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# EURASIAN MATHEMATICAL JOURNAL

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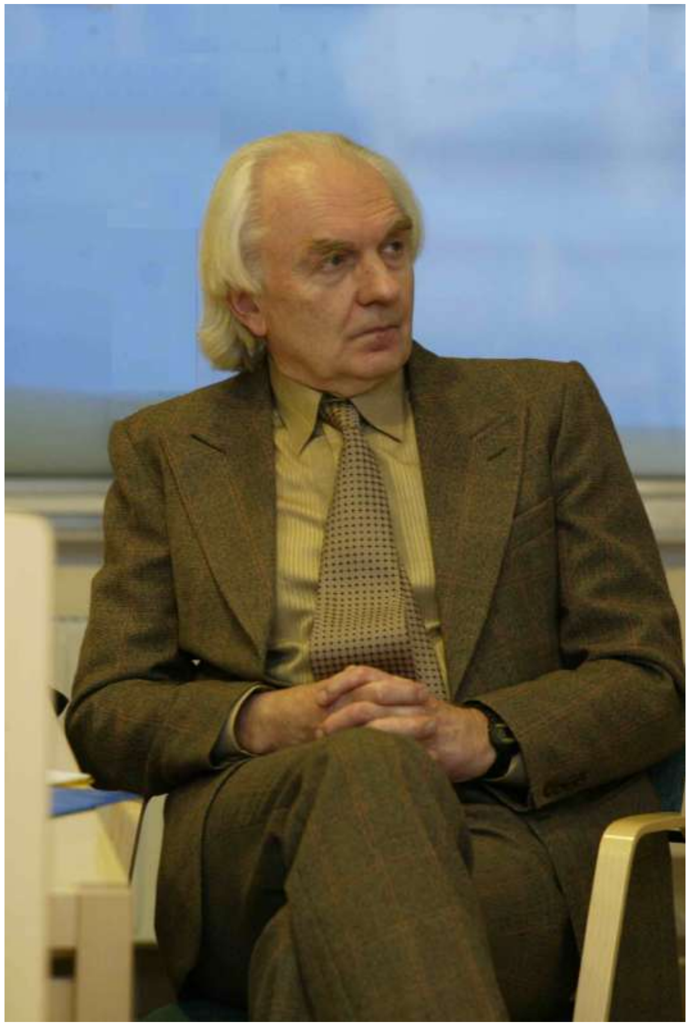
# On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \le p, \theta \le \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter $r$ may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components $r_j$. These spaces consist of functions having common smoothness of order $r$ in the isotropic case (not necessarily integer) and smoothness of orders $r_j$ in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in $L_p$-metrics, and $\theta$ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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# CAFFARELLI-KOHN-NIRENBERG INEQUALITIES FOR BESOV AND TRIEBEL-LIZORKIN-TYPE SPACES

**D. Drihem**

Communicated by D. Suragan

**Key words:** Besov spaces, Triebel-Lizorkin spaces, Morrey spaces, Herz spaces, Caffarelli-Kohn-Nirenberg inequalities.

**AMS Mathematics Subject Classification:** 46B70, 46E35.

**Abstract.** We present some Caffarelli-Kohn-Nirenberg-type inequalities for Herz-type Besov-Triebel-Lizorkin spaces, Besov-Morrey and Triebel-Lizorkin-Morrey spaces. More precisely, we investigate the inequalities

$$\|f\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \leq c\,\|f\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\,\|f\|^{\theta}_{\dot{K}^{\alpha_3,\delta_1}_p A^s_\beta}$$

and

$$\|f\|_{\mathcal{E}^{\sigma}_{p,2,u}} \leq c\,\|f\|^{1-\theta}_{M^{\delta}_{\mu}}\,\|f\|^{\theta}_{\mathcal{N}^{s}_{q,\beta,v}},$$

with some appropriate assumptions on the parameters, where $\dot{k}^{\alpha_1,r}_{v,\sigma}$ are the Herz-type Bessel potential spaces, which are just the Sobolev spaces if $\alpha_1 = 0, 1 < r = v < \infty$ and $\sigma \in \mathbb{N}_0$, and $\dot{K}^{\alpha_3,\delta_1}_p A^s_\beta$ are Besov or Triebel-Lizorkin spaces if $\alpha_3 = 0$ and $\delta_1 = p$. The usual Littlewood-Paley technique, Sobolev and Franke embeddings are the main tools of this paper. Some remarks on Hardy-Sobolev inequalities are given.

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## 1 Introduction

Major results in harmonic analysis and partial differential equations invoke some inequalities. Some examples can be mentioned such as: Caffarelli, Kohn and Nirenberg in [7]. They proved the following useful inequality:

$$\||x|^{\gamma} f\|_{\tau} \leq c\,\left\||x|^{\beta} f\right\|^{\theta}_{q}\,\||x|^{\alpha}\nabla f\|^{1-\theta}_{p}, \quad f \in C^{\infty}_{0}(\mathbb{R}^n), \tag{1.1}$$

where $1 \leq p, q < \infty, \tau > 0, 0 \leq \theta \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ satisfy some suitable conditions and $c > 0$ depends only on these numerical parameters. This inequality plays an important role in theory of PDE's. It was extended to fractional Sobolev spaces in [32]. This estimate can be rewritten in the following form:

$$\|f\|_{\dot{K}^{\gamma,\tau}_{\tau}} \leq c\big\|f\big\|^{\theta}_{\dot{K}^{\beta,q}_{q}}\big\|\nabla f\big\|^{1-\theta}_{\dot{K}^{\alpha,p}_{p}}, \quad f \in C^{\infty}_{0}(\mathbb{R}^n),$$

where $\dot{K}^{\alpha,p}_{q}$ is the Herz space, see Definition 1 below. These function spaces play an important role in Harmonic Analysis. After they have been introduced in [21], the theory of these spaces had a remarkable development, in particular, due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [2], in the semilinear parabolic equations [13], in the summability of Fourier transforms [16], and in the Cauchy problem for Navier-Stokes equations

[45]. For important and latest results for Herz spaces, we refer the reader to the papers [34], [52] and to the monograph [25].

Inequality (1.1) with $\alpha=\beta=\gamma=0$, takes the form

$$\big\|f\big\|_{L^\tau} \leq c\big\|f\big\|_{L^q}^{\theta}\big\|\nabla f\big\|_{L^p}^{1-\theta}, \quad f\in C_0^\infty(\mathbb{R}^n),$$

where $L^p$, $1\leq p\leq\infty$ is the Lebesgue space.

The main purpose of this paper is to present a more general version of such inequalities. More precisely, we extend this estimate to Herz-type Besov-Triebel-Lizorkin spaces, called $\dot{K}_q^{\alpha,p}B_\beta^s$ and $\dot{K}_q^{\alpha,p}F_\beta^s$, which generalize the usual Besov and Triebel-Lizorkin spaces. We mean that

$$\dot{K}_p^{0,p}B_\beta^s = B_{p,\beta}^s \quad \text{and} \quad \dot{K}_p^{0,p}F_\beta^s = F_{p,\beta}^s.$$

In addition $\dot{K}_q^{\alpha,p}F_2^0$ are just the Herz spaces $\dot{K}_q^{\alpha,p}$ when $1<p,q<\infty$ and $-\frac{n}{q}<\alpha<n(1-\frac{1}{q})$. In the same manner, we extend these inequalities to Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Our approach based on the Littlewood-Paley technique of Triebel [44] and some results obtained by the author in [9, 10, 11].

The structure of this paper needs some notation. As usual, $\mathbb{R}^n$ denotes the $n$-dimensional real Euclidean space, $\mathbb{N}$ the set of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For any $u>0, k\in\mathbb{Z}$ we set $C(u)=\{x\in\mathbb{R}^n : \frac{u}{2}<|x|\leq u\}$ and $C_k=C(2^k)$. $\chi_k$, for $k\in\mathbb{Z}$, denote the characteristic function of the set $C_k$. The expression $f\approx g$ means that $Cg\leq f\leq c\,g$ for some $c,C>0$ independent of non-negative functions $f$ and $g$.

For any measurable subset $\Omega\subseteq\mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0<p\leq\infty$ consists of all measurable functions for which

$$\big\|f\big\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p} < \infty, \quad 0<p<\infty$$

and

$$\big\|f\big\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|f(x)| < \infty.$$

If $\Omega=\mathbb{R}^n$, then we put $L^p(\mathbb{R}^n)=L^p$ and $\big\|f\big\|_{L^p(\mathbb{R}^n)}=\big\|f\big\|_p$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used to denote the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f\in\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx, \quad \xi\in\mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on $L^1_{\rm loc}$ by

$$\mathcal{M}f(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy, \quad x\in\mathbb{R}^n$$

and $\mathcal{M}_\tau f = (\mathcal{M}\,|f|^\tau)^{1/\tau}$, $0<\tau<\infty$.

Given two quasi-Banach spaces $X$ and $Y$, we write $X\hookrightarrow Y$ if $X\subset Y$ and the natural embedding of $X$ in $Y$ is continuous. We use $c$ as a generic positive constant, i.e. a constant whose value may be different in different inequalities.

## 2 Function spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_q^{\alpha,p}$.

**Definition 1.** Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_q^p\Big)^{1/p}$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p, q) \geq 1$ then $\dot{K}_q^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ the space $\dot{K}_p^{0,p}$ coincides with the Lebesgue space $L^p$. In addition

$$\dot{K}_p^{\alpha,p} = L^p(\mathbb{R}^n, |\cdot|^{\alpha p}), \quad \text{(Lebesgue space equipped with power weight)},$$

where

$$\|f\|_{L^p(\mathbb{R}^n,|\cdot|^{\alpha p})} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, |x|^{\alpha p} dx\Big)^{1/p}.$$

Note that

$$\dot{K}_q^{\alpha,p} \subset \mathcal{S}'(\mathbb{R}^n)$$

for any $\alpha < n(1 - \frac{1}{q})$, $1 \leq p, q \leq \infty$ or $\alpha = n(1 - \frac{1}{q})$, $p = 1$ and $1 \leq q \leq \infty$. We mean that,

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), f \in \dot{K}_q^{\alpha,p}$$

generates a distribution $T_f \in \mathcal{S}'(\mathbb{R}^n)$. A detailed discussion of the properties of these spaces can be found in [20, 24, 27], and references therein.

The following lemma is the $\dot{K}_q^{\alpha,p}$-version of the Plancherel-Polya-Nikolskij inequality.

**Lemma 2.1.** *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, \tau, q, r \leq \infty$. We suppose that $\alpha_1 + \frac{n}{s} > 0$, $0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$. Then there exists a positive constant $c > 0$ independent of $R$ such that for all $f \in \dot{K}_q^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ with* supp $\mathcal{F}f \subset \{\xi : |\xi| \leq R\}$, *we have*

$$\|f\|_{\dot{K}_s^{\alpha_1,r}} \leq c\, R^{\frac{n}{q} - \frac{n}{s} + \alpha_2 - \alpha_1} \|f\|_{\dot{K}_q^{\alpha_2,\delta}},$$

*where*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1. \end{cases}$$

**Remark 1.** We would like to mention that Lemma 2.1 improves the classical Plancherel-Polya-Nikolskij inequality if $\alpha_1 = \alpha_2 = 0, r = s$ due to the continuous embedding $\ell^q \hookrightarrow \ell^s$.

In the previous lemma we have not treated the case $s < q$. The next lemma gives a positive answer.

**Lemma 2.2.** *Let* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $0 < s, \tau, q, r \leq \infty$. *We suppose that* $\alpha_1 + \frac{n}{s} > 0$, $0 < s \leq q \leq \infty$ *and* $\alpha_2 \geq \alpha_1 + \frac{n}{s} - \frac{n}{q}$. *Then there exists a positive constant* $c$ *independent of* $R$ *such that for all* $f \in \dot{K}_q^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp} \mathcal{F}f \subset \{\xi : |\xi| \leq R\}$, *we have*

$$\|f\|_{\dot{K}_s^{\alpha_1,r}} \leq c\, R^{\frac{n}{q} - \frac{n}{s} + \alpha_2 - \alpha_1} \|f\|_{\dot{K}_q^{\alpha_2,\delta}},$$

*where*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1 + \frac{n}{s} - \frac{n}{q}, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1 + \frac{n}{s} - \frac{n}{q}. \end{cases}$$

The proof of these inequalities is given in [9], Lemmas 3.10 and 3.14. Let $1 < q < \infty$ and $0 < p \leq \infty$. If $f$ is a locally integrable function on $\mathbb{R}^n$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, then

$$\|\mathcal{M}f\|_{\dot{K}_q^{\alpha,p}} \leq c\|f\|_{\dot{K}_q^{\alpha,p}}, \tag{2.1}$$

see [24]. We need the following lemma, which is basically a consequence of Hardy's inequality in the sequence Lebesgue space $\ell^q$.

**Lemma 2.3.** *Let* $0 < a < 1$ *and* $0 < q \leq \infty$. *Let* $\{\varepsilon_k\}_{k\in\mathbb{N}_0}$ *be a sequence of positive real numbers, such that*

$$\|\{\varepsilon_k\}_{k\in\mathbb{N}_0}\|_{\ell^q} = I < \infty.$$

*Then the sequences* $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{N}_0}$ *and* $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{N}_0}$ *belong to* $\ell^q$, *and*

$$\|\{\delta_k\}_{k\in\mathbb{N}_0}\|_{\ell^q} + \|\{\eta_k\}_{k\in\mathbb{N}_0}\|_{\ell^q} \leq c\, I,$$

*with* $c > 0$ *only depending on* $a$ *and* $q$.

Some of our results of this paper are based on the following result, see Tang and Yang [40].

**Lemma 2.4.** *Let* $1 < \beta < \infty, 1 < q < \infty$ *and* $0 < p \leq \infty$. *If* $\{f_j\}_{j=0}^\infty$ *is a sequence of locally integrable functions on* $\mathbb{R}^n$ *and* $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, *then*

$$\Big\| \Big( \sum_{j=0}^\infty (\mathcal{M}f_j)^\beta \Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}} \lesssim \Big\| \Big( \sum_{j=0}^\infty |f_j|^\beta \Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}}.$$

Now, we present the Fourier analytic definition of Herz-type Besov and Triebel-Lizorkin spaces and recall their basic properties. We first need the concept of a smooth dyadic partition of the unity. Let $\varphi_0$ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| \geq \frac{3}{2}$. We put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x)$ for $j = 1, 2, 3, \dots$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity, $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j=0}^\infty \mathcal{F}^{-1}\varphi_j * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Besov and Triebel-Lizorkin spaces.

