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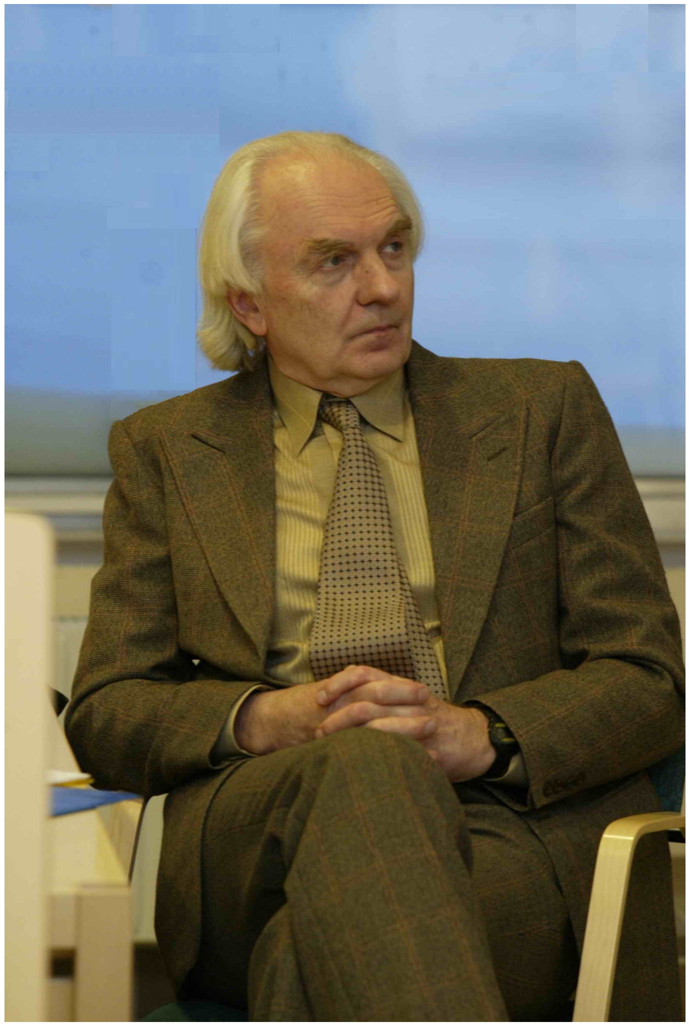
On the 90th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 90th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, leading researcher of the Department of the Theory of Functions of the V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics of the Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \leq p, \theta \leq \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables $x_j, j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 150 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations, in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [18], [12], [15]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [19] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [11], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [21], [22], [23]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [16], [1], [13]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [20] (in Japanese, in 2018 it was translated into English).

A survey of the main facts of the theory of Besov spaces was given in the dedication to the 80th birthday of O.V. Besov [14].

We would that like to add that during the last 10 years Oleg continued active research and published around 25 papers (all of them without co-authors) on various aspects of the theory of function spaces, namely, on the following topics:

Kolmogorov widths of Sobolev classes on an irregular domain (see, for example, [4]),

embedding theorems for weighted Sobolev spaces (see, for example, [5]),

the Sobolev embedding theorem for the limiting exponent (see, for example, [7]),

multiplicative estimates for norms of derivatives on a domain (see, for example, [8]),

interpolation of spaces of functions of positive smoothness on a domain (see, for example, [9]),

embedding theorems for spaces of functions of positive smoothness on irregular domains (see, for example, [10]).

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 30 years, as the head of the seminar first jointly with S.M. Nikol'skii and L.D. Kudryavtsev, then up to the present time on his own.

O.V. Besov participated in numerous research projects supported by grants of several countries, led many of them, and currently is the head of one of them: "Contemporary problems of the theory of function spaces and applications" (project 19-11-00087, Russian Science Foundation).

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 50 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis [6].

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H.G. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan, is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in good physical and mental shape, leads an active life, and continues productive research on the theory of function spaces and lecturing at the Moscow Institute of Physics and Technology.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 90th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov, T.V. Tararykova

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ON ESTIMATES OF NON-INCREASING REARRANGEMENT OF GENERALIZED FRACTIONAL MAXIMAL FUNCTION

N.A. Bokayev, A. Gogatishvili, A.N. Abek

Communicated by V.I. Burenkov

Key words: generalized fractional maximal function, non-increasing rearrangements, generalized Riesz potential.

