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ITERATED DISCRETE HARDY-TYPE INEQUALITIES

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Key words: discrete inequality, Hardy-type operator, weight sequence, sequence space, characterization.

AMS Mathematics Subject Classification: 26D15, 26D20.

Abstract. In this paper, we discuss new discrete inequalities of Hardy-type involving iterated operators. Under some conditions on weight sequences, we establish necessary and sufficient conditions for the validity of these inequalities.

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1 Introduction

Let $0 < q, p, \theta < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of non-negative numbers, $u = \{u_i\}_{i=1}^{\infty}$ and $w = \{w_i\}_{i=1}^{\infty}$ be sequences of positive numbers, which will be called the weight sequences. We consider the Hardy operator H_φ defined for any $f \in l_1$ by

$$(H_\varphi f)_k := \varphi_k \sum_{i=1}^{k} f_i,$$

where $k \in \mathbb{N}$. Let us denote by $l_{p,u}$ the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}} < \infty, \;\; 1 \le p < \infty.$$

For any $f \in l_{p,u}$ we characterize the following iterated discrete Hardy-type inequality with three weights

$$\left(\sum_{n=1}^{\infty} w_n^\theta \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \le C \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}, \tag{1.1}$$

where C is a positive constant independent of f. The dual discrete version of inequality (1.1) has the form

$$\left(\sum_{n=1}^{\infty} w_n^\theta \left(\sum_{k=n}^{\infty} \left| \varphi_k \sum_{i=k}^{\infty} f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \le C \left(\sum_{i=1}^{\infty} |u_i f_i|^p \right)^{\frac{1}{p}}. \tag{1.2}$$

The continuous analogue of inequality (1.1) can be written as follows

$$\left(\int_0^\infty w^\theta(x) \left(\int_0^x \left| \varphi(t) \int_0^t f(s)\, ds \right|^q dt \right)^{\frac{\theta}{q}} dx \right)^{\frac{1}{\theta}} \le C \left(\int_0^\infty |u(x) f(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1.3}$$

The boundedness of the Hardy-type operator in Morrey-type spaces, weighted Sobolev spaces was studied in many papers (see, [6], [9], [15]). In paper [3], the problem of boundedness of the Hardy operator from a Lebesgue space to a local Morrey-type space has been reduced to the validity of inequality (1.3). The results of paper [3] have aroused the interest to study inequalities of form (1.3). We believe that the relation between p and θ is more important than between p and q because we have found out that inequalities of form (1.3) are easier to characterize for $p \le \theta$ rather than for $\theta < p$, as for the standard Hardy inequalities. Paper [14] has covered all possible relations between p, θ and q for characterizations of inequalities of form (1.3), but the obtained results require some auxiliary function and are not given explicitly. Paper [3], where inequality (1.3) was firstly considered and explicitly characterized, has completely covered the case $p \le \theta$, in sense that q can be any positive number, and partially covered the case $\theta < p$ only for $0 < q < \theta$. In paper [12], discrete Hardy-type inequality (1.1) have been characterized for the same relations between p, θ and q, namely, for the cases $p \le \theta < \infty$, $0 < q$ and $\theta < p < \infty$, $0 < q < \theta$. Here we consider the most difficult case $\theta < p < \infty$ and $0 < \theta < q$ or, equivalently, $0 < \theta < \min\{p, q\} < \infty$, which has no explicit characterizations even in the continuous case.

In the relations between p, θ and q listed above, for the continuous case it is assumed that $p > 1$, since for the interval $0 < p < 1$ inequalities of form (1.3) hold only in the trivial cases. For the discrete case the interval $0 < p < 1$ is not excluded, so in this paper we consider the case $0 < \theta < \min\{p, q\} < \infty$ for both $p > 1$ and $0 < p \le 1$. Paper [11] also contains results for inequality (1.2) for the case $0 < p \le 1$, but when $p \le \min\{q, \theta\} < \infty$. In order to complete the relation $p \le \theta$, we include the case $0 < q < p \le \theta < \infty$, $0 < p \le 1$, as an auxiliary result.