**Definition 2.** Let $\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty$ and $0 < \beta \leq \infty$.

(i) The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p} B_\beta^s} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_q^{\alpha,p}}^{\beta}\Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) Let $0 < p, q < \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p} F_\beta^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{K}_q^{\alpha,p} F_\beta^s} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left|\mathcal{F}^{-1}\varphi_j * f\right|^{\beta}\Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}} < \infty,$$

with the obvious modification if $\beta = \infty$.

**Remark 2.** Let $s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. The spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic partition of the unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then they are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [8, 9, 10, 11, 12, 48, 49, 47].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

**Definition 3.** (i) Let $s \in \mathbb{R}$ and $0 < p, \beta \leq \infty$. The Besov space $B_{p,\beta}^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{B_{p,\beta}^s} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_p^{\beta}\Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{F_{p,\beta}^s} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left|\mathcal{F}^{-1}\varphi_j * f\right|^{\beta}\Big)^{1/\beta} \Big\|_p < \infty,$$

with the obvious modification if $\beta = \infty$.

The theory of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$ has been developed in detail in [42, 43] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$,

$$\dot{K}_p^{0,p} B_\beta^s = B_{p,\beta}^s \quad \text{and} \quad \dot{K}_p^{0,p} F_\beta^s = F_{p,\beta}^s.$$

Let $w$ denote a positive, locally integrable function and $0 < p < \infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ consists of all measurable functions such that

$$\big\|f\big\|_{L^p(\mathbb{R}^n,w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, w(x) dx\Big)^{1/p} < \infty.$$

For $\varrho \in [1, \infty)$ we denote by $\mathcal{A}_\varrho$ the Muckenhoupt class of weights, and $\mathcal{A}_\infty = \cup_{\varrho \geq 1} \mathcal{A}_\varrho$. We refer to [17] for the general properties of these classes. Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Besov spaces $B_{p,\beta}^s(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{B_{p,\beta}^s(\mathbb{R}^n,w)} = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{L^p(\mathbb{R}^n,w)}^{\beta}\Big)^{1/\beta}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel-Lizorkin spaces $F^s_{p,\beta}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,\beta}(\mathbb{R}^n,w)} = \Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} \left| \mathcal{F}^{-1}\varphi_j * f \right|^\beta \Big)^{1/\beta} \Big\|_{L^p(\mathbb{R}^n,w)}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

The spaces $B^s_{p,\beta}(\mathbb{R}^n, w) = B^s_{p,\beta}(w)$ and $F^s_{p,\beta}(\mathbb{R}^n, w) = F^s_{p,\beta}(w)$ are independent of the particular choice of the smooth dyadic partition of the unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, \beta \geq 1$). Moreover, for $w \equiv 1 \in \mathcal{A}_\infty$ we obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces. We refer, in particular, to the papers [3, 4, 22] for a comprehensive investigation consists of the weighted spaces. Let $w_\gamma$ be a power weight, i.e., $w_\gamma(x) = |x|^\gamma$ with $\gamma > -n$. Then we have

$$B^s_{p,\beta}(w_\gamma) = \dot{K}^{\frac{\gamma}{p},p}_p B^s_\beta \quad \text{and} \quad F^s_{p,\beta}(w_\gamma) = \dot{K}^{\frac{\gamma}{p},p}_p F^s_\beta,$$

in the sense of equivalent quasi-norms.

**Definition 4.** (i) Let $1 < q < \infty, 0 < p < \infty, -\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$ and $s \in \mathbb{R}$. Then the Herz-type Bessel potential space $\dot{k}^{\alpha,p}_{q,s}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{k}^{\alpha,p}_{q,s}} = \|(1 + |\xi|^2)^{\frac{s}{2}} * f\|_{\dot{K}^{\alpha,p}_q} < \infty.$$

(ii) Let $1 < q < \infty, 0 < p < \infty, -\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$ and $m \in \mathbb{N}$. The homogeneous Herz-type Sobolev space $\dot{W}^{\alpha,p}_{q,m}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{W}^{\alpha,p}_{q,m}} = \sum_{|\beta|\leq m} \Big\| \frac{\partial^\beta f}{\partial^\beta x} \Big\|_{\dot{K}^{\alpha,p}_q} < \infty,$$

where the derivatives must be understood in the sense of distribution.

In the following, we will present the connection between the Herz-type Triebel-Lizorkin spaces and the Herz-type Bessel potential spaces; see [26, 48]. Let $1 < q < \infty, 1 < p < \infty$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$. If $s \in \mathbb{R}$, then

$$\dot{K}^{\alpha,p}_q F^s_2 = \dot{k}^{\alpha,p}_{q,s} \tag{2.2}$$

with equivalent norms. If $s = m \in \mathbb{N}$, then

$$\dot{K}^{\alpha,p}_q F^m_2 = \dot{W}^{\alpha,p}_{q,m} \tag{2.3}$$

with equivalent norms. In particular

$$\dot{K}^{0,p}_p F^m_2 = W^p_m \quad \text{(Sobolev spaces)}$$

and

$$\dot{K}^{\alpha,p}_q F^0_2 = \dot{K}^{\alpha,p}_q \tag{2.4}$$

with equivalent norms. Let $0 < \theta < 1$,

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$$

and

$$s = (1-\theta)s_0 + \theta s_1.$$

For simplicity, in what follows, we use $\dot{K}_q^{\alpha,p} A_\beta^s$ to denote either $\dot{K}_q^{\alpha,p} B_\beta^s$ or $\dot{K}_q^{\alpha,p} F_\beta^s$. As an immediate consequence of Hölder's inequality we have the so-called interpolation inequalities:

$$\|f\|_{\dot{K}_q^{\alpha,p} A_\beta^s} \leq \|f\|^{1-\theta}_{\dot{K}_{q_0}^{\alpha_0,p_0} A_{\beta_0}^{s_0}} \|f\|^{\theta}_{\dot{K}_{q_1}^{\alpha_1,p_1} A_{\beta_1}^{s_1}} \tag{2.5}$$

which hold for all $f \in \dot{K}_{q_0}^{\alpha_0,p_0} A_{\beta_0}^{s_0} \cap \dot{K}_{q_1}^{\alpha_1,p_1} A_{\beta_1}^{s_1}$.

We collect some embeddings on these function spaces as obtained in [9]-[10]. First we have elementary embeddings within these spaces. Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. Then

$$\dot{K}_q^{\alpha,p} B^s_{\min(\beta,p,q)} \hookrightarrow \dot{K}_q^{\alpha,p} F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p} B^s_{\max(\beta,p,q)}. \tag{2.6}$$

**Theorem 2.1.** *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r, \beta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that*

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$

*Let $0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$ or $0 < s \leq q \leq \infty$ and*

$$\alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}. \tag{2.7}$$

(i) *We have the embedding*

$$\dot{K}_q^{\alpha_2,\theta} B_\beta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,r} B_\beta^{s_1},$$

*where*

$$\theta = \begin{cases} r, & \text{if} \quad \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 = \alpha_1, q \leq s, \\ p, & \text{if} \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 > \alpha_1, q \leq s. \end{cases}$$

(ii) *Let $0 < q, s < \infty$. The embedding*

$$\dot{K}_q^{\alpha_2,r} F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\beta^{s_1}$$

*holds if $0 < r \leq p < \infty$, where*

$$\theta = \begin{cases} \beta, & \text{if} \quad 0 < s \leq q < \infty \text{ and } \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}; \\ \infty, & \text{otherwise.} \end{cases}$$

We now present an immediate corollary of the Sobolev embeddings, which are called Hardy-Sobolev inequalities.

**Corollary 2.1.** *Let $1 < q \leq s < \infty$, $1 < q < n$ and $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in \dot{W}_q^1$*

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{|x|^{-\alpha}}\right)^s dx \leq c \left(\sum_{|\beta|=1} \left\|\frac{\partial^\beta f}{\partial^\beta x}\right\|_{\dot{K}_q^{0,s}}\right)^s \leq c \left(\sum_{|\beta|=1} \left\|\frac{\partial^\beta f}{\partial^\beta x}\right\|_q\right)^s.$$

Now we recall the Franke embedding, see [12].

**Theorem 2.2.** *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p, q < \infty, 0 < \theta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that*
$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$
*Let*
$$0 < q < s < \infty \quad \text{and} \quad \alpha_2 \geq \alpha_1,$$
*or*
$$0 < s \leq q < \infty \quad \text{and} \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}.$$
*Then*
$$\dot{K}_q^{\alpha_2,p} B_p^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\theta^{s_1}.$$

**Corollary 2.2.** *Let $1 < q \leq s < \infty$ with $1 < q < n$. Let $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in B_{q,s}^1$*
$$\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{|x|^{-\alpha}} \right)^s dx \leq c \big\| f \big\|_{\dot{K}_q^{0,s} B_s^1}^s \leq c \big\| f \big\|_{B_{q,s}^1}^s.$$

**Remark 3.** We would like to mention that in Theorem 2.1 and Theorem 2.2 the assumptions $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$, (2.7) and $0 < r \leq p < \infty$ are necessary, see [9, 10, 12].

Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a partition of unity. For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote, Peetre maximal function,
$$(\mathcal{F}^{-1}\varphi_j)^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\mathcal{F}^{-1}\varphi_j * f(y)|}{(1 + 2^j |x - y|)^a}, \quad j \in \mathbb{N}_0.$$
We now present a fundamental characterization of the above spaces, which plays an essential role in this paper, see [46, Theorem 1].

**Theorem 2.3.** *Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. Let $a > \frac{n}{\min\left(q, \frac{n}{\alpha + \frac{n}{q}}\right)}$. Then*
$$\big\| f \big\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^\star = \left( \sum_{j=0}^{\infty} 2^{js\beta} \big\| (\mathcal{F}^{-1}\varphi_j)^{*,a} f \big\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta},$$
*is an equivalent quasi-norm in $\dot{K}_q^{\alpha,p} B_\beta^s$. Let $a > \frac{n}{\min\left(\min(q,\beta), \frac{n}{\alpha + \frac{n}{q}}\right)}$. Then*
$$\big\| f \big\|_{\dot{K}_q^{\alpha,p} F_\beta^s}^\star = \Big\| \left( \sum_{j=0}^{\infty} 2^{js\beta} (\mathcal{F}^{-1}\varphi_j)^{*,a} f)^\beta \right)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}},$$
*is an equivalent quasi-norm in $\dot{K}_q^{\alpha,p} F_\beta^s$.*

Let $0 < p, q \leq \infty$. For later use we introduce the following abbreviations:
$$\sigma_q = n \max(\frac{1}{q} - 1, 0) \quad \text{and} \quad \sigma_{p,q} = n \max(\frac{1}{p} - 1, \frac{1}{q} - 1, 0).$$
In the sequel we shall interpret $L_{\text{loc}}^1$ as the set of regular distributions.

**Theorem 2.4.** *Let $0 < p, q, \beta \leq \infty, \alpha > -\frac{n}{q}$, $\alpha_0 = n - \frac{n}{q}$ and $s > \max(\sigma_q, \alpha - \alpha_0)$. Then*
$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow L_{\text{loc}}^1,$$
*where $0 < p, q < \infty$ in the case of Herz-type Triebel-Lizorkin spaces.*

*Proof.* Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic partition of unity. We set

$$\varrho_k = \sum_{j=0}^{k} \mathcal{F}^{-1}\varphi_j * f, \quad k \in \mathbb{N}_0.$$

For technical reasons, we split the proof into two steps.

*Step* 1. We consider the case $1 \leq q \leq \infty$. In order to prove we additionally do it into the four Substeps 1.1, 1.2, 1.3 and 1.4.

*Substep* 1.1. $-\frac{n}{q} < \alpha < \alpha_0$. Since $s > 0$ and $\dot{K}_q^{\alpha,p} \hookrightarrow \dot{K}_q^{\alpha,\max(1,p)}$, we have

$$\sum_{j=0}^{\infty} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_q^{\alpha,\max(1,p)}} \lesssim \big\|f\big\|_{\dot{K}_q^{\alpha,p}A_\beta^s}.$$

Then, the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $g \in \dot{K}_q^{\alpha,\max(1,p)}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\langle f - g, \varphi\rangle = \langle f - \varrho_N, \varphi\rangle + \langle g - \varrho_N, \varphi\rangle, \quad N \in \mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder's inequality there exists a constant $C > 0$ independent of $N$ such that

$$|\langle g - \varrho_N, \varphi\rangle| \leq C\big\|g - \varrho_N\big\|_{\dot{K}_q^{\alpha,\max(1,p)}},$$

which tends to zero as $N \to \infty$. From this and $\dot{K}_q^{\alpha,\max(1,p)} \hookrightarrow L^1_{\mathrm{loc}}$, because of $\alpha < \alpha_0$, we deduce the desired result. In addition, we have

$$\dot{K}_q^{\alpha,p}A_\beta^s \hookrightarrow \dot{K}_q^{\alpha,\max(1,p)}.$$

*Substep* 1.2. $\alpha \geq \alpha_0$ and $1 < q \leq \infty$. Let $1 < q_1 < \infty$ be such that

$$s > \alpha + \frac{n}{q} - \frac{n}{q_1}.$$

We distinguish two cases:

• $q_1 = q$. By Theorem 2.1/(i), we obtain

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_q^{0,q}B_\beta^{s-\alpha} = B_{q,\beta}^{s-\alpha} \hookrightarrow L^1_{\mathrm{loc}}.$$

where the last embedding follows by the fact that

$$B_{q,\beta}^{s-\alpha} \hookrightarrow L^q, \tag{2.8}$$

because of $s - \alpha > 0$. The Herz-type Triebel-Lizorkin case follows by the second embeddings of (2.6).