AMS Mathematics Subject Classification: 42B25, 46E30, 47B38.

Abstract. We give a sharp pointwise estimate of the non-increasing rearrangement of the generalized fractional maximal function $(M_\Phi f)(x)$ via an expression involving the non-increasing rearrangement of f. It is shown that the obtained estimate is more sharp than the inequality which follows from the estimate for the generalized Riesz potential.

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1 Introduction

In this paper, we consider the generalized fractional maximal function

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy,$$

for locally integrable functions f under certain assumptions on the function Φ, where $B(x,r)$ is the ball with the center at the point $x \in \mathbb{R}^n$ and radius $r>0$. When $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$, $n \in \mathbb{N}$ we get the classical fractional maximal function $(M_\alpha f)(x)$. When $\alpha = 0$ we get the Hardy-Littlewood maximal function. Other types of generalized fractional maximal functions were considered in [6], [11-13].

Let $L_0 = L_0(\mathbb{R}^n)$ be the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ and μ_n be the Lebesgue measure on $\mathbb{R}^n$. By L_0^+ we denote the subset of the set L_0 consisting of all non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0,\infty;\downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+. The non-increasing rearrangement f^* is defined by the equality:

$$f^*(t) = \inf\{y \in [0,\infty) : \ \lambda_f(y) \leq t\}, \ \ t \in \mathbb{R}_+ := (0,\infty),$$

where

$$\lambda_f(y) = \mu_n \left\{x \in \mathbb{R}^n : \ |f(x)| > y\right\}, \ \ y \in [0,\infty)$$

is the Lebesgue distribution function. It is known that f^* is a non-negative, non-increasing and right-continuous function on $\mathbb{R}_+$; f^* is equimeasurable with $|f|$, i.e.

$$\mu_1 \left\{t \in \mathbb{R}_+ : \ \ f^*(t) > y\right\} = \mu_n \left\{x \in \mathbb{R}^n : \ \ |f(x)| > y\right\}.$$

Let $f^{\#}:\mathbb{R}^n\to\mathbb{R}^n$ denote the symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right-continuous function (as a function of $r=|x|$, $x\in\mathbb{R}^n$) which is equimeasurable with f. That is

$$f^{\#}(r)=f^*(v_nr^n);\quad f^*(t)=f^{\#}\left(\left(\frac{t}{v_n}\right)^{\frac{1}{n}}\right),\ r,t\in\mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**}:(0,\infty)\to[0,\infty]$ is defined as

$$f^{**}(t)=\frac{1}{t}\int\limits_0^t f^*(\tau)d\tau;\quad t\in\mathbb{R}_+.$$

It is known that f^{**} is a non-increasing function on $\mathbb{R}_+$. For the classical Hardy-Littlewood maximal operator M the rearrangement inequality

$$cf^{**}(t)\le(Mf)^*(t)\le Cf^{**}(t),\quad t\in(0,\infty)$$

holds for some $0<c\le C<\infty$ [2, Chapter 3, Theorem 3.8]. For the classical fractional maximal operator

$$(M_\gamma f)(x):=\sup_{r>0}|B(x,r)|^{\frac{\gamma}{n}-1}\int\limits_{B(x,r)}|f(y)|dy,\quad 0<\gamma<n,$$

in [5] the following estimate was obtained for some $C>0$

$$(M_\gamma f)^*(t)\le C\sup_{t<\tau<\infty}\tau^{\gamma/n}f^{**}(\tau),\quad t\in(0,\infty)$$

for every $f\in L^1_{loc}(\mathbb{R}^n)$. Moreover, this estimate is sharp on the class of all non-negative, radially symmetric non-increasing functions.

