligned}
&\int_{I\cap\Omega[a,\tau(x)]} u^{p'}(y)\left(\int_{\Omega[a,\tau(y)]} u^{p'}d\lambda\right)^{-1/p} d\lambda(y) \\
=\;& \int_{I\cap\Omega[a,\tau(x)]} u^{p'}(y)\left(\int_{I\cap\Omega[a,\tau(y)]} u^{p'}d\lambda\right)^{-1/p} d\lambda(y) \\
\leq\;& c\left(\int_{I\cap\Omega[a,\tau(x)]} u^{p'}d\lambda\right)^{1-1/p} = cU(x)^{1/p'},
\end{aligned}$$

where $U(x)<\infty$ because $x\notin I_\infty$. Then for $x\in I$

$$\begin{aligned}
\int_{\Omega[a,\tau(x)]} fud\lambda &= \int_{I\cap\Omega[a,\tau(x)]} fud\lambda \leq c\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{1/p} U(x)^{1/(p')^2} \\
&\leq cA^{1/p'}\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{1/p} V(x)^{-1/(qp')}
\end{aligned}$$

where $V(x)>0$ by (2.18) and $V(x)<\infty$ because $U(x)>0$, see (2.17). Using this inequality we bound the left-hand side of (2.19) as follows:

$$\begin{aligned}
&\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \qquad (2.23) \\
\leq\;& cA^{1/p'}\left[\int_I\left(\int_{I\cap\Omega[a,\tau(x)]} f^pU^{1/p'}d\lambda\right)^{q/p} V(x)^{-1/p'}v(x)d\mu(x)\right]^{1/q} \\
=\;& cA^{1/p'}\left[\int_I\left(\int_I f^pU^{1/p'}\chi_{\Omega[a,\tau(x)]}d\lambda\right)^{q/p} V(x)^{-1/p'}v(x)d\mu(x)\right]^{1/q} \\
\leq\;& cA^{1/p'}\left[\int_\Omega f(y)^p U(y)^{1/p'}\left(\int_{I\cap\Omega[\tau(y),b]} V^{-1/p'}vd\mu\right)^{p/q} d\lambda(y)\right]^{1/p}.
\end{aligned}$$

The last transition is by Minkowsky's inequality.

By Lemma 2.8 $x\in I$ implies $\Omega[\tau(x),b]\subseteq\Omega\backslash I_0$. Using also (2.18) we see that $V(x)=\int_{I\cap\Omega[\tau(x),b]}vd\mu$ for $x\in I$. Now by Lemma 2.2 for $y\in I$

$$\begin{aligned}
&\int_{I\cap\Omega[\tau(y),b]} vV^{-1/p'}d\mu \\
=\;& \int_{I\cap\Omega[\tau(y),b]} v(x)\left(\int_{I\cap\Omega[\tau(x),b]} vd\mu\right)^{-1/p'} d\mu(x) \\
\leq\;& c\left(\int_{I\cap\Omega[\tau(y),b]} vd\mu\right)^{1-1/p'} = cV(y)^{1/p},
\end{aligned}$$

where $V(y)<\infty$ because $A<\infty$ and $U(x)>0$ on I.

Continuing (2.23) and applying $A<\infty$ together with the last bound we get

$$\begin{aligned}
&\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \\
\leq\;& c_1A^{1/p'}\left(\int_I f^pU^{1/p'}V^{1/q}d\lambda\right)^{1/p} \leq c_1A\left(\int_\Omega f^pd\lambda\right)^{1/p}.
\end{aligned}$$

Necessity. Suppose (2.16) is true. By Lemma 2.8 we have $\Omega[a,\tau(x)]\supset I_0$ for $x\in I$ and

$$(\Omega\backslash I_\infty)\cap\Omega[\tau(x),b]\subset(\Omega\backslash I_\infty)\cap(\Omega\backslash I_0)\subset I.$$