The iterated operator $K^+f(x) = \left(\int\limits_0^x \left|\varphi(t)\int\limits_0^t f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ in inequality (1.3) has the same types of integrals as well as the operator $K^-f(x) = \left(\int\limits_x^\infty \left|\varphi(t)\int\limits_t^\infty f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ in the continuous analogue of dual inequality (1.2). We can also write two inequalities with the iterated operators $T^+f(x) = \left(\int\limits_0^x \left|\varphi(t)\int\limits_t^\infty f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$ and $T^-f(x) = \left(\int\limits_x^\infty \left|\varphi(t)\int\limits_0^t f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}$, which have different types of integrals. In paper [8], the problems of boundedness of the conjugate Hardy operator from a Lebesgue space to a Morrey-type space and boundedness of the Hardy operator from a Lebesgue space to a complementary Morrey-type space have been reduced to the validity of the inequalities with the operators T^+f and T^-f, respectively. The inequalities with the operators T^+f and T^-f have been studied more fully than the inequalities with the operators K^+f and K^-f (see, [2], [4], [5], [10], [13] and [16]). On the contrary, the study of (1.1) and (1.2), which are discrete analogues of the inequalities for the operators K^+f and K^-f, is almost completed in this paper, while the investigation of inequalities for the discrete versions of the operators T^+f and T^-f has only started.

Note that the interest in inequalities with iterated operators has been caused not only by their applicability to Morrey-type spaces shown in [3] and [8], but also by the fact that their characterizations can be applied to obtain characterizations for the bilinear Hardy inequalities (see, [2] and [7]).

The work is organized as follows. Section 2 contains all statements and definitions, which are needed to characterize inequalities (1.1) and (1.2). The main results for $0 < \theta < \min\{p, q\} < \infty$, $p > 1$, are presented in section 3. The main results for $0 < \theta < \min\{p, q\} < \infty$, $0 < p \le 1$, are given in section 4. Section 5 contains the auxiliary result for $0 < q < p \le \theta < \infty$, $0 < p \le 1$.

2 Preliminaries

In the proofs of our main results for the case $0 < q < p \le \theta < \infty$, $0 < p \le 1$, we need the following theorem. This theorem proved in [1, Theorem 1 (iv)] presents characterizations of the following weighted discrete Hardy-type inequality.

Theorem 2.1. *Let $0 < p \le 1$, $p \le q < \infty$. The inequality*

$$\left(\sum_{k=1}^{\infty} v_k^q \Big|\sum_{i=1}^{k} f_i\Big|^q\right)^{\frac{1}{q}} \le C\left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{2.1}$$

holds for some $C > 0$ if and only if $A < \infty$, where

$$A = \sup_{j\ge 1}\left(\sum_{i=j}^{\infty} v_i^q\right)^{\frac{1}{q}} u_j^{-1}.$$

Moreover, $C \approx A$, where C is the best constant in (2.1).

For the proofs we also need the following lemma.

Lemma 2.1. *Let $r > 0$, $1 \le n < N \le \infty$. Then*

$$\sum_{k=n}^{N} a_k \left(\sum_{j=k}^{N} a_j\right)^{r-1} \approx \left(\sum_{i=n}^{N} a_i\right)^{r} \approx \sum_{k=n}^{N} a_k \left(\sum_{j=n}^{k} a_j\right)^{r-1}. \tag{2.2}$$

Convention: The symbol $N \ll M$ means $N \le CM$ with some positive constant C, depending on the parameters p, θ and q. Moreover, the notation $N \approx M$ means $N \ll M \ll N$.

For the estimations we use various classical inequalities such as the Minkowski inequality, the Hölder inequality and the following elementary inequalities.