Definition 1. A function $f:\mathbb{R}_+\to\mathbb{R}_+$ is called *quasi-decreasing (quasi-increasing)* if there exists $C>1$, such that

$$f(t_2)\le Cf(t_1)\text{ if }t_1<t_2$$
$$\big(f(t_1)\le Cf(t_2)\text{ if }t_1<t_2\big).$$

Throughout this work we will denote by C, C_1, C_2 positive constants, generally speaking, different in different places.

By the notation $f(x)\cong g(x)$ we mean that there are constants $C_1>0$, $C_2>0$ such that

$$C_1f(t)\le g(t)\le C_2f(t),\quad t\in\mathbb{R}_+.$$

2 The generalized fractional maximal function and estimate of its non-increasing rearrangement

We define the following classes of functions $A_n(R)$, $B_n(R)$, $D(R)$.

Definition 2. Let $n\in\mathbb{N}$ and $R\in(0;\infty]$. We say that a function $\Phi:(0;R)\to\mathbb{R}_+$ belongs to the class $A_n(R)$ if:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) the function $\Phi(r)r^n$ is quasi-increasing on $(0,R)$.

For example, $\Phi(t) = t^{\alpha-n} \in A_n(\infty), \ \ 0 < \alpha < n.$

Definition 3. [8] Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. A function $\Phi : (0;R) \to \mathbb{R}_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0;R)$;
(2) there exists $C = C(\Phi, n) > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \leq C\Phi(r)r^n, \quad r \in (0,R). \tag{2.1}$$

For example,

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty) \ (0 < \alpha < n); \quad \Phi(\rho) = \ln\frac{eR}{\rho} \in B_n(R).$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \geq n^{-1}\Phi(r)r^n, \ r \in (0,R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r)r^n, \ r \in (0,R), \tag{2.2}$$

$$\Phi \in B_n(R) \Rightarrow \{\ \Phi(r)r^n \text{ is quasi-increasing, } r \in (0,R)\}. \tag{2.3}$$

It follows from (2.3) that for any $\alpha \in [1;\infty)$ there exists $\beta = \beta(\alpha, C, n) \in [1;\infty)$ (where C is the constant from (2.1)) such that [7]:

$$\left\{\rho, r \in (0;R); \alpha^{-1} \leq \frac{\rho}{r} \leq \alpha\right\} \Rightarrow \beta^{-1} \leq \frac{\Phi(\rho)}{\Phi(r)} \leq \beta. \tag{2.4}$$

Note the well-known equivalence result of N.K. Bari and S.B. Stechkin [1]:

$$(2.1) \Leftrightarrow \exists\gamma \in (0;n) \ \text{ such that } \ \Phi(r)r^\gamma \text{ is quasi-increasing on } (0;R).$$

Definition 4. Let $R \in (0;\infty]$. We say that $\Phi : (0;R) \to \mathbb{R}_+$ belongs to the class $D(R)$ if for some $C = C(\Phi) > 0$

$$\int_0^r \frac{dt}{\Phi(t)t} \leq \frac{C}{\Phi(r)} \ \ r \in (0;R). \tag{2.5}$$

Note that relation (2.5) is equivalent to the inequality:

$$\int_0^{r^n} \frac{ds}{\Phi(s^{1/n})s} \leq \frac{nC}{\Phi(r)}, \quad r \in (0;R). \tag{2.6}$$

For example the function $\Phi(t) = t^{\alpha-n} \in D(\infty)$ $(0 < \alpha < n)$. Indeed,

$$\int_0^r \frac{dt}{\Phi(t)t} = \int_0^r \frac{dt}{t^{\alpha-n+1}} = \frac{1}{n-\alpha}r^{n-\alpha} = \frac{1}{n-\alpha}\frac{1}{\Phi(r)}, \ \ r \in \mathbb{R}_+.$$

Lemma 2.1. *Let $n \in \mathbb{N}$, $R \in (0, \infty]$. Then $B_n(R) \subsetneqq A_n(R)$.*

Proof. Let $\Phi \in B_n(R)$ and $r_1 < r_2$. Then by (2.2) for some C_1, $C_2 > 0$, depending on Φ and n,

$$\Phi(r_1) r_1^n \le C_1 \int\limits_0^{r_1} \Phi(t) t^{n-1} dt \le C_1 \int\limits_0^{r_2} \Phi(t) t^{n-1} dt \le C_2 \Phi(r_2) r_2^n,$$

so the function $\Phi(r) r^n$ is quasi-increasing, hence $\Phi \in A_n(R)$.