By (2.18)

$$V(x)=\int_{\Omega[\tau(x),b]}vd\mu=\int_{(\Omega\backslash I_\infty)\cap\Omega[\tau(x),b]}vd\mu\le\int_I vd\mu,\ x\in I.$$

For $x\in I$ put $f=u^{p'-1}\chi_{\Omega[a,\tau(x)]}$. If $y\in\Omega[\tau(x),b]$, then $\tau(y)\ge\tau(x)$ and

$$U(x)=\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\le\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)ud\lambda.$$

Thus, applying also (2.19), we get for $x\in I$

$$\begin{aligned}V(x)^{1/q}U(x)&=\left[\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^q vd\mu\right]^{1/q}\\&\le\left[\int_{\Omega[\tau(x),b]}\left(\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)ud\lambda\right)^q v(y)\,d\mu(y)\right]^{1/q}\\&\le\left[\int_I\left(\int_{\Omega[a,\tau(y)]}\left(f\chi_{\Omega[a,\tau(x)]}\right)d\lambda\right)^q v(y)d\mu(y)\right]^{1/q}\\&\le C\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^{1/p}=CU(x)^{1/p}\end{aligned}$$

which gives $A'\le C$. By Lemma 2.7, (2.18) is true and Lemma 2.8 gives $A=A'\le C$. □

Next we consider the case $q<p$ and define r from $1/r=1/q-1/p$. Denote

$$\begin{aligned}B&=\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]}u^{p'}d\lambda\right)^{r/p'}\left(\int_{\Omega[\tau(x),b]}vd\mu\right)^{r/p}v(x)\,d\mu(x)\right]^{1/r}\\&=\left(\int_\Omega U^{r/p'}V^{r/p}vd\mu\right)^{1/r}.\end{aligned}$$

Lemma 2.9. *a) (2.17) is true. b) $B<\infty$ is equivalent to the combination of (2.18) and*

$$B'=\left(\int_I U^{r/p'}V^{r/p}vd\mu\right)^{1/r}<\infty.$$

Besides, $B=B'$.

Proof. a) The proof of Lemma 2.7 a) does not rely on the inequality $p\le q$ and is valid in the current situation.

b) Let $B<\infty$. Then the fact that $U(x)=\infty$ on I_∞ implies

$$\int_{I_\infty}\left(\int_{\Omega[\tau(x),b]}vd\mu\right)^{r/p}vd\mu=0.$$

We represent

$$I_\infty=\left\{x\in I_\infty:\int_{\Omega[\tau(x),b]}vd\mu=0,\ v(x)\ne0\right\}\cup\{x\in I_\infty:v(x)=0\}=F\cup G.$$

Further, in Lemma 2.4 b) put $E = F$, $d\mu_v = vd\mu$. Then $\bar{E}_x = E \cap \Omega\left[\tau(x), b\right] \subset \Omega\left[\tau(x), b\right]$,

$$\mu_v\left(\bar{L}(x)\right) = \int_{\bar{L}(x)} vd\mu \leq \int_{\Omega[\tau(x),b]} vd\mu = 0, \ x \in F.$$

By Lemma 2.4 $\int_F vd\mu = 0$. Since also $\int_G vd\mu = 0$, (2.18) holds. The definition of I_0 and (2.18) give $B = B'$.

Conversely, let $B' < \infty$ and (2.18) hold. Then in view of (2.17) and (2.18) $B = B' < \infty$. □

Lemma 2.10. *If $B < \infty$, then $A \leq B$ and (2.18) is true.*

Proof. By Lemma 2.2

$$\int_{\Omega[\tau(x),b]} \left(\int_{\Omega[\tau(y),b]} vd\mu\right)^{r/p} v(y)\, d\mu(y) \geq c\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{r/q}.$$

Hence, for any $\tau(x) \in [a, b]$

$$\begin{aligned} B &\geq \left(\int_{\Omega[\tau(x),b]} U^{r/p'} V^{r/p} vd\mu\right)^{1/r} \\ &\geq U(x)^{1/p'} \left[\int_{\Omega[\tau(x),b]} \left(\int_{\Omega[\tau(y),b]} vd\mu\right)^{r/p} v(y)\, d\mu(y)\right]^{1/r} \\ &\geq c^{1/r} U(x)^{1/p'} V(x)^{1/q} = c^{1/r} A(x). \end{aligned}$$