• $1 < q_1 < q \leq \infty$ or $1 < q < q_1 < \infty$. If we assume the first possibility then Theorem 2.1/(i) and Substep 1.1 yield

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_{q_1}^{0,p}B_\beta^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^1_{\mathrm{loc}},$$

since $\alpha + \frac{n}{q} > \frac{n}{q_1}$. The latter possibility follows again by Theorem 2.1/(i). Indeed, we have

$$\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_q^{\alpha_0,p}B_\beta^{s+\alpha_0-\alpha} \hookrightarrow \dot{K}_{q_1}^{0,q_1}B_\beta^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} = B_{q_1,\beta}^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^1_{\mathrm{loc}},$$

where the last embedding follows by the fact that

$$B_{q_1,\beta}^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}} \hookrightarrow L^{q_1}. \tag{2.9}$$

Therefore from (2.6) we obtain the desired embeddings.

*Substep* 1.3. $q = 1$ and $\alpha > 0$. We have

$$\dot{K}_1^{\alpha,p} B_\beta^s \hookrightarrow \dot{K}_1^{0,1} B_\beta^{s-\alpha} = B_{1,\beta}^{s-\alpha} \hookrightarrow L^1,$$

since $s > \alpha$.

*Substep* 1.4. $q = 1$ and $\alpha = 0$. Let $\alpha_3$ be a real number such that $\max(-n, -s) < \alpha_3 < 0$. From Theorem 2.1, we get

$$\dot{K}_1^{0,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3}.$$

We have

$$\sum_{k=0}^{\infty} \big\|\mathcal{F}^{-1}\varphi_k * f\big\|_{\dot{K}_1^{\alpha_3,\max(1,p)}} \lesssim \big\|f\big\|_{\dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3}} \lesssim \big\|f\big\|_{\dot{K}_1^{0,p} A_\beta^s},$$

since $\alpha_3 + s > 0$. Using the same type of arguments as in Substep 1.1 it is easy to see that

$$\dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3} \hookrightarrow \dot{K}_1^{\alpha_3,\max(1,p)} \hookrightarrow L^1_{\text{loc}}.$$

*Step* 2. We consider the case $0 < q < 1$.

*Substep* 2.1. $-\frac{n}{q} < \alpha < 0$. By Lemma 2.1, we obtain

$$\sum_{j=0}^{\infty} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_1^{\alpha,\max(1,p)}} \lesssim \sum_{j=0}^{\infty} 2^{j(\frac{n}{q}-n)} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_q^{\alpha,p}} \lesssim \big\|f\big\|_{\dot{K}_q^{\alpha,p} A_\beta^s},$$

since $s > \frac{n}{q} - n$. The desired embedding follows by the fact that $\dot{K}_1^{\alpha,\max(1,p)} \hookrightarrow L^1_{\text{loc}}$ and the arguments in Substep 1.1. In addition

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha,\max(1,p)}. \tag{2.10}$$

*Substep* 2.2. $\alpha \geq 0$. Let $\alpha_4$ be a real number such that $\max(-n, -s + \frac{n}{q} - n + \alpha) < \alpha_4 < 0$. By Theorem 2.1, we get

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_1^{0,p} A_\beta^{s-\frac{n}{q}+n-\alpha} \hookrightarrow \dot{K}_1^{\alpha_4,p} A_\beta^{s-\frac{n}{q}+n-\alpha+\alpha_4} \hookrightarrow \dot{K}_1^{\alpha_4,\max(1,p)} A_\beta^{s-\frac{n}{q}+n-\alpha+\alpha_4}.$$

As in Substep 1.4, we easily obtain that

$$\dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow\hookrightarrow L^1_{\text{loc}}.$$

Therefore, under the hypothesis of this theorem, every $f \in \dot{K}_q^{\alpha,p} A_\beta^s$ is a regular distribution. $\square$

Let $f$ be an arbitrary function on $\mathbb{R}^n$ and $x, h \in \mathbb{R}^n$. Then

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^{M+1} f(x) = \Delta_h(\Delta_h^M f)(x), \quad M \in \mathbb{N}.$$

These are the well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^j \, C_M^j f(x + (M-j)h), \quad x \in \mathbb{R}^n,$$

where $C_M^j$ are the binomial coefficients. By ball means of differences we mean the quantity

$$d_t^M f(x) = t^{-n} \int_{|h|\leq t} \big|\Delta_h^M f(x)\big| \, dh = \int_B \big|\Delta_{th}^M f(x)\big| \, dh, \quad x \in \mathbb{R}^n.$$

Here $B = \{y \in \mathbb{R}^n : |h| \leq 1\}$ is the unit ball of $\mathbb{R}^n$ and $t > 0$ is a real number. We set

$$\|f\|^*_{\dot{K}^{\alpha,p}_q B^s_\beta} = \|f\|_{\dot{K}^{\alpha,p}_q} + \Big(\int_0^\infty t^{-s\beta} \|d^M_t f\|^\beta_{\dot{K}^{\alpha,p}_q} \frac{dt}{t}\Big)^{1/\beta}$$

and

$$\|f\|^*_{\dot{K}^{\alpha,p}_q F^s_\beta} = \|f\|_{\dot{K}^{\alpha,p}_q} + \Big\| \Big(\int_0^\infty t^{-s\beta} (d^M_t f)^\beta \frac{dt}{t}\Big)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_q}.$$

The following theorem play a central role in our paper.

**Theorem 2.5.** *Let* $0 < p, q, \beta \leq \infty, \alpha > -\frac{n}{q}$, $\alpha_0 = n - \frac{n}{q}$ *and* $M \in \mathbb{N}\backslash\{0\}$.

(i) *Assume that*

$$\max(\sigma_q, \alpha - \alpha_0) < s < M.$$

*Then* $\|\cdot\|^*_{\dot{K}^{\alpha,p}_q B^s_\beta}$ *is an equivalent quasi-norm on* $\dot{K}^{\alpha,p}_q B^s_\beta$.

(ii) *Let* $0 < p < \infty$ *and* $0 < q < \infty$. *Assume that*

$$\max(\sigma_{q,\beta}, \alpha - \alpha_0) < s < M.$$

*Then* $\|\cdot\|^*_{\dot{K}^{\alpha,p}_q F^s_\beta}$ *is an equivalent quasi-norm on* $\dot{K}^{\alpha,p}_q F^s_\beta$.

*Proof.* We split the proof into three steps.

*Step* 1. We will prove that

$$\|f\|_{\dot{K}^{\alpha,p}_q} \lesssim \|f\|_{\dot{K}^{\alpha,p}_q A^s_\beta}$$

for all $f \in \dot{K}^{\alpha,p}_q A^s_\beta$. We employ the same notations as in Theorem 2.4. Recall that

$$\varrho_k = \sum_{j=0}^{k} \mathcal{F}^{-1}\varphi_j * f, \quad k \in \mathbb{N}_0.$$

Obviously $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $f$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\{\varrho_k\}_{k\in\mathbb{N}_0} \subset \dot{K}^{\alpha,p}_q$ for any $0 < p, q \leq \infty$ and any $\alpha > -\frac{n}{q}$. Furthermore, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ is a Cauchy sequence in $\dot{K}^{\alpha,p}_q$ and hence it converges to a function $g \in \dot{K}^{\alpha,p}_q$, and

$$\|g\|_{\dot{K}^{\alpha,p}_q} \lesssim \|f\|_{\dot{K}^{\alpha,p}_q A^s_\beta}.$$

Let us prove that $g = f$ almost everywhere. We will do this in four cases.

*Case* 1. $-\frac{n}{q} < \alpha < \alpha_0$ and $1 \leq q \leq \infty$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We write

$$\langle f - g, \varphi\rangle = \langle f - \varrho_N, \varphi\rangle + \langle g - \varrho_N, \varphi\rangle, \quad N \in \mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder's inequality there exists a constant $C > 0$ independent of $N$ such that

$$|\langle g - \varrho_N, \varphi\rangle| \leq C\|g - \varrho_N\|_{\dot{K}^{\alpha,\max(1,p)}_q},$$

which tends to zero as $N \to \infty$. Then, with the help of Substep 1.1 of the proof of Theorem 2.4, we have $g = f$ almost everywhere.

*Case* 2. $\alpha \geq \alpha_0$ and $1 < q \leq \infty$. Let $1 < q_1 < \infty$ be as in Theorem 2.4. From (2.8) and (2.9), we derive in this case, that every $f \in \dot{K}^{\alpha,p}_q A^s_\beta$ is a regular distribution, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $f$ in $L^{q_1}$ and

$$\|f\|_{q_1} \lesssim \|f\|_{\dot{K}^{\alpha,p}_q A^s_\beta}.$$

Indeed, from the embeddings (2.9) and since $f \in B^{\frac{n}{q_1}-\alpha-\frac{n}{q}+s}_{q_1,\beta}$, it follows that $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to a function $h \in L^{q_1}$. Similarly as in Case 1, we conclude that $f = h$ almost everywhere. It remains to prove that $g = f$ almost everywhere. We have

$$\big\|f-g\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \le \big\|f-\varrho_k\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q} + \big\|g-\varrho_k\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q}, \quad k \in \mathbb{N}_0$$

and

$$\big\|f-\varrho_k\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \le \sum_{j=k+1}^{\infty} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q} \le \big\|f\big\|^{\sigma}_{\dot{K}^{\alpha,p}_q A^s_\beta} \sum_{j=k+1}^{\infty} 2^{-js\sigma},$$

where $\sigma = \min(1,p,q)$. Letting $k$ tends to infinity, we get $g = f$ almost everywhere. For the latter case $1 < q_1 < q \le \infty$, we have

$$\dot{K}^{\alpha,p}_q A^s_\beta \hookrightarrow \dot{K}^{0,\max(1,p)}_{q_1} A^{s-\alpha-\frac{n}{q}+\frac{n}{q_1}}_{\beta}.$$

As in Case 1, $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to a function $h \in \dot{K}^{0,\max(1,p)}_{q_1}$. Then again, similarly to the arguments in Case 1 it is easy to check that $f = h$ almost everywhere. Therefore, we can conclude that $g = f$ almost everywhere.

*Case* 3. $q = 1$ and $\alpha \ge 0$.

*Subcase* 3.1. $q = 1$ and $\alpha > 0$. We have

$$\dot{K}^{\alpha,p}_1 B^s_\beta \hookrightarrow L^1,$$

since $s > \alpha$, see Theorem 2.4, Substep 1.3. Now one can continue as in Case 2.

*Subcase* 3.2. $q = 1$ and $\alpha = 0$. Let $\alpha_3$ be a real number such that $\max(-n,-s) < \alpha_3 < 0$. By Theorem 2.1, we get

$$\dot{K}^{0,p}_1 A^s_\beta \hookrightarrow \dot{K}^{\alpha_3,p}_1 A^{s+\alpha_3}_\beta.$$

We have

$$\sum_{k=0}^{\infty} \big\|\mathcal{F}^{-1}\varphi_k * f\big\|_{\dot{K}^{\alpha_3,\max(1,p)}_1} \lesssim \big\|f\big\|_{\dot{K}^{\alpha_3,p}_1 A^{s+\alpha_3}_\beta} \lesssim \big\|f\big\|_{\dot{K}^{0,p}_1 A^s_\beta},$$

since $\alpha_3 + s > 0$. Hence the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $f$ in $\dot{K}^{\alpha_3,\max(1,p)}_1$, see Case 1. As in Case 2, we obtain $g = f$ almost everywhere.

*Case* 4. $0 < q < 1$.

*Subcase* 4.1. $-\frac{n}{q} < \alpha < 0$. From the embedding (2.10) and the fact that $s > \frac{n}{q} - n$, the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converge to $f$ in $\dot{K}^{\alpha,\max(1,p)}_1$. As above we prove that $g = f$ almost everywhere.

*Subcase* 4.2. $\alpha \ge 0$. Recall that

$$\dot{K}^{\alpha,p}_q A^s_\beta \hookrightarrow \dot{K}^{\alpha_4,\max(1,p)}_1 A^{s-\frac{n}{q}+n-\alpha+\alpha_4}_\beta,$$

see Substep 2.2 of the proof of Theorem 2.4. As in Subcase 3.2 the sequence $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $f$ in $\dot{K}^{\alpha_4,\max(1,p)}_1$. By the same arguments as above one can conclude that: $g = f$ almost everywhere.

*Step* 2. In this step we prove that

$$\big\|f\big\|^{**}_{\dot{K}^{\alpha,p}_q F^s_\beta} = \Big\|\Big(\int_0^\infty t^{-s\beta}(d^M_t f)^\beta \frac{dt}{t}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,p}_q} \lesssim \big\|f\big\|_{\dot{K}^{\alpha,p}_q F^s_\beta}, \quad f \in \dot{K}^{\alpha,p}_q F^s_\beta.$$

Thus, we need to prove that

$$\Big\|\Big(\sum_{k=-\infty}^{\infty} 2^{sk\beta}|d^M_{2^{-k}} f|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,p}_q}$$

does not exceed $c\big\|f\big\|_{\dot{K}_q^{\alpha,p}F_\beta^s}$. In order to prove this we additionally consider two Substeps 2.1 and 2.2. The estimate for the space $\dot{K}_q^{\alpha,p}B_\beta^s$ is similar.

*Substep* 2.1. We will estimate

$$\Big\|\Big(\sum_{k=0}^{\infty}2^{sk\beta}|d_{2^{-k}}^M f|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}_q^{\alpha,p}}.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic partition of unity. Obviously we need to estimate

$$\Big\{2^{ks}\sum_{j=0}^{k}d_{2^{-k}}^M(\mathcal{F}^{-1}\varphi_j*f)\Big\}_{k\in\mathbb{N}_0} \tag{2.11}$$

and

$$\Big\{2^{ks}\sum_{j=k+1}^{\infty}d_{2^{-k}}^M(\mathcal{F}^{-1}\varphi_j*f)\Big\}_{k\in\mathbb{N}_0}. \tag{2.12}$$

Recall that

$$d_{2^{-k}}^M(\mathcal{F}^{-1}\varphi_j*f)\lesssim 2^{(j-k)M}(\mathcal{F}^{-1}\varphi_j)^{*,a}f\,(x)$$

if $a>0$, $0\le j\le k, k\in\mathbb{N}_0$ and $x\in\mathbb{R}^n$, see, e.g., [13], where the implicit constant is independent of $j,k$ and $x$. We choose $a>\frac{n}{\min\left(\min(q,\beta),\frac{n}{\alpha+\frac{n}{q}}\right)}$. Since $s<M$, (2.11) in $\ell^\beta$-quasi-norm does not exceed

$$\Big(\sum_{j=0}^{\infty}2^{js\beta}((\mathcal{F}^{-1}\varphi_j)^{*,a}f)^\beta\Big)^{1/\beta}. \tag{2.13}$$

By Theorem 2.3, the $\dot{K}_q^{\alpha,p}$-quasi-norm of (2.13) is bounded by $c\big\|f\big\|_{\dot{K}_q^{\alpha,p}F_\beta^s}$.