The function $\Phi(t) = t^{-n} \ln(1+t)^{\alpha}$, with $\alpha > 0$, belongs to $A_n(R)$ and $\Phi \notin B_n(R)$. Indeed

$$\begin{aligned} \sup_{r>0} \frac{1}{\Phi(r)} \int\limits_0^r \Phi(t) t^{n-1} dt &= \sup_{r>0} \frac{1}{\ln(1+r)^{\alpha}} \int\limits_0^r \ln(1+t)^{\alpha} t^{-1} dt \\ &\ge \sup_{r>0} \frac{1}{\ln(1+r)^{\alpha}} \int\limits_0^r \ln(1+t)^{\alpha} (1+t)^{-1} dt \\ &= \frac{1}{1+\alpha} \sup_{r>0} \ln(1+r) = \infty. \end{aligned}$$

□

Definition 5. Let $\Phi \in A_n(\infty)$. The *generalized fractional maximal function $M_\Phi f$* is defined for a function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)| dy,$$

where $B(x,r)$ is the open ball with the center at the point $x \in \mathbb{R}^n$ and radius $r > 0$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0; n)$ we obtain the classical fractional maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{B(x,r)} |f(y)| dy.$$

Let $E \equiv E(\mathbb{R}^n)$ be a rearrangement invariant space. We introduce the space of generalized fractional maximal functions $M_E^\Phi = M_E^\Phi(\mathbb{R}^n)$ as the set of all functions u, for which there is a function $f \in E(\mathbb{R}^n)$ such that for almost all $x \in \mathbb{R}^n$

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n);\ \ M_\Phi f = u \quad \text{a.e.}\} < \infty.$$

The generalized Riesz potential was considered in [3-4], [7-10] as the convolution operator

$$(I_G f)(x) = (G * f)(x) = \int\limits_{\mathbb{R}^n} G(x-y) f(y) dy, \ \ f \in E(\mathbb{R}^n),$$

where the kernel $G(x)$ satisfies the following condition: for some $\Phi \in B_n(\infty)$

$$G(x) \cong \Phi(|x|), \ \ x \in \mathbb{R}^n, \tag{2.7}$$

where the equivalence constants depend only on Φ and on n. The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha-n}, \ \ \alpha \in (0; n).$$

In the following lemma, we prove that the generalized fractional maximal function $M_\Phi f(x)$ is estimated by the generalized Riesz potential.

Lemma 2.2. *Let $\Phi \in B_n(0,\infty)$ and $G(x) = \Phi(|x|)$, $x \in \mathbb{R}^n$. Then*

$$(M_\Phi f)(x) \le (I_G|f|)(x), \quad x \in \mathbb{R}^n$$

for all $f \in E(\mathbb{R}^n)$.

Proof. Indeed,

$$\begin{aligned}(I_G|f|)(x) &= (G * |f|)(x) = \int\limits_{R^n} \Phi(|x-y|)|f(y)|dy = \sup_{r>0} \int\limits_{B(x,r)} \Phi(|x-y|)|f(y)|dy \\ &\ge \sup_{r>0} \operatorname*{ess\,inf}_{y\in B(x,r)} \Phi(|x-y|) \int\limits_{B(x,r)} |f(y)|dy \\ &= \sup_{r>0} \operatorname*{ess\,inf}_{z\in B(0,r)} \Phi(|z|) \int\limits_{B(x,r)} |f(y)|dy = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy = (M_\Phi f)(x).\end{aligned}$$

□

Lemma 2.3. (Hardy-Littlewood inequality, [2]). *If f and g belong to $L_0(\mathbb{R}^n)$, then*

$$\int\limits_{\mathbb{R}^n} |fg|d\mu_n \le \int\limits_0^\infty f^*(s)g^*(s)ds.$$

Lemma 2.4. *Let $\Phi \in B_n(\infty)$, $f \in L^1_{loc}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$*

$$(M_\Phi f)(x) \le C \sup_{r>0} r\Phi(r^{1/n}) f^{**}(r),$$

where $C > 0$ depends only on Φ and n.