Hence, $c^{1/r} A \leq B < \infty$ and (2.18) follows by Lemma 2.8. □

Denote

$$h(x) = \chi_I(x) \left(\int_{\Omega[a,\tau(x)]} U^{r/q'} V^{r/p} u^{p'} d\lambda\right)^{q/r}$$

and define a measure on $\mathfrak{M}_\mu$ by $d\tilde{\mu} = \chi_I v h^{-p/q} d\mu$. Assuming that $B < \infty$ we plan to derive the bound (2.19) from

$$\begin{aligned} &\left[\int_I \left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\, d\mu(x)\right]^{1/q} \\ \leq\ & c_1 B \left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p} \end{aligned} \tag{2.24}$$

and

$$\left[\int_\Omega \left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p} \leq c_2 \left(\int_\Omega f^p d\lambda\right)^{1/p}. \tag{2.25}$$

Lemma 2.11. *If $B < \infty$ and $I \neq \varnothing$, then*

$$\int_{\{x \in I: h(x)=0\}} vd\mu = \int_{\{x \in I: h(x)=\infty\}} vd\mu = 0 \tag{2.26}$$

and (2.24) holds.

Proof. h is Borel measurable. For $x \in I$ by Lemma 2.1

$$\begin{aligned} h(x) &\geq V(x)^{q/p}\left[\int_{\Omega[a,\tau(x)]}\left(\int_{\Omega[a,\tau(y)]} u^{p'}d\lambda\right)^{r/q'} u(y)^{p'}\,d\lambda(y)\right]^{q/r} \\ &\geq cV(x)^{q/p}\,U(x)^{q/p'}. \end{aligned}$$

Here $U(x) \neq 0$, so $h(x) = 0$ implies $\int_{\Omega[\tau(x),b]} v d\mu = 0$ and by Lemma 2.4 we get the first equation in (2.26).

Changing the order of integration we see that

$$\begin{aligned} & \int_I vh^{r/q}d\mu = \int_\Omega vh^{r/q}d\mu \qquad (2.27)\\ \leq\ & \int_\Omega\left(\int_{\Omega[a,\tau(x)]} U^{r/q'}V^{r/p}u^{p'}\chi_{\Omega[a,\tau(x)]}d\lambda\right)v(x)\,d\mu(x) \\ =\ & \int_\Omega U(y)^{r/q'}\,V(y)^{r/p}\,u(y)^{p'}\left(\int_{\Omega[\tau(y),b]} vd\mu\right)d\lambda(y) \\ =\ & \int_\Omega U^{r/q'}V^{r/q}u^{p'}d\lambda \\ & \text{(using the left inequality in (2.3) and changing integration order)} \\ \leq\ & c_1\int_\Omega U(y)^{r/q'}\,u(y)^{p'}\left(\int_{\Omega[\tau(y),b]} vV^{r/p}d\mu\right)d\lambda(y) \\ =\ & c_1\int_\Omega\left(\int_{\Omega[a,\tau(x)]} U^{r/q'}u^{p'}d\lambda\right)V(x)^{r/p}\,v(x)\,d\mu(x) \\ & \text{(by Lemma 2.1)} \\ \leq\ & c_2\int_\Omega U^{r/p'}V^{r/p}vd\mu = c_2B^r. \end{aligned}$$

This bound implies, in particular, the second equality in (2.26).