Now, we estimate (2.12). We can distinguish two cases as follows:

- *Case* 1. $\min(q,\beta)\le 1$. If $-\frac{n}{q}<\alpha<n(1-\frac{1}{q})$, then $s>\frac{n}{\min(q,\beta)}-n$. We choose

$$\max\left(0,1-\frac{s\min(q,\beta)}{n}\right)<\lambda<\min(q,\beta), \tag{2.14}$$

which is possible because of

$$s>\frac{n}{\min(q,\beta)}-n=\frac{n}{\min(q,\beta)}\left(1-\min(q,\beta)\right).$$

Let $\frac{n}{\min(q,\beta)}<a<\frac{s}{1-\lambda}$. Then $s>a(1-\lambda)$. Now, assume that $\alpha\ge n(1-\frac{1}{q})$. Therefore

$$s>\max\left(\frac{n}{\min(q,\beta)}-n,\frac{n}{q}+\alpha-n\right).$$

If $\min(q,\beta)\le\frac{n}{\frac{n}{q}+\alpha}$, then we choose $\lambda$ as in (2.14). If $\min(q,\beta)>\frac{n}{\frac{n}{q}+\alpha}$, then we choose

$$\max\left(0,1-\frac{s}{\frac{n}{q}+\alpha}\right)<\lambda<\frac{n}{\frac{n}{q}+\alpha}, \tag{2.15}$$

which is possible because of

$$s > \frac{n}{q} + \alpha - n = \left(\frac{n}{q} + \alpha\right)\left(1 - \frac{n}{\frac{n}{q} + \alpha}\right).$$

In this case, we choose $\frac{n}{q} + \alpha < a < \frac{s}{1-\lambda}$. We set

$$J_{2,k}(f) = 2^{ks} \sum_{j=k+1}^{\infty} d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f), \quad k \in \mathbb{N}_0.$$

Recalling the definition of $d^M_{2^{-k}}(\phi_j * f)$, we have

$$\begin{aligned} d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f) &= \int_B \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right| dh \\ &\leq \int_B \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right|^{\lambda} dh \sup_{h\in B} \left|\Delta^M_{2^{-k}h}(\mathcal{F}^{-1}\phi_j * f)\right|^{1-\lambda}. \end{aligned} \tag{2.16}$$

Observe that

$$\left|\mathcal{F}^{-1}\phi_j * f(x + (M-i)2^{-k}h)\right| \leq c2^{(j-k)a}\phi_j^{*,a} f\,(x)\,, \quad |h| \leq 1 \tag{2.17}$$

and

$$\int_B \left|\mathcal{F}^{-1}\phi_j * f(x + (M-i)2^{-k}h)\right|^{\lambda} dh \leq c\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda})(x). \tag{2.18}$$

if $j > k, i \in \{0, ..., M\}$ and $x \in \mathbb{R}^n$. Therefore

$$d^M_{2^{-k}}(\mathcal{F}^{-1}\phi_j * f) \leq c2^{(j-k)a(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda})$$

for any $j > k$, where the positive constant $c$ is independent of $j$ and $k$. Hence

$$J_{2,k}(f) \leq c2^{ks} \sum_{j=k+1}^{\infty} 2^{(j-k)a(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}).$$

Using Lemma 2.3, we obtain that (2.12) in $\ell^{\beta}$-quasi-norm can be estimated from above by

$$\begin{aligned} & c\left(\sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^{(1-\lambda)\beta}(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}))^{\beta}\right)^{1/\beta} \\ \lesssim & \left(\sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^{\beta}\right)^{(1-\lambda)/\beta} \left(\sum_{j=0}^{\infty} 2^{js\beta}(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda}))^{\beta/\lambda}\right)^{\lambda/\beta}. \end{aligned}$$

Applying the $\dot{K}^{\alpha,p}_q$-quasi-norm and using Hölder's inequality we obtain that

$$\Big\| \left(\sum_{j=0}^{\infty} (J_{2,k}(f))^{\beta}\right)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_q}$$

is bounded by

$$c\Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} (\phi_j^{*,a} f)^{\beta} \Big)^{(1-\lambda)/\beta} \Big\|_{\dot{K}^{\alpha(1-\lambda),\frac{p}{1-\lambda}}_{\frac{q}{1-\lambda}}}$$
$$\times \Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} \left(\mathcal{M}(|\mathcal{F}^{-1}\phi_j * f|^{\lambda})\right)^{\beta/\lambda} \Big)^{\lambda/\beta} \Big\|_{\dot{K}^{\alpha\lambda,\frac{p}{\lambda}}_{\frac{q}{\lambda}}}$$
$$\lesssim \Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} (\phi_j^{*,a} f)^{\beta} \Big)^{1/\beta} \Big\|^{1-\lambda}_{\dot{K}^{\alpha,p}_q} \Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\phi_j * f|^{\beta} \Big)^{1/\beta} \Big\|^{\lambda}_{\dot{K}^{\alpha,p}_q}$$
$$\lesssim \|f\|_{\dot{K}^{\alpha,p}_q F^s_\beta},$$

where we have used Lemma 2.4 and Theorem 2.3.

- *Case 2.* $\min(q,\beta) > 1$. Assume that $\alpha \geq n(1-\frac{1}{q})$. Then we choose $\lambda$ as in (2.15) and $\frac{n}{q} + \alpha < a < \frac{s}{1-\lambda}$. If $-\frac{n}{q} < \alpha < n(1-\frac{1}{q})$, then we choose $\lambda = 1$. The desired estimate can be done in the same manner as in Case 1.

*Substep 2.2.* We will estimate

$$\Big\| \Big( \sum_{k=-\infty}^{-1} 2^{sk\beta} |d^M_{2^{-k}} f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}^{\alpha,p}_q}.$$

We employ the same notations as in Subtep 1.1. Define

$$H_{k,2}(f)(x) = \int_B \Big| \sum_{j=0}^{\infty} \Delta^M_{z2^{-k}} (\mathcal{F}^{-1}\varphi_j * f)(x) \Big| dz, \quad k \leq 0, x \in \mathbb{R}^n.$$

As in the estimation of $J_{2,k}$, we obtain that

$$H_{2,k}(f) \lesssim 2^{-ka(1-\lambda)} \sup_{j\in\mathbb{N}_0} \Big( \left(2^{js} (\mathcal{F}^{-1}\varphi_j)^{*,a} f\right)^{1-\lambda} \mathcal{M}\left(2^{js} |\mathcal{F}^{-1}\varphi_j * f|\right)^{\lambda} \Big)$$

and this yields that

$$\Big( \sum_{k=-\infty}^{-1} 2^{sk\beta} |H_{2,k}|^{\beta} \Big)^{1/\beta} \lesssim \sup_{j\in\mathbb{N}_0} \Big( \left(2^{js} (\mathcal{F}^{-1}\varphi_j)^{*,a} f\right)^{1-\lambda} \mathcal{M}\left(2^{js} |\mathcal{F}^{-1}\varphi_j * f|\right)^{\lambda} \Big).$$

By the same arguments as used in Subtep 2.1 we obtain the desired estimate.

*Step 3.* Let $f \in \dot{K}^{\alpha,p}_q A^s_\beta$. We will prove that

$$\|f\|_{\dot{K}^{\alpha,p}_q A^s_\beta} \lesssim \|f\|^*_{\dot{K}^{\alpha,p}_q A^s_\beta}.$$

As the proof for $\dot{K}^{\alpha,p}_q B^s_\beta$ is similar, we only consider $\dot{K}^{\alpha,p}_q F^s_\beta$. Let $\Psi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq \frac{3}{2}$, and in addition radially symmetric. We use an observation made by Nikol'skij [33] (see also [37] and [42, Section 3.3.2]). We put

$$\psi(x) = (-1)^{M+1} \sum_{i=0}^{M-1} (-1)^i C^M_i \Psi(x\,(M-i)).$$

The function $\psi$ satisfies $\psi(x) = 1$ for $|x| \leq \frac{1}{M}$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{2}$. Then, taking $\varphi_0(x) = \psi(x), \varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$ for $j = 2, 3, ...$, we obtain that $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a smooth dyadic partition of unity. This yields that

$$\Big\| \Big( \sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\varphi_j * f|^{\beta} \Big)^{1/\beta} \Big\|_{\dot{K}_q^{\alpha,p}}$$

is a quasi-norm equivalent in $\dot{K}_q^{\alpha,p}F_{\beta}^{s}$. Let us prove that the last expression is bounded by

$$C\big\|f\big\|^{*}_{\dot{K}_q^{\alpha,p}F_{\beta}^{s}}. \tag{2.19}$$

We observe that

$$\mathcal{F}^{-1}\varphi_0 * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Psi(z)\, \Delta_{-z}^{M} f(x) dz + f(x) \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Psi(z)\, dz$$

Moreover, it holds for $x \in \mathbb{R}^n$ and $j = 1, 2, ...$

$$\mathcal{F}^{-1}\varphi_j * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \Delta_{2^{-j}y}^{M} f(x)\, \widetilde{\Psi}(y)\, dy,$$

with $\widetilde{\Psi} = \mathcal{F}^{-1}\Psi - 2^{-n}\mathcal{F}^{-1}\Psi(\cdot/2)$. Now, for $j \in \mathbb{N}_0$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |\Delta_{2^{-j}y}^{M} f(x)||\widetilde{\Psi}(y)| dy \\ &= \int_{|y|\leq 1} |\Delta_{2^{-j}y}^{M} f(x)||\widetilde{\Psi}(y)| dy + \int_{|y|>1} |\Delta_{2^{-j}y}^{M} f(x)||\widetilde{\Psi}(y)| dy. \end{aligned} \tag{2.20}$$

Thus, we need only to estimate the second term of (2.20). We write

$$\begin{aligned} &2^{sj} \int_{|y|>1} |\Delta_{2^{-j}y}^{M} f(x)||\widetilde{\Psi}(y)| dy \\ &= 2^{sj} \sum_{k=0}^{\infty} \int_{2^k<|y|\leq 2^{k+1}} |\Delta_{2^{-j}y}^{M} f(x)||\widetilde{\Psi}(y)| dv \\ &\leq c2^{sj} \sum_{k=0}^{\infty} 2^{nj-Nk} \int_{2^{k-j}<|h|\leq 2^{k-j+1}} |\Delta_{h}^{M} f(x)| dh \end{aligned} \tag{2.21}$$

where $N > 0$ is at our disposal and we have used the properties of the function $\widetilde{\Psi}$, $|\widetilde{\Psi}(x)| \leq c(1+|x|)^{-N}$, for any $x \in \mathbb{R}^n$ and any $N > 0$. Without lost of generality, we may assume $1 \leq \beta \leq \infty$. Now, the right-hand side of (2.21) in $\ell^{\beta}$-norm is bounded by

$$c \sum_{k=0}^{\infty} 2^{-Nk} \Big( \sum_{j=0}^{\infty} 2^{(s+n)j\beta} \Big( \int_{|h|\leq 2^{k-j+1}} |\Delta_h^M f(x)| dh \Big)^{\beta} \Big)^{1/\beta}. \tag{2.22}$$

After a change of variable $j - k - 1 = v$, we estimate (2.22) by

$$c \sum_{k=0}^{\infty} 2^{(s+n-N)k} \Big( \sum_{v=-k-1}^{\infty} 2^{sv\beta} \big( d_{2^{-v}}^{M} f(x) \big)^{\beta} \Big)^{1/\beta} \lesssim \Big( \sum_{v=-\infty}^{\infty} 2^{sv\beta} \big( d_{2^{-v}}^{M} f(x) \big)^{\beta} \Big)^{1/\beta},$$

where we choose $N > n + s$. Taking the $\dot{K}_q^{\alpha,p}$-quasi-norm we obtain the desired estimate (2.19). □

We would like to mention that

$$\|f(\lambda\cdot)\|^{*}_{\dot{K}^{\alpha,p}_{q}B^{s}_{\beta}} \approx \lambda^{-\alpha-\frac{n}{q}}\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}} + \lambda^{s-\alpha-\frac{n}{q}}\Big(\int_0^\infty t^{-s\beta}\big\|d^M_t f\big\|^{\beta}_{\dot{K}^{\alpha,p}_{q}}\frac{dt}{t}\Big)^{\frac{1}{\beta}} \tag{2.23}$$

and

$$\|f(\lambda\cdot)\|^{*}_{\dot{K}^{\alpha,p}_{q}F^{s}_{\beta}} \approx \lambda^{-\alpha-\frac{n}{q}}\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}} + \lambda^{s-\alpha-\frac{n}{q}}\Big\|\Big(\int_0^\infty t^{-s\beta}(d^M_t f)^{\beta}\frac{dt}{t}\Big)^{\frac{1}{\beta}}\Big\|_{\dot{K}^{\alpha,p}_{q}}$$

for any $\lambda > 0, 0 < p \leq \infty, 0 < q \leq \infty, \alpha > -\frac{n}{q}, \max(\sigma_q, \alpha - \alpha_0) < s < M$ $(0 < p, q < \infty$ and $\max(\sigma_{q,\beta}, \alpha - \alpha_0) < s < M$ in the $\dot{K}F$-case) and $M \in \mathbb{N}$.

Let $\varphi^j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x)$ for $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. In view of [48] we have the following equivalent norm of $\dot{K}^{\alpha,p}_{q}$. Let $1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Then

$$\Big\|\Big(\sum_{j=-\infty}^{\infty}\big|\mathcal{F}^{-1}\varphi^j * f\big|^2\Big)^{1/2}\Big\|_{\dot{K}^{\alpha,p}_{q}} \approx \big\|f\big\|_{\dot{K}^{\alpha,p}_{q}}, \tag{2.24}$$

holds for all $f \in \dot{K}^{\alpha,p}_{q}$.

Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. We set

$$\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}\dot{B}^{s}_{\beta}} = \Big(\sum_{j=-\infty}^{\infty}2^{js\beta}\big\|\mathcal{F}^{-1}\varphi^j * f\big\|^{\beta}_{\dot{K}^{\alpha,p}_{q}}\Big)^{1/\beta}$$

and

$$\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}\dot{F}^{s}_{\beta}} = \Big\|\Big(\sum_{j=-\infty}^{\infty}2^{js\beta}\big|\mathcal{F}^{-1}\varphi^j * f\big|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,p}_{q}}.$$

**Proposition 2.1.** *Let* $s > \max(\sigma_q, \alpha - n + \frac{n}{q}), 0 < p, q < \infty, 0 < \beta \leq \infty$ *and* $\alpha > -\frac{n}{q}$.

(i) *Let* $s > \max(\sigma_q, \alpha - n + \frac{n}{q})$ *and* $f \in \dot{K}^{\alpha,p}_{q}B^{s}_{\beta}$. *Then*

$$\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}B^{s}_{\beta}} \approx \big\|f\big\|_{\dot{K}^{\alpha,p}_{q}} + \big\|f\big\|_{\dot{K}^{\alpha,p}_{q}\dot{B}^{s}_{\beta}},$$

(ii) *Let* $s > \max(\sigma_{q,\beta}, \alpha - n + \frac{n}{q})$ *and* $f \in \dot{K}^{\alpha,p}_{q}F^{s}_{\beta}$. *Then*

$$\big\|f\big\|_{\dot{K}^{\alpha,p}_{q}F^{s}_{\beta}} \approx \big\|f\big\|_{\dot{K}^{\alpha,p}_{q}} + \big\|f\big\|_{\dot{K}^{\alpha,p}_{q}\dot{F}^{s}_{\beta}}.$$

*Proof.* As the proof for (i) is similar, we only consider (ii). We use the following Marschall's inequality which is given in [28, Proposition 1.5], see also [14]. Let $A > 0, R \geq 1$. Let $b \in \mathcal{D}(\mathbb{R}^n)$ and a function $g \in C^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\mathcal{F}g \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq AR\} \quad \text{and} \quad \operatorname{supp} b \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}.$$

Then

$$\big|\mathcal{F}^{-1}b * g(x)\big| \leq c(AR)^{\frac{n}{t}-n}\,\|b\|_{\dot{B}^{\frac{n}{t}}_{1,t}}\,\mathcal{M}_t(g)(x)$$

for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where $c$ is independent of $A$, $R$, $x$, $b$, $j$ and $g$. Here $\dot{B}^{\frac{n}{t}}_{1,t}$ denotes the homogeneous Besov spaces. We have

$$\mathcal{F}^{-1}\varphi^j * f = \mathcal{F}^{-1}\varphi^j * \mathcal{F}^{-1}\varphi_0 * f, \quad -j \in \mathbb{N}.$$

Therefore,

$$\left|\mathcal{F}^{-1}\varphi^j * f(x)\right| \le c\left\|\varphi^j\right\|_{\dot{B}^{\frac{n}{t}}_{1,t}} \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f)(x) \le c2^{j(n-\frac{n}{t})}\mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f)(x), \quad x \in \mathbb{R}^n,$$

where the positive constant $c$ is independent of $j$ and $x$. If we choose $\frac{n}{s+n} < t < \min(1, q, \beta, \frac{n}{\alpha+\frac{n}{q}})$ then

$$\Big(\sum_{j=-\infty}^{-1} 2^{js\beta}\left|\mathcal{F}^{-1}\varphi^j * f\right|^\beta\Big)^{1/\beta} \lesssim \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 * f).$$

Taking the $\dot{K}^{\alpha,p}_q$-quasi-norm and using (2.1) we obtain

$$\Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{js\beta}\left|\mathcal{F}^{-1}\varphi^j * f\right|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,p}_q} \lesssim \left\|f\right\|_{\dot{K}^{\alpha,p}_q F^s_\beta}.$$

Because of $s > \max(\sigma_q, \alpha - n + \frac{n}{q})$ the series $\sum_{j=0}^{\infty}\mathcal{F}^{-1}\varphi_j * f$ converges not only in $\mathcal{S}'(\mathbb{R}^n)$ but almost everywhere in $\mathbb{R}^n$. Then

$$\left\|f\right\|_{\dot{K}^{\alpha,p}_q} \lesssim \left\|\mathcal{F}^{-1}\varphi_0 * f\right\|_{\dot{K}^{\alpha,p}_q} + \Big(\sum_{j=1}^{\infty}\left\|\mathcal{F}^{-1}\varphi_j * f\right\|^{\min(1,p,q)}_{\dot{K}^{\alpha,p}_q}\Big)^{1/\min(1,p,q)}.$$

Therefore $\left\|f\right\|_{\dot{K}^{\alpha,p}_q} + \left\|f\right\|_{\dot{K}^{\alpha,p}_q \dot{F}^s_\beta}$ can be estimated from above by $c\left\|f\right\|_{\dot{K}^{\alpha,p}_q F^s_\beta}$. Obviously

$$\mathcal{F}^{-1}\varphi_0 * f = \sum_{j=0}^{N}\mathcal{F}^{-1}\varphi_j * f - \sum_{j=1}^{N}\mathcal{F}^{-1}\varphi_j * f = g_N + h_N, \quad N \in \mathbb{N}.$$

We have

$$\left\|h_N\right\|_{\dot{K}^{\alpha,p}_q} \le \Big(\sum_{j=1}^{\infty}\left\|\mathcal{F}^{-1}\varphi^j * f\right\|^{\min(1,p,q)}_{\dot{K}^{\alpha,p}_q}\Big)^{1/\min(1,p,q)}, \quad N \in \mathbb{N}.$$

By Lebesgue's dominated convergence theorem, it follows that $\left\|g_N - f\right\|_{\dot{K}^{\alpha,p}_q}$ tends to zero as $N$ tends to infinity. Therefore $\left\|\mathcal{F}^{-1}\varphi_0 * f\right\|_{\dot{K}^{\alpha,p}_q}$ can be estimated from above by the quasi-norm

$$c\left\|f\right\|_{\dot{K}^{\alpha,p}_q} + c\left\|f\right\|_{\dot{K}^{\alpha,p}_q \dot{F}^s_\beta}.$$

□

**Proposition 2.2.** *Let $s > 0, 1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Let*

$$\mathcal{S}_0(\mathbb{R}^n) = \big\{f \in \mathcal{S}(\mathbb{R}^n) : \operatorname{supp}\mathcal{F}f \subset \mathbb{R}^n \backslash \{0\}\big\}.$$

*Then $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{k}^{\alpha,p}_{q,s}$.*

*Proof.* Let $\varphi_0 = \varphi$ be as above. As in [44] it suffices to approximate $f \in \mathcal{S}(\mathbb{R}^n)$ in $\dot{W}^{\alpha,p}_{q,k}$, $k \in \mathbb{N}$, by functions belonging to $\mathcal{S}_0(\mathbb{R}^n)$. We have

$$|D^\alpha\mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)| = 2^{-jn}|\tilde{\varphi}_j * D^\alpha f| \le 2^{-jn}\mathcal{M}(\tilde{\varphi}_j),$$

where $\tilde{\varphi}_j = \mathcal{F}^{-1}\varphi(2^{-j}\cdot), j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. From (2.1) we obtain

$$\left\|D^\alpha\mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)\right\|_{\dot{K}^{\alpha,p}_q} \le c2^{-jn}\left\|\tilde{\varphi}_j\right\|_{\dot{K}^{\alpha,p}_q} \le c2^{j(\frac{n}{q}-n+\alpha)},$$

where the positive constant $c$ is independent of $j$. Since $\alpha < n - \frac{n}{q}$, we obtain that $f - \mathcal{F}^{-1}(\varphi(2^j\cdot)\mathcal{F}f)$ approximate $f \in \mathcal{S}(\mathbb{R}^n)$ in $\dot{W}^{\alpha,p}_{q,k}$, $k \in \mathbb{N}$. □

**Proposition 2.3.** *Let $s > 0, 1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Let $f \in \dot{k}^{\alpha,p}_{q,s}$. Then*

$$\|f\|_{\dot{k}^{\alpha,p}_{q,s}} \approx \|f\|_{\dot{K}^{\alpha,p}_{q}} + \|(-\Delta)^{\frac{s}{2}} f\|_{\dot{K}^{\alpha,p}_{q}},$$

*where*

$$(-\Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f).$$

*Proof.* Let $f \in \mathcal{S}_0(\mathbb{R}^n)$. We apply Marschall's inequality to $g_j = \mathcal{F}^{-1}(\varphi^j |x|^s \mathcal{F} f), j \in \mathbb{Z}$ and $b_j(x) = 2^{js}|x|^{-s}\psi^j(x), j \in \mathbb{Z}, x \in \mathbb{R}^n$ where

$$\varphi^j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x), \quad \psi^j = \varphi^{j-1} + \varphi^j + \varphi^{j+1}, \quad j \in \mathbb{Z}, x \in \mathbb{R}^n.$$

Then

$$\left|\mathcal{F}^{-1} b_j * g_j(x)\right| \leq c \|b_j\|_{B^n_{1,1}} \mathcal{M}(\mathcal{F}^{-1}(\varphi^j |\xi|^s \mathcal{F} f))(x) \leq c\mathcal{M}(\mathcal{F}^{-1}(\varphi^j |\xi|^s \mathcal{F} f))(x)$$

for any $j \in \mathbb{Z}$ and any $x \in \mathbb{R}^n$, where $c$ is independent of $j$. Let $j \in \mathbb{Z}$. In view of the fact that

$$\mathcal{F}^{-1}\varphi^j * f = \mathcal{F}^{-1}(\varphi^j \mathcal{F} f) = 2^{-js}\mathcal{F}^{-1}(2^{js}|\xi|^{-s}\psi^j |x|^s \varphi^j \mathcal{F} f) = 2^{-js}\mathcal{F}^{-1}(b_j |\xi|^s \varphi^j \mathcal{F} f),$$

by Lemma 2.4 and (2.24) we obtain

$$\begin{aligned} \Big\| \Big( \sum_{j=-\infty}^{\infty} 2^{2sj} \left|\mathcal{F}^{-1}\varphi^j * f\right|^2 \Big)^{1/2} \Big\|_{\dot{K}^{\alpha,p}_{q}} &\lesssim \Big\| \Big( \sum_{j=-\infty}^{\infty} \left|\mathcal{F}^{-1}(\varphi^j |\xi|^s \mathcal{F} f)\right|^2 \Big)^{1/2} \Big\|_{\dot{K}^{\alpha,p}_{q}} \\ &\lesssim \left\|\mathcal{F}^{-1}(|\xi|^s \mathcal{F} f)\right\|_{\dot{K}^{\alpha,p}_{q}}. \end{aligned}$$

The same arguments can be used to prove the opposite inequality in view of the fact that

$$\mathcal{F}^{-1}(\varphi^j |\xi|^s \mathcal{F} f) = \mathcal{F}^{-1}(2^{-js}\psi^j |\xi|^s 2^{js} \varphi^j \mathcal{F} f) = \mathcal{F}^{-1}(b_j 2^{js} \varphi^j \mathcal{F} f), \quad j \in \mathbb{Z}.$$

The rest follows by Propositions 2.1 and 2.2. □

**Definition 5.** Let $0 < u \leq p < \infty$. The Morrey space $M^p_u$ is defined to be the set of all $u$-locally Lebesgue-integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{M^p_u} = \sup |B|^{\frac{1}{p}-\frac{1}{u}} \|f\chi_B\|_u < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

**Remark 4.** The Morrey spaces $M^p_u$ which are quasi-Banach spaces, Banach spaces for $u \geq 1$, were introduced by Morrey to study the regularity of solutions to some PDE's, see [31]. For the theory of Morrey spaces, general Morrey-type spaces, and their applications see the book [1] and survey papers [5, 6, 18, 23, 35, 38, 39].

One can easily see that $M^p_p = L^p$ and that for $0 < u \leq v \leq p < \infty$,

$$M^p_v \hookrightarrow M^p_u.$$

The Sobolev-Morrey spaces are defined as follows.

**Definition 6.** Let $1 < u \leq p < \infty$ and $m = 1, 2, ....$ The Sobolev-Morrey space $M^{m,p}_u$ is defined to be the set of all $u$-locally Lebesgue-integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{M^{m,p}_u} = \|f\|_{M^p_u} + \sum_{|\alpha| \leq m} \|D^\alpha f\|_{M^p_u} < \infty.$$

Let now recall the definition of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of the unity, see Section 2.

**Definition 7.** Let $s \in \mathbb{R}, 0 < u \leq p < \infty$ and $0 < q \leq \infty$.The Besov-Morrey space $\mathcal{N}^s_{p,q,u}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}^s_{p,q,u}} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\varphi_j * f\big\|^q_{M^p_u}\Big)^{1/q} < \infty.$$

In the limiting case $q = \infty$ the usual modification is required.
The Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{p,q,u}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{E}^s_{p,q,u}} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} \big|\mathcal{F}^{-1}\varphi_j * f\big|^q\Big)^{1/q}\Big\|_{M^p_u} < \infty.$$

In the limiting case $q = \infty$ the usual modification is required.

We have

$$\mathcal{E}^m_{p,2,u} = M^{m,p}_u, \quad m \in \mathbb{N}, \quad 1 < u \leq p < \infty$$

and the norms of these spaces are equivalent, see [38, Theorem 3.1]. In particular, we have that

$$\mathcal{E}^0_{p,2,u} = M^p_u, \quad 1 < u \leq p < \infty, \tag{2.25}$$

also in the sense of with equivalent norms, see [29, Proposition 4.1].