If $a_i > 0$, $i = 1, 2, ..., k$, then

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \le \sum_{i=1}^{k} a_i^{\alpha}, \quad 0 < \alpha \le 1, \tag{2.3}$$

and

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \ge \sum_{i=1}^{k} a_i^{\alpha}, \quad \alpha \ge 1. \tag{2.4}$$

3 Main results for $0 < \theta < \min\{p, q\} < \infty$, $p > 1$

Theorem 3.1. *Let $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. Then inequality (1.2) holds if and only if $B_1 < \infty$, where*

$$B_1 = \left[\sum_{i=1}^{\infty} u_i^{-p'}\left(\sum_{j=i}^{\infty} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{i} w_n^{\theta}\left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_1$, where C is the best constant in (1.2).

Proof. Necessity. Suppose that inequality (1.2) holds with the best constant $C > 0$. Let us show that $B_1 < \infty$. For an arbitrary $1 \leq r < N < \infty$ we take a test sequence $\widetilde{f}_r = \{\widetilde{f}_{r,i}\}_{i=1}^{\infty}$ such that

$$\widetilde{f}_{r,i} = \begin{cases} 0, & 1 \leq i < r, \quad i > N, \\ u_i^{-p'} \left(\sum\limits_{j=i}^{N} u_j^{-p'}\right)^{\frac{\theta-1}{p-\theta}} \left(\sum\limits_{n=r}^{i} w_n^{\theta} \left(\sum\limits_{s=n}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}, & r \leq i \leq N < \infty. \end{cases}$$

Then

$$\|\widetilde{f}_r\|_{p,u} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_r \cdot u_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=r}^{N} u_i^{-p'} \left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{s=n}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{1}{p}} =: \widetilde{B}^{\frac{1}{p}} < \infty. \quad (3.1)$$

Substituting $\widetilde{f}_r$ in the left-hand side $I = I(f)$ of inequality (1.2), we derive that

$$I(\widetilde{f}) = \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left|\varphi_k \sum_{i=k}^{\infty} \widetilde{f}_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \geq \left(\sum_{n=r}^{N} w_n^{\theta} \left(\sum_{k=n}^{N} \varphi_k^q \left(\sum_{i=k}^{N} \widetilde{f}_i\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}.$$

By applying Lemma 2.1, we obtain

$$I(\widetilde{f}) \gg \left(\sum_{n=r}^{N} w_n^{\theta} \left(\sum_{k=n}^{N} \varphi_k^q \sum_{i=k}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)}\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}.$$

Next, changing the orders of sums and using Lemma 2.1, we get

$$I(\widetilde{f}) \gg \left(\sum_{n=r}^{N} w_n^{\theta} \sum_{i=n}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)}\right.$$

$$\left.\times \sum_{k=n}^{i} \varphi_k^q \left(\sum_{m=i}^{N} \widetilde{f}_m \left(\sum_{s=m}^{N} \widetilde{f}_s\right)^{(q-1)} \sum_{z=n}^{m} \varphi_z^q\right)^{\frac{\theta-q}{q}}\right)^{\frac{1}{\theta}}$$

$$= \left(\sum_{i=r}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(q-1)} \left(\sum_{m=i}^{N} \widetilde{f}_m \left(\sum_{s=m}^{N} \widetilde{f}_s\right)^{(q-1)}\right)^{\frac{\theta-q}{q}} \sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}$$

$$\gg \left(\sum_{i=r}^{N} \widetilde{f}_i \left(\sum_{j=i}^{N} \widetilde{f}_j\right)^{(\theta-1)} \sum_{n=r}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}. \quad (3.2)$$

First we estimate

$$\sum_{j=i}^{N} \widetilde{f}_j = \sum_{j=i}^{N} u_j^{-p'} \left(\sum_{s=j}^{N} u_s^{-p'}\right)^{\frac{\theta-1}{p-\theta}} \left(\sum_{n=r}^{j} w_n^{\theta} \left(\sum_{k=n}^{j} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}$$

$$\gg \left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p-1}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta}\left(\sum_{k=n}^{i}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}. \tag{3.3}$$