Using (2.26), (2.27) and Hölder's inequality with the exponents r/q and p/q we complete the proof of (2.24):

$$\begin{aligned} & \left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \\ =\ & \left[\int_I h(x)\,h(x)^{-1}\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \\ \leq\ & \left(\int_\Omega vh^{r/q}d\mu\right)^{1/r}\left[\int_I h(x)^{-p/q}\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p v(x)\,d\mu(x)\right]^{1/p} \\ \leq\ & c_2^{1/r}B\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^p d\tilde{\mu}(x)\right]^{1/p}. \end{aligned}$$

□

Lemma 2.12. *If* $B < \infty$ *and* $I \neq \varnothing$, *then*

$$\sup_{x\in I}\tilde{\mu}\left(\Omega\left[\tau(x),b\right]\right)^{1/p}U(x)^{1/p'} \leq c \qquad (2.28)$$

and (2.25) *is true.*

Proof. Let $x \in I$. By Lemma 2.10 we know that $A < \infty$. Therefore $U(x) > 0$ implies that the integral $\int_{\Omega[\tau(x),b]} v d\mu$ is finite. If it is zero, then $\tilde{\mu}(\Omega[\tau(x), b]) = 0$. Suppose that integral is not zero. Using the inequalities $h(y) \geq h(x)$ for $\tau(y) \geq \tau(x)$ and $V(z) \geq V(x)$ for $\tau(z) \leq \tau(x)$, we have for $x \in I$

$$
\begin{aligned}
\tilde{\mu}(\Omega[\tau(x), b]) &= \int_{\Omega[\tau(x),b]} \chi_I v h^{-p/q} d\mu \leq \int_{\Omega[\tau(x),b]} v d\mu h(x)^{-p/q} \\
&\leq V(x) \left(\int_{\Omega[a,\tau(x)]} U^{r/q'} V^{r/p} u^{p'} d\lambda \right)^{-p/r} \\
&\quad \text{(applying Lemma 2.1 with } \gamma = r/q') \\
&\leq V(x) \left(V(x)^{r/p} \int_{\Omega[a,\tau(x)]} U^{r/q'} u^{p'} d\lambda \right)^{-p/r} \leq cU(x)^{-p/p'} < \infty.
\end{aligned}
$$

This proves (2.28).

Further, put $i = \inf\{\tau(x) : x \in I\}$. If the infimum is attained on I, then $I \subset \Omega[i, b]$, $U(i) > 0$ and

$$
\tilde{\mu}(\Omega) = \int_I v h^{-p/q} d\mu = \int_{\Omega[i,b]} \chi_I v h^{-p/q} d\mu = \tilde{\mu}(\Omega[i, b]) < \infty
$$

by (2.28). If $i \notin I$, then $I \subseteq \Omega(i, b]$ and $\tilde{\mu}(\Omega[a, i] \cap I) = 0$. Take any sequence $\{i_n\} \subset I$ such that $i_n \downarrow i$ and put $E_n = \Omega[a, i] \cup \Omega[i_n, b]$. Then $\Omega = \cup E_n$ and $\tilde{\mu}(E_n) < \infty$ for all n. Hence $\tilde{\mu}$ is σ-finite on Ω. Besides, (2.18) implies $\tilde{\mu}(I_\infty) = 0$. Therefore (2.28) and Theorem 2.1 give for any n

$$
\begin{aligned}
&\left[\int_{E_n} \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^p d\tilde{\mu}(x) \right]^{1/p} \\
&= \left[\int_{E_n \cap I} \left(\int_{\Omega[a,\tau(x)]} f u d\lambda \right)^p d\tilde{\mu}(x) \right]^{1/p} \leq c \left(\int_\Omega f^p d\lambda \right)^{1/p}.
\end{aligned}
$$

Here c does not depend on n. With fixed f, we can let $n \to \infty$ and finish the proof. □

Theorem 2.2. *Let* $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$. (2.16) *holds if and only if* $B < \infty$, *and* $c_1 B \leq C \leq c_2 B$.

Proof. Sufficiency. Let $B < \infty$. If $I = \varnothing$, by Lemma 2.9 we see that in fact $B = B' = 0$. On the other hand, the best constant in (2.16) in this case is also 0 because of (2.17) and (2.18). If $I \neq \varnothing$, the sufficiency follows by Lemmas 2.11 and 2.12.