**Theorem 2.6.** *Let $s_i \in \mathbb{R}, 0 < q_i \leq \infty, 0 < u_i \leq p_i < \infty, i = 1, 2$. There is a continuous embedding*

$$\mathcal{E}^{s_1}_{p_1,q_1,u_1} \hookrightarrow \mathcal{E}^{s_2}_{p_2,q_2,u_2}$$

*if, and only if,*

$$p_1 \leq p_2 \quad \text{and} \quad \frac{u_2}{p_2} \leq \frac{u_1}{p_1}$$

*and*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{or} \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2} \quad \text{and} \quad p_1 \neq p_2.$$

For the proof of these Sobolev embeddings, see [19, Theorem 3.1].

**Remark 5.** A detailed study of Besov-Morrey and Triebel-Lizorkin-Morrey spaces including their history and properties can be found in [19, 29, 30, 38, 51] and references therein.

## 3 Caffarelli-Kohn-Nirenberg inequalities

As mentioned in the introduction, Caffarelli-Kohn-Nirenberg inequalities play a crucial role to study regularity and integrability for solutions of nonlinear partial differential equations, see [15, 50]. The main aim of this section is to extend these inequalities to more general function spaces. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity and

$$Q_J f = \sum_{j=0}^{J} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}, f \in \mathcal{S}'(\mathbb{R}^n).$$

## 3.1 CKN inequalities in Herz-type Besov and Triebel-Lizorkin spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in the spaces $\dot{K}_q^{\alpha,p}A_\beta^s$. The main results of this section are based on the following proposition.

**Proposition 3.1.** *Let $\alpha_1, \alpha_2 \in \mathbb{R}, \sigma \geq 0, 1 < r, v < \infty, 0 < \tau, u \leq \infty$ and*

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}.$$

(i) *Assume that $1 < u \leq v < \infty$ and $\alpha_2 \geq \alpha_1$. Then for all $f \in \dot{K}_u^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ and all $J \in \mathbb{N}$,*

$$\big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)} \big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}, \tag{3.1}$$

*where*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1 \end{cases}$$

*and the positive constant $c$ is independent of $J$.*

(ii) *Assume that $1 < v \leq u < \infty$ and $\alpha_2 \geq \alpha_1 + \frac{n}{v} - \frac{n}{u}$. Then for all $f \in \dot{K}_u^{\alpha_2,\delta} \cap \mathcal{S}'(\mathbb{R}^n)$ and all $J \in \mathbb{N}$,* (3.1) *holds where the positive constant $c$ is independent of $J$ and*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1 + \frac{n}{v} - \frac{n}{u}, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1 + \frac{n}{v} - \frac{n}{u}. \end{cases}$$

*Proof.* We only give the proof for (i), the case of (ii) being similar. Let $\sigma = \theta m + (1-\theta)0$, $\alpha \in \mathbb{N}^n$ with $0 < \theta < 1$ and $|\alpha| \leq m$. From (2.5) we have

$$\big\|Q_J f\big\|_{\dot{K}_v^{\alpha_1,r}A_2^\sigma} \leq \big\|Q_J f\big\|_{\dot{K}_v^{\alpha_1,r}A_2^0}^{1-\theta} \big\|Q_J f\big\|_{\dot{K}_v^{\alpha_1,r}A_2^m}^{\theta}.$$

Observe that

$$\dot{K}_v^{\alpha_1,r}A_2^\sigma = \dot{k}_{v,\sigma}^{\alpha_1,r}, \quad \dot{K}_v^{\alpha_1,r}A_2^m = \dot{W}_{v,m}^{\alpha_1,r}, \quad \text{and} \quad \dot{K}_v^{\alpha_1,r}A_2^0 = \dot{K}_v^{\alpha_1,r},$$

see (2.2), (2.3) and (2.4). It follows that

$$\big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq \big\|Q_J f\big\|_{\dot{K}_v^{\alpha_1,r}}^{1-\theta} \big\|Q_J f\big\|_{\dot{W}_{v,m}^{\alpha_1,r}}^{\theta},$$

where the positive constant $c$ is independent of $J$. Observe that

$$Q_J f = 2^{Jn}\mathcal{F}^{-1}\varphi_0(2^J \cdot) * f.$$

Therefore,

$$D^\alpha(Q_J f) = 2^{J(|\alpha|+n)}\omega_J * f = 2^{J|\alpha|}\tilde{Q}_J f, \quad |\alpha| \leq m$$

with $\omega_J(x) = D^\alpha(\mathcal{F}^{-1}\varphi_0)(2^J x)$, $x \in \mathbb{R}^n$. Recall that

$$|\tilde{Q}_J f| \lesssim \mathcal{M}(f).$$

Applying Lemma 2.1 and estimate (2.1), we obtain

$$\begin{aligned} \big\|D^\alpha(Q_J f)\big\|_{\dot{K}_v^{\alpha_1,r}} &\leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+|\alpha|)} \big\|\tilde{Q}_J f\big\|_{\dot{K}_u^{\alpha_2,\delta}} \\ &\leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+m)} \big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}} \end{aligned}$$

for any $|\alpha| \leq m$. □

**Remark 6.** With $\alpha_1 = \alpha_2 = 0$ and $r = v$ estimate (3.1) can be rewritten as

$$\begin{aligned}\big\|Q_J f\big\|_{H_v^\sigma} &\leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\sigma)}\big\|f\big\|_{\dot{K}_u^{0,v}} \\ &\leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\sigma)}\big\|f\big\|_u,\end{aligned}$$

where the second estimate follows by the embedding $L^u \hookrightarrow \dot{K}_u^{0,v}$, for $1 < u \leq v < \infty$, which has been proved by Triebel in [44, Proposition 4.5].

Now we are in position to state the main results of this section.

**Theorem 3.1.** *Let* $0 < p, \tau, \beta, \varrho < \infty$, $1 < r, v, u < \infty, \sigma \geq 0$,

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p}, \quad v \geq \max(p, u), \tag{3.2}$$

$$s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0 \tag{3.3}$$

*and*

$$\sigma - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta\Big(s - \frac{n}{p}\Big) + \alpha_1 - \big((1-\theta)\alpha_2 + \theta\alpha_3\big), \quad 0 < \theta < 1. \tag{3.4}$$

*Assume that* $s > \sigma_{p,\beta}$ *in the* $\dot{K}F$*-case.*

(i) *Let* $\alpha_1 \leq \alpha_2 \leq \alpha_3$*. There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\delta} \cap \dot{K}_p^{\alpha_3,\delta_1} B_\beta^s$,

$$\big\|f\big\|_{\dot{K}_v^{\alpha_1,r}\dot{F}_2^\sigma} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\delta_1}\dot{B}_\beta^s}^{\theta} \tag{3.5}$$

*with*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1. \end{cases} \quad \text{and} \quad \delta_1 = \begin{cases} r, & \text{if} \quad \alpha_3 = \alpha_1, \\ \varrho, & \text{if} \quad \alpha_3 > \alpha_1. \end{cases}$$

(ii) *Let* $\frac{1}{r} \leq (1-\theta)\frac{n}{u} + \theta\frac{n}{p}$ *and*

$$\alpha_1 = (1-\theta)\alpha_2 + \theta\alpha_3.$$

*There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,u}F_\infty^0 \cap \dot{K}_p^{\alpha_3,p}A_\infty^s$,

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,u}F_\infty^0}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,p}A_\infty^s}^{\theta}.$$

*Proof. Proof of* (i). For technical reasons, we split the proof into two steps.

*Step* 1. We consider the case $p \leq u$. Let

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Then it follows that

$$\begin{aligned} f &= \sum_{j=0}^{J} \mathcal{F}^{-1}\varphi_j * f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f \\ &= Q_J f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}. \end{aligned}$$

Hence

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq \big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} + \Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}. \tag{3.6}$$

Using Proposition 3.1, it follows that

$$\big\|Q_J f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u}. \tag{3.7}$$

From the embedding

$$\dot{K}^{\alpha_1,r}_v B^{\sigma}_1 \hookrightarrow \dot{k}^{\alpha_1,r}_{v,\sigma}, \tag{3.8}$$

see (2.6), the last norm in (3.6) can be estimated by

$$\begin{aligned} c\sum_{j=J+1}^{\infty} 2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}^{\alpha_1,r}_v} &\lesssim \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1+\sigma)}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} \\ &\lesssim 2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}, \end{aligned} \tag{3.9}$$

by Lemma 2.1, where the last estimate follows by (3.3). By substituting (3.7) and (3.9) into (3.6) we obtain

$$\begin{aligned} \big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} &\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u} + 2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta} \\ &= c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\Big(\big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u} + 2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}\Big), \end{aligned}$$

with some positive constant $c$ independent of $J$. Again from, Lemma 2.1, it follows that

$$\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta \hookrightarrow \dot{K}^{\alpha_2,\delta}_u, \tag{3.10}$$

since $s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>0$. We choose $J\in\mathbb{N}$ such that

$$2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)} \approx \big\|f\big\|_{\dot{K}^{\alpha_2,\delta}_u}\big\|f\big\|^{-1}_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}.$$

We obtain

$$\big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\big\|f\big\|^{\theta}_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}.$$

By (3.3) one has $s>\max\big(\sigma_p,\alpha_3-n+\frac{n}{p}\big)$ and by the fact that $-\frac{n}{u}<\alpha_2<n-\frac{n}{u}$,

$$\sigma > \max\Big(0,\alpha_1+\frac{n}{v}-n\Big)$$

and Theorem 2.5, or Proposition 2.1, can be used. Therefore

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^{\sigma}_2} \lesssim \big\|f\big\|_{\dot{k}^{\alpha_1,r}_{v,\sigma}}$$

and

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^{\sigma}_2} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\Big(\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} + \big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p \dot{B}^s_\beta}\Big)^{\theta}.$$

By replacing $f(\cdot)$ by $f(\lambda\cdot)$ we obtain

$$\big\|f\big\|_{\dot{K}^{\alpha_1,r}_v \dot{F}^{\sigma}_2} \lesssim \big\|f\big\|^{1-\theta}_{\dot{K}^{\alpha_2,\delta}_u}\Big(\lambda^{-s}\big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p} + \big\|f\big\|_{\dot{K}^{\alpha_3,\delta_1}_p \dot{B}^s_\beta}\Big)^{\theta}.$$

Taking $\lambda$ large enough we obtain (3.5) but with $p\le u$.

*Step* 2. We consider the case $u<p$. We choose $\lambda>0$ large enough such that

$$\frac{\big\|f(\lambda\cdot)\big\|_{\dot{K}^{\alpha_2,\delta}_u}}{\big\|f(\lambda\cdot)\big\|_{\dot{K}^{\alpha_3,\delta_1}_p B^s_\beta}} \le 1, \tag{3.11}$$

which is possible because of $s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>0$, see (2.23). As in Step 1, with $f(\lambda\cdot)$ in place of $f(\cdot)$ and (3.11) in place of (3.10), we obtain the desired estimate. The proof of (i) is complete.

*Proof of* (ii). Observe that

$$\frac{n}{v_1}=\frac{n}{v}+\theta s-\sigma=(1-\theta)\frac{n}{u}+\theta\frac{n}{p}$$

and $\frac{\sigma}{s}\leq\theta<1$. Therefore

$$\dot{K}_{v_1}^{\alpha_1,r}F_\infty^{\theta s}\hookrightarrow\dot{k}_{v,\sigma}^{\alpha_1,r},$$

see Theorems 2.1. From (2.2), (2.4) and (2.5), we obtain

$$\big\|f\big\|_{\dot{K}_{v_1}^{\alpha_1,r}F_\infty^{\theta s}}\leq\big\|f\big\|_{\dot{K}_u^{\alpha_2,u}F_\infty^0}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,p}F_\infty^{\theta s}}^{\theta}.$$

We have

$$\dot{K}_p^{\alpha_3,p}A_\beta^s\hookrightarrow\dot{K}_p^{\alpha_3,p}F_\infty^{\theta s}.$$

This completes the proof of (ii). □

**Remark 7.** (i) Taking $\alpha_1=\alpha_2=\alpha_3=0$ and $r=v$ we obtain

$$\begin{aligned}\big\|f\big\|_{\dot{H}_v^\sigma}&\leq c\big\|f\big\|_{\dot{K}_u^{0,v}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{0,v}\dot{B}_\beta^s}^{\theta}\\&\leq c\big\|f\big\|_u^{1-\theta}\big\|f\big\|_{\dot{B}_{p,\beta}^s}^{\theta}\end{aligned}$$

for all $f\in L_u\cap B_{p,\beta}^s$, because of $L_u\hookrightarrow\dot{K}_u^{0,v}$ and $\dot{B}_{p,\beta}^s=\dot{K}_p^{0,p}\dot{B}_{p,\beta}^s\hookrightarrow\dot{K}_p^{0,v}\dot{B}_\beta^s$, which has been proved by Triebel in [44, Theorem 4.6].

(ii) Under the hypothesis of Theorem 3.1/(ii), with $0<p<\frac{n}{s-\frac{\sigma}{\theta}}$ and $\frac{1}{r}\leq(1-\theta)\frac{n}{u}+\theta(\frac{n}{p}-s+\frac{\sigma}{\theta})$, we have

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}\leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,u}F_2^0}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}}A_\kappa^s}^{\theta}$$

for all $f\in\dot{K}_u^{\alpha_2,u}F_2^0\cap\dot{K}_p^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}}A_\kappa^s$, where

$$\kappa=\begin{cases}\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}, & \text{if}\quad A=B,\\ \infty, & \text{if}\quad A=F.\end{cases}$$

Indeed, observe that

$$\frac{n}{v}=(1-\theta)\frac{n}{u}+\theta\Big(\frac{n}{p}-s+\frac{\sigma}{\theta}\Big)=(1-\theta)\frac{n}{u}+\theta\frac{n}{u_1}$$

and $\frac{\sigma}{\theta}-s\leq0$. Therefore, from (2.2), (2.4) and (2.5), we obtain

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}\leq\big\|f\big\|_{\dot{K}_u^{\alpha_2,u}F_2^0}^{1-\theta}\big\|f\big\|_{\dot{K}_{u_1}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}}F_2^{\frac{\sigma}{\theta}}}^{\theta}.$$

The result follows by the embedding

$$\dot{K}_p^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}}A_\kappa^s\hookrightarrow\dot{K}_{u_1}^{\alpha_3,\frac{1}{\frac{n}{p}-s+\frac{\sigma}{\theta}}}F_2^{\frac{\sigma}{\theta}},$$

see Theorems 2.1 and 2.2.