Proof. By using Lemma 2.3 and (2.4) we have

$$\begin{aligned}(M_\Phi f)(x) &= \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy \le \sup_{r>0} \Phi(r) \int\limits_0^{|B(x,r)|} f^*(t)dt \\ &= \sup_{r>0} \Phi(r) \int\limits_0^{v_n r^n} f^*(t)dt = \sup_{s>0} s\Phi\left(\left(\frac{s}{v_n}\right)^{\frac{1}{n}}\right) \frac{1}{s} \int\limits_0^s f^*(t)dt \\ &\le C \sup_{s>0} s\Phi(s^{1/n}) f^{**}(s),\end{aligned}$$

where $C > 0$ depends only on Φ and n. □

Theorem 2.1. *Let $\Phi \in B_n(\infty)$. Then there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \quad t \in (0,\infty), \tag{2.8}$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Theorem 2.2. *Let $\Phi \in A_n(\infty)$. Inequality (2.8) is sharp in the sense that for every $\varphi \in L_0^+(0,\infty;\downarrow)$ there exists a function $f \in L^+(\mathbb{R}^n)$ such that $f^* = \varphi$ almost everywhere on $(0,\infty)$ and*

$$(M_\Phi f)^*(t) \geq C_1 \sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s), \quad t \in (0,\infty), \tag{2.9}$$

where C_1 is a positive constant which depends only on Φ and n.

Remark 1. For $\Phi(r) = r^{\alpha-n}$, $0 < \alpha < n$, hence for the fractional maximal operator M_α, Theorems 2.1 *and* 2.2 *were proved in* [5].

Theorem 2.3. *Let $\Phi \in B_n(\infty)$. Then there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^{**}(t) \leq C \sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s), \quad t \in (0,\infty) \tag{2.10}$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

Remark 2. It is known that the generalized Riesz potential satisfies the O'Neil estimate for non-increasing rearrangement of the convolution

$$(G * f)^{**}(t) \leq C_0 \left(\frac{1}{t}\int_0^t G^*(s)ds \int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau \right),$$

where $C_0 > 0$ depends only on Φ and n. [14, Lemma 1.5].

Then by Lemma 2.2 for some $C > 0$ depending only on Φ and n we get

$$(M_\Phi f)^{**}(t) \leq C \left(\frac{1}{t}\int_0^t G^*(s)ds \int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau \right). \tag{2.11}$$

Assume that $\Phi \in B_n(\infty)\bigcap D(\infty)$ and a function f on $\mathbb{R}^n$ is such that

$$f^*(t) \cong \frac{1}{t\Phi(t^{\frac{1}{n}})}.$$

We show that for such function f the right-hand side of (2.10) is finite while the right-hand side of (2.11) is not. Indeed, by (2.6) we have

$$\begin{aligned}\sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}})f^{**}(s) &\leq C_1 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}})\frac{1}{s}\int_0^s \frac{1}{t\Phi(t^{\frac{1}{n}})}dt \\ &\leq C_2 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}})\frac{1}{s}\frac{1}{\Phi(s^{\frac{1}{n}})} < \infty,\end{aligned}$$

where $C_1, C_2 > 0$ depends only on Φ and n.

For the second term on the right-hand side of inequality (2.11) we get

$$\int_t^\infty G^*(\tau)f^*(\tau)d\tau \geq C_3 \int_t^\infty \Phi\Big(\big(\frac{\tau}{v_n}\big)^{\frac{1}{n}}\Big)\frac{1}{\tau\Phi\Big(\big(\frac{\tau}{v_n}\big)^{\frac{1}{n}}\Big)}d\tau = C_3\int_t^\infty \frac{1}{\tau}d\tau = \infty,$$

where $C_3 > 0$ depends only on Φ and n.