Necessity. Suppose (2.16) is true with $C < \infty$ and, hence, (2.17) and (2.18) hold. Since μ is σ-finite, there is a sequence $\{E_n\}$ of sets such that $\Omega = \cup E_n$ and $\mu(E_n) < \infty$. We can assume that $E_n \subset E_{n+1}$ and $E_n \cap I \neq \varnothing$ for all n. Let $\{s_n\} \subset I$ be such that $s_n \uparrow s = \sup\{\tau(x) : x \in I\}$ and for $n \in N$ define

$$
F_n = \begin{cases} E_n \cap I, & \text{if } s \in \tau(I) \\ E_n \cap I \cap \Omega[a, s_n], & \text{if } s \notin \tau(I). \end{cases}
$$

Then $\{F_n\}$ satisfies $\cup F_n = I$, $F_n \subset F_{n+1}$, $\mu(F_n) < \infty$.

Put $v_n = \min\{v, n\}$, $d\mu_n = v_n \chi_{F_n} d\mu$,

$$
\begin{aligned}
B_n &= \left(\int_I U(x)^{r/p'} \mu_n(\Omega[\tau(x), b])^{r/p} d\mu_n(x) \right)^{1/r} \\
&= \left[\int_{F_n} U(x)^{r/p'} \left(\int_{\Omega[\tau(x),b]} v_n \chi_{F_n} d\mu \right)^{r/p} v_n(x) d\mu(x) \right]^{1/r}.
\end{aligned}
$$

If $F_n = \varnothing$, then $B_n = 0$. If $F_n \neq \varnothing$, then with $\alpha_n = \sup F_n \in I$ we have

$$B_n \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} \left[\int_{F_n} \left(\int_{\Omega[\tau(x),b]} v_n \chi_{F_n} d\mu\right)^{r/p} v_n(x)\, d\mu(x)\right]^{1/r}.$$

By Lemma 2.2 and the definition of v_n

$$B_n \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} \mu_n(F_n)^{1/q} \leq \left(\int_{\Omega(\alpha_n)} u^{p'} d\lambda\right)^{1/p'} (n\mu(F_n))^{1/q} < \infty,$$

because $I \cap I_\infty = \varnothing$.

Put

$$f(y) = \mu_n(\Omega[\tau(x), b])^{r/(pq)} U(y)^{r/(pq')} u(y)^{p'-1} \chi_I(y).$$

Then

$$\begin{aligned}
& \left[\int_I \left(\int_{\Omega[a,\tau(x)]} f u d\lambda\right)^q d\mu_n(x)\right]^{1/q} \qquad (2.29)\\
= \; & \left[\int_I \left(\int_{\Omega[a,\tau(x)]\cap I} \mu_n(\Omega[\tau(y), b])^{\frac{r}{pq}} U(y)^{\frac{r}{pq'}} u(y)^{p'} d\lambda(y)\right)^q d\mu_n(x)\right]^{\frac{1}{q}}\\
\geq \; & \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/(pq')} u^{p'} d\lambda\right)^q d\mu_n(x)\right]^{1/q}\\
& \text{(applying (2.17) and Lemma 2.1)}\\
= \; & \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]} U^{r/(pq')} u^{p'} d\lambda\right)^q d\mu_n(x)\right]^{1/q}\\
\geq \; & c\left(\int_I \mu_n(\Omega[\tau(x), b])^{r/p} U(x)^{r/p'} d\mu_n(x)\right)^{1/q} = cB_n^{r/q}.
\end{aligned}$$