**Theorem 3.2.** *Let* $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}, 0<p,\tau,\beta,\varrho\leq\infty,\ 1<r,v,u<\infty,$

$$s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>-\frac{n}{v}+\alpha_2-\alpha_1+\frac{n}{u}>0$$

*and*

$$\frac{n}{v}=(1-\theta)\frac{n}{u}+\theta\Big(\frac{n}{p}-s\Big)-\alpha_1+(1-\theta)\alpha_2+\theta\alpha_3,\quad 0<\theta<1.$$

*Assume that* $0<p,\tau<\infty$ *and* $s>\sigma_{p,\beta}$ *in the* $\dot{K}F$*-case.*

*Let* $\delta$ *and* $\delta_1$ *be as in Theorem* 3.1/(*i*). *Let* $\alpha_1\leq\alpha_2\leq\alpha_3$, $v\geq\max(u,p)$, $\alpha_1>-\frac{n}{v}$, $-\frac{n}{u}<\alpha_2<n-\frac{n}{u}$ *and* $\alpha_3>-\frac{n}{p}$. *We have*

$$\big\|f\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\delta_1}A_\beta^s}^{\theta},$$

*for all* $f\in\dot{K}_u^{\alpha_2,\delta}\cap\dot{K}_p^{\alpha_3,\delta_1}A_\beta^s$.

*Proof.* We employ the same notation and conventions as in Theorem 3.1. As in Proposition 3.1

$$\big\|Q_Jf\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}},\quad J\in\mathbb{N}.$$

Therefore,

$$\big\|f\big\|_{\dot{K}_v^{\alpha_1,r}}\lesssim 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}+\sum_{j=J+1}^{\infty}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}_v^{\alpha_1,r}},\quad J\in\mathbb{N}.$$

Repeating the same arguments of Theorem 3.1 we obtain the desired estimate. □

**Remark 8.** Under the same hypothesis of Theorem 3.2, with $1<p<\infty, -\frac{n}{p}<\alpha_3<n-\frac{n}{p}, r=v$ and $\beta=2$, we obtain

$$\begin{aligned}\big\||\cdot|^{\alpha_1}f\big\|_v&\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,v}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,v}F_2^s}^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\big\|f\big\|_{\dot{k}_{p,s}^{\alpha_3,v}}^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\big\|f\big\|_{\dot{k}_{p,s}^{\alpha_3,p}}^{\theta}\end{aligned}$$

for all $f\in L^u(\mathbb{R}^n,|\cdot|^{\alpha_2u})\cap\dot{k}_{p,s}^{\alpha_3,p}$, because of

$$\dot{K}_u^{\alpha_2,u}\hookrightarrow\dot{K}_u^{\alpha_2,v}\quad\text{and}\quad\dot{k}_{p,s}^{\alpha_3,p}\hookrightarrow\dot{k}_{p,s}^{\alpha_3,v}.$$

In particular, if $s=m\in\mathbb{N}$, we obtain

$$\begin{aligned}\big\||\cdot|^{\alpha_1}f\big\|_v&\lesssim\big\|f\big\|_{\dot{K}_u^{\alpha_2,v}}^{1-\theta}\Big(\sum_{|\beta|\leq m}\Big\|\frac{\partial^\beta f}{\partial^\beta x}\Big\|_{\dot{K}_p^{\alpha_3,v}}\Big)^{\theta}\\&\lesssim\big\||\cdot|^{\alpha_2}f\big\|_u^{1-\theta}\Big(\sum_{|\beta|\leq m}\Big\||\cdot|^{\alpha_3}\frac{\partial^\beta f}{\partial^\beta x}\Big\|_p\Big)^{\theta}\end{aligned}\tag{3.12}$$

for all $f\in L^u(\mathbb{R}^n,|\cdot|^{\alpha_2u})\cap W_p^m(\mathbb{R}^n,|\cdot|^{\alpha_3u})$. As in [44, Theorem 4.6] replace $f$ in (3.12) by $f(\lambda\cdot)$ with $\lambda>0$, the sum $\sum_{|\beta|\leq m}\cdots$ can be replaced by $\sum_{0<|\beta|\leq m}\cdots$.

By Proposition 2.3 and Theorem 3.1/(i) we obtain the following statement.

**Theorem 3.3.** *Let* $1 < p, \varrho < \infty, 0 < \tau \leq \infty$, $1 < r, v, u < \infty, \sigma \geq 0$, (3.2), (3.3) *and* (3.4) *with* $\alpha_3 < n - \frac{n}{p}$. *Let* $\alpha_1 \leq \alpha_2 \leq \alpha_3$. *There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\delta} \cap \dot{k}_{p,s}^{\alpha_3,\delta_1}$,

$$\big\|(-\Delta^{\frac{\sigma}{2}})f\big\|_{\dot{K}_v^{\alpha_1,r}} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,\delta}}^{1-\theta}\big\|(-\Delta^{\frac{s}{2}})f\big\|_{\dot{K}_p^{\alpha_3,\delta_1}}^{\theta}$$

*with*

$$\delta = \begin{cases} r, & \text{if} \quad \alpha_2 = \alpha_1, \\ \tau, & \text{if} \quad \alpha_2 > \alpha_1. \end{cases} \quad \text{and} \quad \delta_1 = \begin{cases} r, & \text{if} \quad \alpha_3 = \alpha_1, \\ \varrho, & \text{if} \quad \alpha_3 > \alpha_1. \end{cases}$$

Further we study the case when $p \leq v < u$ in Theorem 3.1.

**Theorem 3.4.** *Let* $0 < p, \tau < \infty, 0 < \beta, \kappa \leq \infty, 1 < r, v < \infty, \sigma \geq 0, 1 < u < \infty$,

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p},$$

$$s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0$$

*and*

$$\sigma - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta\Big(s - \frac{n}{p}\Big) + \alpha_1 - ((1-\theta)\alpha_2 + \theta\alpha_3), \quad 0 < \theta < 1.$$

(i) *Let* $p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u}$ *and* $\alpha_3 = \alpha_2$. *There is a constant* $c > 0$ *such that for all* $f \in \dot{K}_u^{\alpha_2,\tau} \cap \dot{K}_p^{\alpha_3,\tau}F_\beta^s$,

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}}^{1-\theta}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\tau}F_\beta^s}^{\theta}. \tag{3.13}$$

(ii) *Let* $p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u}$ *and* $\alpha_3 > \alpha_2$. *There is a constant* $c > 0$ *such that* (3.13) *holds for all* $f \in \dot{K}_u^{\alpha_2,\tau} \cap \dot{K}_p^{\alpha_3,\kappa}F_\beta^s$ *with* $\dot{K}_p^{\alpha_3,\kappa}F_\beta^s$ *in place of* $\dot{K}_p^{\alpha_3,\tau}F_\beta^s$.

*Proof.* Recall that, as in Theorem 3.1, one has the estimate

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq \big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} + \Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}, \quad J \in \mathbb{N}.$$

From Proposition 3.1/(ii),

$$\big\|Q_J f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} \leq c2^{J(\frac{n}{u} - \frac{n}{v} + \alpha_2 - \alpha_1 + \sigma)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}},$$

which is possible since

$$\frac{n}{v} + \alpha_1 - \alpha_2 \leq \frac{n}{u} < \frac{n}{v}.$$

Using again embedding (3.8) and Lemma 2.1, we get

$$\begin{aligned}\Big\|\sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}} &\lesssim \sum_{j=J+1}^{\infty} 2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_v^{\alpha_1,r}} \\ &\lesssim \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_3 - \alpha_1 + \sigma)}\big\|\mathcal{F}^{-1}\varphi_j * f\big\|_{\dot{K}_p^{\alpha_3,\vartheta}},\end{aligned}$$

where

$$\vartheta = \begin{cases} \tau, & \text{if} \quad \alpha_3 = \alpha_2, \\ \kappa, & \text{if} \quad \alpha_3 > \alpha_2. \end{cases}$$

Therefore, $\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}$ can be estimated by

$$
\begin{aligned}
&c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}}+2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\sigma)}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\vartheta}F_\beta^s}\\
&=c2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\Big(\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}}+2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\big\|f\big\|_{\dot{K}_p^{\alpha_3,\vartheta}F_\beta^s}\Big),
\end{aligned}
$$

where the positive constant $c>0$ is independent of $J$. Observe that

$$\dot{K}_p^{\alpha_3,\vartheta}F_\beta^s\hookrightarrow \dot{K}_u^{\alpha_2,\tau},$$

since $s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>0$. We choose $J\in\mathbb{N}$ such that

$$2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)}\approx\big\|f\big\|_{\dot{K}_u^{\alpha_2,\tau}}\big\|f\big\|^{-1}_{\dot{K}_p^{\alpha_3,\vartheta}F_\beta^s},$$

we obtain the desired estimate. □

By combining Theorem 3.2 with Theorem 3.4 we obtain the following statement.

**Theorem 3.5.** *Under the hypothesis of Theorem* 3.4 *with* $\alpha_1>-\frac{n}{v}$ *and* $\sigma=0$, *the estimates of Theorem* 3.4 *hold with* $\dot{K}_v^{\alpha_1,r}$ *replaced by* $\dot{k}_{v,\sigma}^{\alpha_1,r}$.

Finally we study the case of $v\le\min(p,u)$.

**Theorem 3.6.** *Let* $1<r<\infty, 0<p,\beta,\tau\le\infty, 1<v\le\min(p,u), \alpha_2-\alpha_1>\frac{n}{v}-\frac{n}{\max(p,u)}, \alpha_3\ge\alpha_2, \sigma\ge 0$,

$$-\frac{n}{v}<\alpha_1<n-\frac{n}{v},\qquad -\frac{n}{u}<\alpha_2<n-\frac{n}{u},\qquad \alpha_3>-\frac{n}{p}$$

*and*

$$s-\frac{n}{p}+\frac{n}{u}+\alpha_2-\alpha_3>\sigma-\frac{n}{v}+\alpha_2-\alpha_1+\frac{n}{u}>0.$$

*Assume that* $0<p,\tau<\infty$ *and* $s>\sigma_{p,\beta}$ *in the* $\dot{K}F$*-case. There is a constant* $c>0$ *such that for all* $f\in\dot{K}_u^{\alpha_2,\tau}\cap\dot{K}_p^{\alpha_3,\tau}A_\beta^s$,

$$\big\|f\big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}\le c\big\|f\big\|^{1-\theta}_{\dot{K}_u^{\alpha_2,\tau}}\big\|f\big\|^{\theta}_{\dot{K}_p^{\alpha_3,\tau}A_\beta^s}$$

*with*

$$\sigma-\frac{n}{v}=-(1-\theta)\frac{n}{u}+\theta\Big(s-\frac{n}{p}\Big)+\alpha_1-\big((1-\theta)\alpha_2+\theta\alpha_3\big).$$

*Proof.* By similarity, we only consider the case of the spaces $\dot{K}_p^{\alpha_3,\tau}B_\beta^s$. We split the proof into two steps.

*Step 1.* We consider the case $p\le u$. We employ the same notation as in Theorem 3.1. In view of Theorem 3.4 we need only to estimate

$$\Big\|\sum_{j=J+1}^{\infty}\mathcal{F}^{-1}\varphi_j*f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}},\quad J\in\mathbb{N}.$$

Using embedding (3.8) and Lemma 2.2, we obtain

$$
\begin{aligned}
\Big\|\sum_{j=J+1}^{\infty}\mathcal{F}^{-1}\varphi_j*f\Big\|_{\dot{k}_{v,\sigma}^{\alpha_1,r}}&\lesssim\sum_{j=J+1}^{\infty}2^{j\sigma}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}_v^{\alpha_1,r}}\\
&\lesssim\sum_{j=J+1}^{\infty}2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)}\big\|\mathcal{F}^{-1}\varphi_j*f\big\|_{\dot{K}_p^{\alpha_3,\tau}}.
\end{aligned}
$$

which is possible since

$$\frac{n}{v}+\alpha_1-\alpha_2<\frac{n}{p}\le\frac{n}{v}.$$

Repeating the same arguments as in the proof of Theorem 3.1 we obtain the desired estimate.

*Step* 2. We consider the case $u<p$. Applying a combination of the arguments used in the corresponding step of the proof of Theorem 3.1 and those used in the first step above, we arrive at the desired estimate. □

Similarly we obtain the following statement.

**Theorem 3.7.** *Under the hypothesis of Theorem 3.6 with $\sigma=0$, we have*

$$\|f\|_{\dot{K}_v^{\alpha_1,r}}\lesssim\|f\|^{1-\theta}_{\dot{K}_u^{\alpha_2,\tau}}\|f\|^{\theta}_{\dot{K}_p^{\alpha_3,\tau}A^s_\varrho}$$

*for all $f\in\dot{K}_u^{\alpha_2,\tau}\cap\dot{K}_p^{\alpha_2,\tau}A^s_\varrho$.*

**Remark 9.** Under the same hypothesis of Theorems 3.5 and 3.7, with $r=v$, $\sigma=0$, $\tau=\max(u,p)$ and $\beta=2$, we, to a certain extent, improve Caffarelli-Kohn-Nirenberg inequality (1.1).

## 3.2 CKN inequalities in Besov-Morrey and Triebel-Lizorkin-Morrey spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in $\mathcal{E}^s_{p,q,u}$ and $\mathcal{N}^s_{p,q,u}$ spaces. The main results of this section are based on the following statement.

**Lemma 3.1.** *Let $1<u\le p<\infty, 1<s\le q<\infty$ and $R>0$.*

(i) *Assume that $1\le v\le u$. There exists a constant $c>0$ independent of $R$ such that for all $f\in M_v^{\frac{v}{u}p}\cap M_s^q$ with* supp $\mathcal{F}f\subset\{\xi:|\xi|\le R\}$, *we have*

$$\|f\|_{M_u^p}\le cR^{\frac{n}{q}-\frac{vn}{qu}}\|f\|^{1-\frac{v}{u}}_{M_s^q}\|f\|^{\frac{v}{u}}_{M_v^{\frac{v}{u}p}}.$$

(ii) *Assume that $\frac{u}{p}\le\frac{s}{q}$ and $q\le p$. There exists a constant $c>0$ independent of $R$ such that for all $f\in M_s^q$ with* supp $\mathcal{F}f\subset\{\xi:|\xi|\le R\}$, *we have*

$$\|f\|_{M_u^p}\le cR^{\frac{n}{q}-\frac{n}{p}}\|f\|_{M_s^q}.$$

*Proof.* We split the proof in two steps.