Theorem 2.4. *Let $\Phi \in B_n(\infty) \cap D(\infty)$, then for every $f \in L^1_{loc}(\mathbb{R}^n)$ there exists a positive constant C, depending only on Φ and n, such that*

$$(M_\Phi f)^*(t) \leq C\Big(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^*(\tau)\Big), \quad t \in (0,\infty). \tag{2.12}$$

3 Proofs of the results of Section 2

3.1 Proof of Theorem 2.1

Fix $t \in (0;\infty)$ and let $f \in L^1_{loc}(\mathbb{R}^n)$. We may assume that

$$\sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s) < \infty,$$

otherwise (2.8) holds trivially. Then by Lemma 2.3

$$\int_E |f(x)|dx \le \int_0^t f^*(y)dy < \infty$$

for every set $E \subset \mathbb{R}^n$ of measure at most t. In particular, if we put

$$E = \{x \in \mathbb{R}^n : |f(x)| > f^*(t)\},$$

then $|E| \le t$ since $\lambda_f(f^*(t)) \le t$ ([2], Chapter 2, (1.18)) and so f is integrable over E. We define the functions:

$$g_t(x) = \max\{|f(x)| - f^*(t), 0\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n,$$
$$h_t(x) = \min\{|f(x)|, f^*(t)\} \operatorname{sgn} f(x), \quad x \in \mathbb{R}^n.$$

Then $f = g_t + h_t$ and

$$\begin{aligned} g_t^*(\tau) &= \chi_{(0,t)}(\tau)(f^*(\tau) - f^*(t)), \quad \tau \in (0,\infty), \\ h_t^*(\tau) &= \min\{f^*(\tau), f^*(t)\}, \quad \tau \in (0,\infty). \end{aligned} \tag{3.1}$$

Therefore

$$\|g_t\|_1 = \int_0^\infty g_t^*(\tau)d\tau = \int_0^t \Big(f^*(\tau) - f^*(t)\Big)d\tau \le \int_0^t f^*(\tau)d\tau. \tag{3.2}$$

From inequality (3.2) it follows that

$$(M_\Phi g_t)^*(\tau) \le \Phi(\tau^{1/n})\|g_t\|_1, \quad \tau \in (0;\infty). \tag{3.3}$$

By Lemma 2.4 and by (3.1), we have

$$\begin{aligned} (M_\Phi h_t)^*(\tau) &\le C \sup_{0<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) h_t^{**}(\tau) \\ &= C \max\Big\{ \sup_{0<\tau<t} \tau \cdot \Phi(\tau^{1/n}) f^{**}(t), \sup_{t\le\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau)\Big\} \\ &= C \max\Big\{ t \cdot \Phi(t^{1/n}) f^{**}(t), \sup_{t\le\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau)\Big\} \\ &\le C \sup_{t<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau). \end{aligned} \tag{3.4}$$

Hence

$$\sup_{0<\tau<\infty} (M_\Phi h_t)^*(\tau) \le C \sup_{t<\tau<\infty} \tau \cdot \Phi(\tau^{1/n}) f^{**}(\tau).$$

Using inequality ([2])

$$\big(M_\Phi f\big)^*(t) \le \big(M_\Phi g_t\big)^*\Big(\frac{t}{2}\Big) + \big(M_\Phi h_t\big)^*\Big(\frac{t}{2}\Big)$$

and by (3.4), based on (3.3), (3.2) and (2.3) we get

$$\begin{aligned}
\big(M_\Phi f\big)^*(t) &\le C\left(\Phi\Big((\frac{t}{2})^{1/n}\Big)\|g_t\|_1 + (M_\Phi h_t)^*(\tau)\right)\\
&\le C_1\left(\Phi(t^{1/n})\int\limits_0^t f^*(u)du + \sup_{t<\tau<\infty}\tau\Phi(\tau^{1/n})f^{**}(\tau)\right)\\
&\le C_1\left(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<\tau<\infty}\tau\Phi(\tau^{1/n})f^{**}(\tau)\right)\\
&\le C_2\sup_{t<\tau<\infty}\tau\Phi(\tau^{1/n})f^{**}(\tau).
\end{aligned}$$