On the other hand, by Lemma 2.2

$$\begin{aligned}
& \left(\int_\Omega f^p d\lambda\right)^{1/p} = \left[\int_I \mu_n(\Omega[\tau(y), b])^{r/q} U(y)^{r/q'} u(y)^{p'} d\lambda(y)\right]^{1/q} \qquad (2.30)\\
\leq \; & c_2 \left[\int_\Omega \left(\int_{\Omega[\tau(y),b]} \mu_n(\Omega[\tau(x), b])^{\frac{r}{p}} d\mu_n(x)\right) U(y)^{\frac{r}{q'}} u(y)^{p'} \chi_I(y)\, d\lambda(y)\right]^{\frac{1}{p}}\\
& \text{(changing integration order)}\\
= \; & c_2 \left[\int_\Omega \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/q'} u^{p'} d\lambda\right) d\mu_n(x)\right]^{1/p}\\
& \text{(using } \operatorname{supp}\mu_n \subset F_n \subset I\text{)}\\
\leq \; & c_2 \left[\int_I \mu_n(\Omega[\tau(x), b])^{r/p} \left(\int_{\Omega[a,\tau(x)]\cap I} U^{r/q'} u^{p'} d\lambda\right) d\mu_n(x)\right]^{1/p}\\
& \text{(by Lemma 2.1)}\\
\leq \; & c_3 \left(\int_I \mu_n(\Omega[\tau(x), b])^{r/p} U(x)^{r/p'} d\mu_n(x)\right)^{1/p} = B_n^{r/p}.
\end{aligned}$$

Putting together (2.29), (2.19) and (2.30) we see that

$$\begin{aligned} B_n^{r/q} &\le c_4\left[\int_I\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q d\mu_n(x)\right]^{1/q} \\ &\le c_5 C\left(\int_\Omega f^p d\lambda\right)^{1/p} \le c_5 C B_n^{r/p} \end{aligned}$$

or $B_n \le c_5 C$. Since $\cup F_n = I$, $F_n \subset F_{n+1}$, we have $v_n\chi_{F_n} \uparrow v$ as $n \to \infty$, so the statement follows by the monotone convergence theorem. □

The general result is stated next.

Theorem 2.3. *Suppose* $1 < p < \infty$, $0 < q < \infty$, $1/r = 1/q - 1/p$. *Let* $\nu = \nu_a + \nu_s$ *be the Lebesgue decomposition of* ν *with respect to* λ, *where* ν_a *is absolutely continuous with respect to* λ *and* ν_s *and* λ *are mutually singular. Denote* $\frac{d\nu_a}{d\lambda}$ *the Radon-Nikodym derivative of* ν_a *with respect to* λ.

a) If $p \le q$, *then the inequality*

$$\begin{aligned} \left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} &\le C\left(\int_\Omega f^p w d\nu\right)^{1/p}, \\ f &\in \{\mathfrak{M}_\lambda\}^+, \end{aligned} \tag{2.31}$$

holds if and only if $\mathcal{A} = \sup_{x\in\Omega}\mathcal{A}(x) < \infty$, *where*

$$\mathcal{A}(x) = \left[\int_{\Omega[a,\tau(x)]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'} d\lambda\right]^{1/p'}\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{1/q}.$$

Moreover, $c_1\mathcal{A} \le C \le c_2\mathcal{A}$.

b) If $q < p$, *then* (2.31) *is true if and only if* $\mathcal{B} < \infty$, *where*

$$\mathcal{B} = \left\{\int_\Omega\left[\int_{\Omega[a,\tau(x)]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'} d\lambda\right]^{r/p'}\left(\int_{\Omega[\tau(x),b]} vd\mu\right)^{r/p} v(x)\,d\mu(x)\right\}^{1/r}.$$

Besides, $c_1\mathcal{B} \le C \le c_2\mathcal{B}$.

Proof. By Lemma 2.6, (2.31) is equivalent to

$$\left[\int_\Omega\left(\int_{\Omega[a,\tau(x)]} fud\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \le C\left(\int_\Omega f^p w\frac{d\nu_a}{d\lambda}d\lambda\right)^{1/p},\ f \in \{\mathfrak{M}_\lambda\}^+.$$

This inequality, in turn, by Lemma 2.5 is equivalent to

$$\begin{aligned} \left\{\int_\Omega\left[\int_{\Omega[a,\tau(x)]} fu\left(w\frac{d\nu_a}{d\lambda}\right)^{-1/p} d\lambda\right]^q v(x)\,d\mu(x)\right\}^{1/q} &\le C\left(\int_\Omega f^p d\lambda\right)^{1/p}, \\ f &\in \{\mathfrak{M}_\lambda\}^+. \end{aligned}$$