Now, we put (3.3) into (3.2), then substitute $\widetilde{f}_r$ and find

$$I(\widetilde{f}) \gg \left(\sum_{i=r}^{N} u_i^{-p'}\left(\sum_{j=i}^{N} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta}\left(\sum_{k=n}^{i}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{1}{\theta}} = \widetilde{B}^{\frac{1}{\theta}}. \tag{3.4}$$

From (3.1), (3.4) and (1.2) it follows that

$$\widetilde{B}^{\frac{p-\theta}{p\theta}} \ll C, \quad \text{for all } \ 1 \le r < N < \infty. \tag{3.5}$$

Since $r \ge 1$ is arbitrary, taking the supremum on both sides of inequality (3.5) with respect to r (C is independent of r) and passing to the limit $N \to \infty$, we get that

$$B_1 \ll C < \infty. \tag{3.6}$$

Sufficiency. Suppose that $B_1 < \infty$. Now, we prove that inequality (1.2) holds. Let $0 \le f \in l_{p,u}$ be such that $\sum\limits_{i=1}^{\infty} f_i < \infty$.

Let

$$k_1 := \sup\{k \in \mathbb{Z} : \sum_{i=1}^{\infty} f_i \le 2^{-k}\},$$

then

$$2^{-k_1-1} < \sum_{i=1}^{\infty} f_i \le 2^{-k_1}.$$

We consider the sequence $\{j_k\}$, where j_k are defined by

$$j_k := \min\{j \ge 1 : \sum_{i=j}^{\infty} f_i \le 2^{-k_1-k+1}\}.$$

We note that

$$j_1 := \min\{j \ge 1 : \sum_{i=j}^{\infty} f_i \le 2^{-k_1}\} = 1.$$

For all $k \ge 1$ it yields that

$$\sum_{i=j_k}^{\infty} f_i \le 2^{-k_1-k+1} < \sum_{i=j_k-1}^{\infty} f_i. \tag{3.7}$$

Therefore, the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k\ge 1} [j_k, j_{k+1}-1]\,.$$

Furthermore,

$$2^{-k_1-m+1} < \sum_{i=j_m-1}^{\infty} f_i = \sum_{i=j_m-1}^{j_{m+1}-1} f_i + \sum_{i=j_{m+1}}^{\infty} f_i$$

$$< \sum_{i=j_m-1}^{j_{m+1}-1} f_i + 2^{-k_1-(m+1)+1}, \quad m \geq 2.$$

$$2^{-k_1-m} < \sum_{i=j_m-1}^{j_{m+1}-1} f_i, \quad m \geq 2.$$

$$2^{-k_1-m+2} < 4 \sum_{i=j_m-1}^{j_{m+1}-1} f_i, \quad m \geq 2.$$

Substituting m by $m+1$, we obtain

$$2^{-k_1-m+1} < 4 \sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i, \quad m \geq 1. \tag{3.8}$$

Hence, taking into account (3.7), we get

$$I^\theta(f) := \sum_{n=1}^{\infty} w_n^\theta \left(\sum_{s=n}^{\infty} \left| \varphi_s \sum_{i=s}^{\infty} f_i \right|^q \right)^{\frac{\theta}{q}}$$

$$\leq \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{m=k}^{\infty} \sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_m}^{\infty} f_i \right)^q \right)^{\frac{\theta}{q}}.$$

Therefore, using (3.7) and (3.8), we have

$$I^\theta(f) \leq 4^\theta \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{m=k}^{\infty} \sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \right)^{\frac{\theta}{q}}.$$

Using inequality (2.3), we get

$$I^\theta(f) \leq 4^\theta \sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \sum_{m=k}^{\infty} \left(\sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta.$$