*Step* 1. We will prove (i). Let $B$ be a ball in $\mathbb{R}^n$. Then

$$\big\||B|^{\frac{1}{p}-\frac{1}{u}}f\chi_B\big\|^u_u=u\int_0^\infty t^{u-1}|\{x\in B:|f(x)|\,|B|^{\frac{1}{p}-\frac{1}{u}}>t\}|dt<\infty.$$

We have

$$|f(x)|\le cR^{\frac{n}{q}}\|f\|_{M_s^q},\quad x\in\mathbb{R}^n,$$

see [36, Proposition 2.1] where $c>0$ independent of $R$. Let $p_0=\frac{v}{u}$. Clearly

$$\begin{aligned}|f(x)|&=|f(x)|^{p_0}|f(x)|^{1-p_0}\\&\lesssim|f(x)|^{p_0}\big(R^{\frac{n}{q}}\|f\|_{M_s^q}\big)^{1-p_0}\\&=c|f(x)|^{p_0}d^{1-p_0},\end{aligned}$$

which yields that

$$\big\| |B|^{\frac{1}{p}-\frac{1}{u}} f\chi_B\big\|_u^u \leq u\int_0^\infty t^{u-1}|\{x\in B: |f(x)|\,|B|^{\frac{1}{pp_0}-\frac{1}{v}} > cd^{1-\frac{1}{p_0}}t^{\frac{1}{p_0}}\}|dt$$
$$= cud^{u-v}\int_0^\infty \lambda^{v-1}|\{x\in B: |f(x)|\,|B|^{\frac{u}{pv}-\frac{1}{v}} > \lambda\}|d\lambda,$$

after the change the variable $\lambda^{p_0}c^{-p_0}d^{1-p_0}=t$. The last expression is clearly bounded by

$$cd^{u-v}\big\|f\big\|^v_{M_v^{\frac{pv}{u}}} \leq cR^{n\frac{u-v}{q}}\big\|f\big\|^v_{M_v^{\frac{pv}{u}}}\big\|f\big\|^{u-v}_{M_s^q}.$$

*Step* 2. We will prove (ii). If $p=q$, then $u\leq s$ and the estimate follows by Hölder's inequality. Assume that $q<p$ and we choose $v>0$ such that $\max(1,\frac{qu}{p})<v\leq u<\frac{pu}{q}$. By Step 1

$$R^{\frac{n}{q}-\frac{vn}{qu}}\big\|f\big\|^{\frac{v}{u}}_{M_v^{\frac{v}{u}p}} = R^{\frac{n}{q}-\frac{n}{p}}\big\|R^{\frac{nu}{pv}-\frac{n}{q}}f\big\|^{\frac{v}{u}}_{M_v^{\frac{v}{u}p}}.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of the unity. Observe that

$$\mathcal{F}^{-1}\varphi_j * f = 0 \quad \text{if} \quad R<2^{j-1}, \quad j\in\mathbb{N}_0.$$

This observation together with (2.25) yield

$$\big\|R^{\frac{nu}{pv}-\frac{n}{q}}f\big\|_{M_v^{\frac{v}{u}p}} \approx \Big\|\Big(\sum_{j\in\mathbb{N}_0,2^{j-1}\leq R}^\infty R^{\frac{2nu}{pv}-\frac{2n}{q}}\big|\mathcal{F}^{-1}\varphi_j * f\big|^2\Big)^{1/2}\Big\|_{M_v^{\frac{v}{u}p}}$$
$$\lesssim \big\|f\big\|_{\mathcal{E}^{\frac{nu}{pv}-\frac{n}{q}}_{\frac{v}{u}p,2,v}} \lesssim \big\|f\big\|_{M_s^q},$$

which follows by Sobolev embedding, see Theorem 2.6,

$$M_s^q = \mathcal{E}^0_{q,2,s} \hookrightarrow \mathcal{E}^{\frac{nu}{pv}-\frac{n}{q}}_{\frac{v}{u}p,2,v},$$

since

$$-\frac{n}{q} = \frac{nu}{pv}-\frac{n}{q}-\frac{nu}{pv}, \quad q<\frac{vp}{u} \quad \text{and} \quad \frac{u}{p}\leq\frac{s}{q}.$$

□

**Proposition 3.2.** *Let $1<u\leq p<\infty, 1<q<\infty$ and $s>0$.*

(i) *Let $f\in\mathcal{N}^s_{p,q,u}$. Then*

$$\big\|f\big\|_{\mathcal{N}^s_{p,q,u}} \approx \big\|f\big\|_{M_u^p} + \big\|f\big\|_{\dot{\mathcal{N}}^s_{p,q,u}}, \tag{3.14}$$

*where*

$$\big\|f\big\|_{\dot{\mathcal{N}}^s_{p,q,u}} = \Big\|\Big(\sum_{j=-\infty}^\infty 2^{qjs}\big|\mathcal{F}^{-1}\varphi^j * f\big|^q\Big)^{1/q}\Big\|_{M_u^p}.$$

(ii) *Let $f\in\mathcal{E}^s_{p,q,u}$. Then*

$$\big\|f\big\|_{\mathcal{E}^s_{p,q,u}} \approx \big\|f\big\|_{M_u^p} + \big\|f\big\|_{\dot{\mathcal{E}}^s_{p,q,u}} \tag{3.15}$$

*where*

$$\big\|f\big\|_{\dot{\mathcal{E}}^s_{p,q,u}} = \Big\|\Big(\sum_{j=-\infty}^\infty 2^{qjs}\big|\mathcal{F}^{-1}\varphi^j * f\big|^q\Big)^{1/q}\Big\|_{M_u^p}.$$

*Proof.* By similarity, we prove only (ii). We have as in the proof of Proposition 2.1 that

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,u}} \lesssim \|f\|_{\mathcal{E}^s_{p,q,u}}.$$

The only distinction of the proof of Proposition 2.1 is the fact that we use [41, Lemma 2.5]. Since $s > 0$ we observe

$$\|f\|_{M^p_u} \approx \|f\|_{\mathcal{E}^0_{p,2,u}} \lesssim \|f\|_{\mathcal{E}^s_{p,q,u}}.$$

Now we prove the opposite inequality. Obviously $\|\mathcal{F}^{-1}\varphi_0 * f\|_{M^p_u}$ can be estimated from above by $\|f\|_{M^p_u}$. □

**Theorem 3.8.** *Let $1 < u \le p < \infty$ and $1 < v \le q < \infty$. Assume that $\frac{u}{p} \le \frac{v}{q}, q \le p$ and $\sigma \ge 0$. Then for all $f \in M^q_v$ and all $J \in \mathbb{N}$,*

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le c 2^{Jn(\frac{1}{q}-\frac{1}{p})+\sigma} \|f\|_{M^q_v},$$

*where $c$ is a positive constant independent of $f$ and $J$.*

*Proof.* Let $\sigma = \theta m + (1-\theta)0$, $\alpha \in \mathbb{N}^n$ with $0 < \theta < 1$ and $|\alpha| \le m$. We have

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le \|Q_J f\|^{1-\theta}_{\mathcal{E}^0_{p,2,u}} \|Q_J f\|^\theta_{\mathcal{E}^m_{p,2,u}}.$$

Observe that

$$\mathcal{E}^m_{p,2,u} = M^{m,p}_u \quad \text{and} \quad \mathcal{E}^0_{p,2,u} = M^p_u,$$

which yield that

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \le \|Q_J f\|^{1-\theta}_{M^p_u} \|Q_J f\|^\theta_{M^{m,p}_u},$$

where the positive constant $c$ is independent of $J$. Lemma 3.1 yields that

$$\|D^\alpha(Q_J f)\|_{M^p_u} \lesssim 2^{Jn(\frac{1}{q}-\frac{1}{p})+|\alpha|} \|f\|_{M^q_v}.$$

Therefore,

$$\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \lesssim 2^{Jn(\frac{1}{q}-\frac{1}{p})+\sigma} \|f\|_{M^q_v}.$$

□

Now we are in position to state the main result of this section.

**Theorem 3.9.** *Let $1 < u \le p < \infty, 1 < \mu \le \delta < \infty, 1 < \beta < \infty$, $\sigma \ge 0$ and $1 < v \le q < \infty$. Assume that*

$$\frac{u}{p} \le \frac{\mu}{\delta} \le \frac{v}{q}, \quad s > 0 \quad \text{and} \quad p \ge \delta \ge q.$$

*Let*

$$s - \frac{n}{q} > \sigma - \frac{n}{p} \quad \text{and} \quad \sigma - \frac{n}{p} = -(1-\theta)\frac{n}{\delta} + \theta\Big(s - \frac{n}{q}\Big), \quad 0 < \theta < 1.$$

*Then*

$$\|f\|_{\dot{\mathcal{E}}^\sigma_{p,2,u}} \lesssim \|f\|^{1-\theta}_{M^\delta_\mu} \|f\|^\theta_{\dot{\mathcal{N}}^s_{q,\beta,v}}, \quad \sigma > 0 \tag{3.16}$$

*and*

$$\|f\|_{M^p_u} \lesssim \|f\|^{1-\theta}_{M^\delta_\mu} \|f\|^\theta_{\dot{\mathcal{N}}^s_{q,\beta,v}} \tag{3.17}$$

*for all $f \in M^\delta_\mu \cap \mathcal{N}^s_{q,\beta,v}$.*

*Proof.* We have

$$f = Q_J f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f, \quad J \in \mathbb{N}.$$

Hence

$$\|f\|_{\mathcal{E}^{\sigma}_{p,2,u}} \leq \|Q_J f\|_{\mathcal{E}^{\sigma}_{p,2,u}} + \Big\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1}\varphi_j * f \Big\|_{\mathcal{E}^{\sigma}_{p,2,u}}. \tag{3.18}$$

By applying Theorem 3.8, it follows that

$$\|Q_J f\|_{\mathcal{E}^{\sigma}_{p,2,u}} \lesssim 2^{Jn(\frac{1}{\delta}-\frac{1}{p})+\sigma J}\|f\|_{M^{\delta}_{\mu}}.$$

From the embedding $\mathcal{N}^{\sigma}_{p,1,u} \hookrightarrow \mathcal{N}^{\sigma}_{p,\min(2,u),u} \hookrightarrow \mathcal{E}^{\sigma}_{p,2,u}$ and Lemma 3.1 the last term in (3.18) can be estimated by

$$\begin{aligned} c\sum_{j=J+1}^{\infty} 2^{j\sigma}\|\mathcal{F}^{-1}\varphi_j * f\|_{M^p_u} &\lesssim \sum_{j=J+1}^{\infty} 2^{jn(\frac{1}{q}-\frac{1}{p})+j\sigma}\|\mathcal{F}^{-1}\varphi_j * f\|_{M^q_v} \\ &\lesssim 2^{J(\frac{n}{q}-\frac{n}{p}+\sigma-s)}\|f\|_{\mathcal{N}^s_{q,\infty,v}}, \end{aligned}$$

since $s-\frac{n}{q} > \sigma - \frac{n}{p}$. Therefore,

$$\begin{aligned} \|f\|_{\mathcal{E}^{\sigma}_{p,2,u}} &\leq c2^{J(\frac{n}{\delta}-\frac{n}{p})+\sigma J}\|f\|_{M^{\delta}_{\mu}} + 2^{J(\frac{n}{q}-\frac{n}{p}+\sigma-s)}\|f\|_{\mathcal{N}^s_{q,\infty,v}} \\ &= c2^{J(\frac{n}{\delta}-\frac{n}{p})+\sigma J}\left(\|f\|_{M^{\delta}_{\mu}} + 2^{J(\frac{n}{q}-\frac{n}{\delta}-s)}\|f\|_{\mathcal{N}^s_{q,\infty,v}}\right), \end{aligned}$$

where the positive constant $c$ is independent of $J$. We wish to choose $J \in \mathbb{N}$ such that

$$\|f\|_{M^{\delta}_{\mu}} \approx 2^{J(\frac{n}{q}-\frac{n}{\delta}-s)}\|f\|_{\mathcal{N}^s_{q,\infty,v}},$$

which is possible since $\mathcal{N}^s_{q,\infty,v} \hookrightarrow M^{\delta}_{\mu}$. Indeed, from Theorem 2.6 and (2.25), we get

$$\mathcal{N}^s_{q,\infty,v} \hookrightarrow \mathcal{E}^s_{q,\infty,v} \hookrightarrow \mathcal{E}^0_{\delta,2,\mu} = M^{\delta}_{\mu},$$

becuase of $s-\frac{n}{q} > \sigma-\frac{n}{p} \geq -\frac{n}{\delta}$. Thus

$$\|f\|_{\mathcal{E}^{\sigma}_{p,2,u}} \lesssim \|f\|^{1-\theta}_{M^{\delta}_{\mu}}\|f\|^{\theta}_{\mathcal{N}^s_{q,\infty,v}}.$$

Using (3.14) and (3.15) we arrive at the inequality

$$\|f\|_{\dot{\mathcal{E}}^{\sigma}_{p,2,u}} \lesssim \|f\|^{1-\theta}_{M^{\delta}_{\mu}}\left(\|f\|_{M^q_v} + \|f\|_{\dot{\mathcal{N}}^s_{q,\infty,v}}\right)^{\theta}.$$

In this estimate replacing $f(\cdot)$ by $f(\lambda\cdot)$ and using (3.14) we obtain

$$\|f\|_{\dot{\mathcal{E}}^{\sigma}_{p,2,u}} \lesssim \|f\|^{1-\theta}_{M^{\delta}_{\mu}}\left(\lambda^{-s}\|f\|_{M^q_v} + \|f\|_{\dot{\mathcal{N}}^s_{q,\infty,v}}\right)^{\theta}.$$

Taking $\lambda$ sufficiently large we obtain (3.16)-(3.17). □

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