Application of Theorems 2.1 and 2.2 completes the proof. □

3 Results for the dual operator

The dual operator is defined by

$$T^*f(x) = \int_{\{y\in\Omega:\tau(y)\geq\tau(x)\}} f(y)\,d\lambda(y)\,,\ x\in\Omega.$$

The analogues of Theorems 2.1 and 2.2 are in the next theorem.

Let

$$\begin{aligned} U^*(x) &= \int_{\Omega[\tau(x),b]} u^{p'}d\lambda,\ V^*(x) = \int_{\Omega[a,\tau(x)]} v d\mu, \\ A^*(x) &= V^*(x)^{1/q} U^*(x)^{1/p'},\ x\in\Omega,\ A^* = \sup_{x\in\Omega} A^*(x), \\ B^* &= \left[\int_\Omega \left(\int_{\Omega[\tau(x),b]} u^{p'}d\lambda\right)^{r/p'} \left(\int_{\Omega[a,\tau(x)]} v d\mu\right)^{r/p} v(x)\,d\mu(x)\right]^{1/r} \end{aligned}$$

and consider the inequality

$$\left[\int_\Omega \left(\int_{\Omega[\tau(x),b]} fu d\lambda\right)^q v(x)d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p d\lambda\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+. \tag{3.1}$$

Theorem 3.1. *Suppose $1<p<\infty$, $0<q<\infty$. a) If $p\leq q$, then inequality (3.1) holds if and only if $A^*<\infty$, with the equivalence $c_1A^*\leq C\leq c_2A^*$.*

b) If $q<p$, then (3.1) holds if and only if $B^*<\infty$, and $c_1B^*\leq C\leq c_2B^*$.

The general result looks as follows.

Theorem 3.2. *Suppose $1<p<\infty$, $0<q<\infty$, $1/r=1/q-1/p$. Let $\nu=\nu_a+\nu_s$ be the Lebesgue decomposition of ν with respect to λ, where ν_a is absolutely continuous with respect to λ and ν_s and λ are mutually singular. Denote $\frac{d\nu_a}{d\lambda}$ the Radon-Nikodym derivative of ν_a with respect to λ.*

a) If $p\leq q$, then the inequality

$$\left[\int_\Omega \left(\int_{\Omega[\tau(x),b]} fu d\lambda\right)^q v(x)\,d\mu(x)\right]^{1/q} \leq C\left(\int_\Omega f^p w d\nu\right)^{1/p},\ f\in\{\mathfrak{M}_\lambda\}^+, \tag{3.2}$$

holds if and only if $\mathcal{A}^*=\sup_{x\in\Omega}\mathcal{A}^*(x)<\infty$, where

$$\mathcal{A}^*(x) = \left[\int_{\Omega[\tau(x),b]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'} d\lambda\right]^{1/p'} \left(\int_{\Omega[a,\tau(x)]} v d\mu\right)^{1/q}.$$

Moreover, $c_1\mathcal{A}^*\leq C\leq c_2\mathcal{A}^*$.

b) If $q<p$, then (3.2) is true if and only if $\mathcal{B}^*<\infty$, where

$$\mathcal{B}^* = \left\{\int_\Omega \left[\int_{\Omega[\tau(x),b]} u^{p'}\left(w\frac{d\nu_a}{d\lambda}\right)^{1-p'} d\lambda\right]^{r/p'} \left(\int_{\Omega[a,\tau(x)]} v d\mu\right)^{r/p} v(x)\,d\mu(x)\right\}^{1/r}.$$

Besides, $c_1\mathcal{B}^*\leq C\leq c_2\mathcal{B}^*$.

Acknowledgments

The author thanks the anonymous referee for detailed remarks which improved the exposition.

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19676673).

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Received: 18.02.2023