Next, changing the orders of sums, we have

$$I^\theta(f) \leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=\max\{n,j_m\}}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}$$

$$= 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \left(\sum_{k=1}^{m-1} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_m}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right.$$

$$\left. + \sum_{n=j_m}^{j_{m+1}-1} w_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)$$

$$\leq 4^{\theta}\sum_{m=1}^{\infty}\left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i\right)^{\theta}\sum_{k=1}^{m}\sum_{n=j_k}^{j_{k+1}-1} w_n^{\theta}\left(\sum_{s=n}^{j_{m+1}-1}\varphi_s^q\right)^{\frac{\theta}{q}}.$$

Hence,

$$I^{\theta}(f)\leq 4^{\theta}\sum_{m=1}^{\infty}\left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i\right)^{\theta}\sum_{n=1}^{j_{m+1}-1} w_n^{\theta}\left(\sum_{s=n}^{j_{m+1}-1}\varphi_s^q\right)^{\frac{\theta}{q}}. \tag{3.9}$$

Using the Hölder inequality with powers p and p' in (3.9), we have

$$I^{\theta}(f)\leq 4^{\theta}\sum_{m=1}^{\infty}\left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1}|f_iu_i|^p\right)^{\frac{\theta}{p}}\left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1}u_i^{-p'}\right)^{\frac{\theta}{p'}}$$

$$\times\sum_{n=1}^{j_{m+1}-1} w_n^{\theta}\left(\sum_{s=n}^{j_{m+1}-1}\varphi_s^q\right)^{\frac{\theta}{q}}. \tag{3.10}$$

For the outer sum in (3.10) again using the Hölder inequality with the parameters $\frac{p}{\theta}$ and $\frac{p}{p-\theta}$, we get

$$I^{\theta}(f)\leq 4^{\theta}\left(\sum_{m=1}^{\infty}\sum_{i=j_{m+1}-1}^{j_{m+2}-1}|f_iu_i|^p\right)^{\frac{\theta}{p}}\left(\sum_{m=1}^{\infty}\left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1}u_i^{-p'}\right)^{\frac{\theta(p-1)}{(p-\theta)}}\right.$$

$$\left.\times\left(\sum_{n=1}^{j_{m+1}-1} w_n^{\theta}\left(\sum_{s=n}^{j_{m+1}-1}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{p-\theta}{p}}. \tag{3.11}$$

Now, applying Lemma 2.1 to (3.11), we find that

$$I^{\theta}(f)\ll 2^{\theta(2+\frac{1}{p})}\left(\sum_{i=1}^{\infty}|f_iu_i|^p\right)^{\frac{\theta}{p}}\left(\sum_{m=1}^{\infty}\sum_{i=j_{m+1}-1}^{j_{m+2}-1}u_i^{-p'}\left(\sum_{j=i}^{j_{m+2}-1}u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}\right.$$

$$\left.\times\left(\sum_{n=1}^{i} w_n^{\theta}\left(\sum_{s=n}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{p-\theta}{p}} = 2^{\theta(2+\frac{1}{p})}\left(\sum_{m=1}^{\infty}\left(u_{j_{m+1}-1}^{-p'}\right.\right.$$

$$\times\left(\sum_{j=j_{m+1}-1}^{j_{m+2}-1}u_i^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}\left(\sum_{n=1}^{j_{m+1}-1} w_n^{\theta}\left(\sum_{s=n}^{j_{m+1}-1}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}$$

$$\left.\left.+\sum_{i=j_{m+1}}^{j_{m+2}-1}u_i^{-p'}\left(\sum_{j=i}^{j_{m+2}-1}u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}\left(\sum_{n=1}^{i} w_n^{\theta}\left(\sum_{s=n}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)\right)^{\frac{p-\theta}{p}}\|f\|_{p,u}^{\theta}$$

$$\leq 2^{\theta(2+\frac{1}{\theta})}\left(\left(\sum_{i=1}^{\infty}u_i^{-p'}\left(\sum_{j=i}^{\infty}u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}\left(\sum_{n=1}^{i} w_n^{\theta}\left(\sum_{s=n}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{p-\theta}{p\theta}}\right)^{\theta}$$

$$\times\|f\|_{p,u}^{\theta} \leq 2^{\theta(2+\frac{1}{\theta})}B_1^{\theta}\|f\|_{p,u}^{\theta}.$$