□

3.2 Proof of Theorem 2.2

Let $\varphi \in L_0^+(0,\infty;\downarrow)$, we put

$$f(x) = \varphi(v_n|x|^n), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then $f^* = \varphi$ almost everywhere on $(0,\infty)$. For given $y \in \mathbb{R}^n$ we denote

$$B(|y|) = B(0,|y|),$$

for every $x, y \in \mathbb{R}^n$ such that $|y| > |x|$ we have

$$(M_\Phi f)(x) = \sup_{r>0}\Phi(t)\int\limits_{B(x,t)} f(z)dz \ge C_1\Phi(|y|)\int\limits_{B(|y|)} f(z)dz. \tag{3.5}$$

Since the definition of f and spherical coordinates give

$$\int\limits_{B(|y|)} f(z)dz = \int\limits_0^{|y|}\int\limits_{\{|z|=r\}}\varphi(v_n r^n)dvdr = \int\limits_0^{|y|}\varphi(v_n r^n)v_n n r^{n-1}dr = \int\limits_0^{v_n|y|^n}\varphi(\tau)d\tau. \tag{3.6}$$

From (3.5) and (3.6) we have

$$(M_\Phi f)(x) \ge C_1\Phi(|y|)\int\limits_0^{v_n|y|^n} f^*(\tau)d\tau = C_1 H(v_n|y|^n),$$

where $H(t) = \Phi(|t|)\int\limits_0^{v_n|t|^n} f^*(\tau)d\tau$. Consequently,

$$(M_\Phi f)^*(x) \ge C_1 \sup_{\tau > v_n|x|^n} H(\tau),$$

whence (2.9) follows on taking rearrangements. □

3.3 Proof of Theorem 2.3

By using Theorem 2.1 and Lemma 2.1 we get

$$\begin{aligned}
(M_\Phi f)^{**}(t) &= \frac{1}{t}\int_0^t (M_\Phi f)^*(s)ds \le \frac{C}{t}\int_0^t \left(\sup_{s<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\right) ds \\
&\le \frac{C}{t}\int_0^t \left(\sup_{s<\tau<t} \tau\Phi(\tau^{1/n})f^{**}(\tau) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau)\right) ds \\
&= \frac{C}{t}\int_0^t \left(\sup_{s<\tau<t} \Phi(\tau^{1/n})\int_0^\tau f^*(u)du\right) ds + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&\le \frac{C}{t}\int_0^t \Phi(s^{1/n})ds\int_0^t f^*(u)du + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&= Cf^{**}(t)\int_0^{t^{1/n}} \Phi(s)s^{n-1}ds + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \\
&\le Ct\Phi(t^{1/n})f^{**}(t) + C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau) \le 2C\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^{**}(\tau).
\end{aligned}$$

□

3.4 Proof of Theorem 2.4

It is clear that

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(\tau)d\tau = \frac{1}{s}\Big(\int_0^t f^*(\tau)d\tau + \int_t^s f^*(\tau)d\tau\Big)$$

holds for $t < s < \infty$. Then by Theorem 2.1 and taking into account that Φ is non-increasing we have

$$\begin{aligned}
(M_\Phi f)^*(t) &\le C\sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s) \\
&= C\sup_{t<s<\infty} \Phi(s^{1/n})\left(\int_0^t f^*(\tau)d\tau + \int_t^s f^*(\tau)d\tau\right) \\
&\le C\left(\Phi(t^{1/n})\int_0^t f^*(\tau)d\tau + \sup_{t<s<\infty} \Phi(s^{1/n})\int_t^s f^*(\tau)d\tau\right) \\
&\le C\left(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<s<\infty} \Phi(s^{1/n})\sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^*(\tau)\int_t^s \frac{d\tau}{\tau\Phi(\tau^{1/n})}\right) \\
&= C\left(t\Phi(t^{1/n})f^{**}(t) + \sup_{t<\tau<\infty} \tau\Phi(\tau^{1/n})f^*(\tau)\sup_{t<s<\infty} \Phi(s^{1/n})\int_0^s \frac{d\tau}{\tau\Phi(\tau^{1/n})}\right),
\end{aligned}$$

therefore (2.12) follows from (2.6).

□

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