Hence,

$$I(f) \ll B_1\|f\|_{p,u} \tag{3.12}$$

and $C \ll B_1$, where C is the best constant in (1.2). Inequalities (3.6) and (3.12) give that $C \approx B_1$. □

Theorem 3.2. *Let* $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. *Then inequality* (1.1) *holds if and only if* $B_2 < \infty$, *where*

$$B_2 = \left[\sum_{i=1}^{\infty} u_i^{-p'}\left(\sum_{j=1}^{i} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}\left(\sum_{n=i}^{\infty} w_n^{\theta}\left(\sum_{k=i}^{n}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_2$, *where* C *is the best constant in* (1.1).

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1.

4 Main results for $0 < \theta < \min\{p, q\} < \infty$, $0 < p \leq 1$

Theorem 4.1. *Let* $0 < \theta < \min\{p, q\} < \infty$, $0 < p \leq 1$. *Then inequality* (1.2) *holds if and only if* $B_3 < \infty$, *where*

$$B_3 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}}\left(\sum_{n=1}^{i} w_n^{\theta}\left(\sum_{k=n}^{i}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_3$, *where* C *is the best constant in* (1.2).

Proof. Necessity. Suppose that inequality (1.2) holds with the best constant $C > 0$. Let $1 \leq r < N < \infty$. We take a test sequence $\widetilde{f}_r = \{\widetilde{f}_{r,i}\}_{i=1}^{\infty}$ such that $\widetilde{f}_{r,i} = 0$ for $1 \leq i < r$, $i > N$ and $\widetilde{f}_{r,i} = u_i^{-\frac{p}{p-\theta}}\left(\sum\limits_{n=r}^{i} w_n^{\theta}\left(\sum\limits_{s=n}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}$ for $r \leq i \leq N < \infty$.

Then

$$\|\widetilde{f}_r\|_{p,u} = \left(\sum_{i=1}^{\infty}|\widetilde{f}_r \cdot u_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=r}^{N} u_i^{-\frac{p\theta}{p-\theta}}\left(\sum_{n=r}^{i} w_n^{\theta}\left(\sum_{s=n}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right)^{\frac{1}{p}} =: \mathcal{B}^{\frac{1}{p}} < \infty. \tag{4.1}$$

In the same way as in the proof of Theorem 3.1, we substitute $\widetilde{f}_r$ in the left-hand side of inequality (1.2) and obtain inequality (3.2). Now, let us estimate

$$\sum_{j=i}^{N}\widetilde{f}_j \geq u_i^{-\frac{p}{p-\theta}}\left(\sum_{n=r}^{i} w_n^{\theta}\left(\sum_{k=n}^{i}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{p-\theta}}. \tag{4.2}$$

We put (4.2) into (3.2), then we have

$$I(\widetilde{f}) \gg \left(\sum_{i=r}^{N} u_i^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=r}^{i} w_n^{\theta} \left(\sum_{s=n}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{1}{\theta}} = \mathcal{B}^{\frac{1}{\theta}}. \tag{4.3}$$

From (4.1), (4.3) and (1.2), as a result we get

$$\mathcal{B}^{\frac{p-\theta}{p\theta}} \ll C, \text{for all } \ 1 \leq r < N < \infty.$$

Since $r \geq 1$ is arbitrary, passing to the limit $N \to \infty$, we have

$$B_3 \ll C < \infty. \tag{4.4}$$

Sufficiency. We start to prove the sufficient part of Theorem 4.1 in the same way as the sufficient part of Theorem 3.1. Since in this case $0 < p \leq 1$, we can not use the Hölder inequality in (3.9). Therefore, we continue the proof in the following way

$$I^{\theta}(f) \leq 4^{\theta} \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i u_i u_i^{-1} \right)^{p\frac{\theta}{p}} \sum_{n=1}^{j_{m+1}-1} w_n^{\theta} \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

Applying (2.3) with $0 < p \leq 1$, we obtain that

$$I^{\theta}(f) \leq 4^{\theta} \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i u_i|^p \right)^{\frac{\theta}{p}}$$

$$\times \sup_{j_{m+1}-1 \leq k \leq j_{m+2}-1} u_k^{-\theta} \sum_{n=1}^{j_{m+1}-1} w_n^{\theta} \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

Using the Hölder inequality for the outer sum, we get

$$I^{\theta}(f) \leq 2^{\theta(2+\frac{1}{p})} \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{\theta}{p}}$$

$$\times \left(\sum_{m=1}^{\infty} \sum_{k=j_{m+1}-1}^{j_{m+2}-1} u_k^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=1}^{j_{m+1}-1} w_n^{\theta} \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}}$$

$$\leq 2^{\theta(2+\frac{1}{\theta})} \left(\sum_{k=1}^{\infty} u_k^{-\frac{p\theta}{p-\theta}} \left(\sum_{n=1}^{k} w_n^{\theta} \left(\sum_{s=n}^{k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right)^{\frac{p-\theta}{p}} \|f\|_{p,u}^{\theta}.$$

Hence,

$$I^{\theta}(f) \leq 2^{\theta(2+\frac{1}{\theta})} B_3^{\theta} \|f\|_{p,u}^{\theta},$$

so that

$$I(f) \ll B_3 \|f\|_{p,u}. \tag{4.5}$$

Therefore, from inequalities (4.4) and (4.5), we get $C \approx B_3$, where C is the best constant in (1.2). $\square$

Theorem 4.2. *Let $0 < \theta < \min\{p,q\} < \infty$, $0 < p \leq 1$. Then inequality* (1.1) *holds if and only if $B_4 < \infty$, where*

$$B_4 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_4$, where C is the best constant in (1.1).

The proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Remark 1. Theorems 3.1 and 4.1 mean that inequality (1.2) holds for both cases $0 < \theta < q < p < \infty$ and $0 < \theta < p < q < \infty$.

5 Auxiliary result for $0 < q < p \leq \theta < \infty$, $0 < p \leq 1$

Theorem 5.1. *Let $0 < q < p \leq \theta < \infty, 0 < p \leq 1$. Then inequality* (1.1) *holds if and only if $B = \max\{B_5, B_6\} < \infty$, where*

$$B_5 = \sup_{i \geq 1} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_i^{-1},$$

$$B_6 = \sup_{i \geq 1} \left(\sum_{n=i}^{\infty} w_n^{\theta} \right)^{\frac{1}{\theta}} \left(\sum_{k=1}^{i} \varphi_k^q \right)^{\frac{1}{q}} \sup_{j \leq i} u_j^{-1}.$$

Moreover, $C \approx B$, where C is the best constant in (1.1).

Proof. Necessity. Assume that inequality (1.1) holds with the best constant $C > 0$. First, we prove that $B_5 < \infty$. Let $j \geq 1$. We take a test sequence $\widetilde{f}_j = \{\widetilde{f}_{j,i}\}_{i=1}^{\infty}$ such that $\widetilde{f}_{j,i} = u_i^{-1}$ for $i = j$ and $\widetilde{f}_{j,i} = 0$ for $i \neq j$. Then

$$\|\widetilde{f}_j\|_{p,u} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_j \cdot u_i|^p \right)^{\frac{1}{p}} = 1. \tag{5.1}$$

Substituting $\widetilde{f}_j$ in left-hand side of inequality (1.1), we deduce that

$$I(\widetilde{f}) := \left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} \widetilde{f}_{j,i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \left| \varphi_k \sum_{i=1}^{k} \widetilde{f}_{j,i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\geq \left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1}. \tag{5.2}$$

From (5.1), (5.2) and (1.1) it follows that

$$\left(\sum_{n=j}^{\infty} w_n^{\theta} \left(\sum_{k=j}^{n} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_j^{-1} \leq C, \quad \forall j \geq 1.$$

Since $j \geq 1$ is arbitrary, we have

$$B_5 = \sup_{j\geq 1}\left(\sum_{n=j}^{\infty} w_n^{\theta}\left(\sum_{k=j}^{n}\varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} u_j^{-1} \leq C < \infty. \tag{5.3}$$

Now, let us show that $B_6 < \infty$. For $1 < r \leq j < \infty$, we take a test sequence $\widetilde{v}_k = \{\widetilde{v}_{k,r}\}_{r=1}^{\infty}$ such that $\widetilde{v}_{k,r} = u_r^{-1}$ for $r = k$ and $\widetilde{v}_{k,r} = 0$ for $r \neq k$. Then

$$\|\widetilde{v}_r\|_{p,u} = 1. \tag{5.4}$$

Substituting $\widetilde{v}_k$ in the left-hand side of inequality (1.1), we find that

$$I(\widetilde{v}) \geq \left(\sum_{n=j}^{\infty} w_n^{\theta}\left(\sum_{k=1}^{n}\left|\varphi_k\sum_{i=1}^{k}\widetilde{v}_{i,r}\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \geq \left(\sum_{n=j}^{\infty} w_n^{\theta}\left(\sum_{k=1}^{j}\left|\varphi_k\sum_{i=1}^{k}\widetilde{v}_{i,r}\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}$$

$$\geq \left(\sum_{n=j}^{\infty} w_n^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{k=1}^{j}\varphi_k^q\right)^{\frac{1}{q}} u_r^{-1}, \quad \forall r \leq j. \tag{5.5}$$

From (5.4), (5.5) and (1.1), we obtain

$$\left(\sum_{n=j}^{\infty} w_n^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{k=1}^{j}\varphi_k^q\right)^{\frac{1}{q}} u_r^{-1} \leq C, \quad \forall r \leq j.$$

$$\left(\sum_{n=j}^{\infty} w_n^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{k=1}^{j}\varphi_k^q\right)^{\frac{1}{q}} \sup_{r\leq j} u_r^{-1} \leq C, \quad \forall j \geq 1.$$

Therefore,

$$B_6 = \sup_{j\geq 1}\left(\sum_{n=j}^{\infty} w_n^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{k=1}^{j}\varphi_k^q\right)^{\frac{1}{q}} \sup_{r\leq j} u_r^{-1} \leq C < \infty. \tag{5.6}$$

Sufficiency. Let $B < \infty$. Without loss of generality, we assume that $0 \leq f \in l_{p,u}$.

Let $\inf \emptyset = \infty$ and

$$k_{\infty} = \inf\left\{k \in \mathbb{Z} : \sum_{s=1}^{\infty}\left(\varphi_s\sum_{i=1}^{s} f_i\right)^q < 2^{q(k+1)}\right\}.$$

Assume that $k \leq k_{\infty}$ if $k_{\infty} < \infty$ and

$$j_k = \inf\left\{j \geq 1 : \sum_{s=1}^{j}\left(\varphi_s\sum_{i=1}^{s} f_i\right)^q \geq 2^{qk}\right\}.$$

Then

$$\sum_{s=1}^{j_k-1}\left(\varphi_s\sum_{i=1}^{s} f_i\right)^q < 2^{qk} \leq \sum_{s=1}^{j_k}\left(\varphi_s\sum_{i=1}^{s} f_i\right)^q.$$

Therefore, the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k\geq 1}\left[j_k, j_{k+1}-1\right].$$

Since in this case $0 < q < 1$, we have

$$2^{q(k-1)} = \frac{2^{qk} - 2^{q(k-1)}}{2^q - 1} \leq \frac{1}{2^q - 1}\left(\sum_{s=1}^{j_k}\left(\varphi_s \sum_{i=1}^{s} f_i\right)^q\right.$$

$$\left.- \sum_{s=1}^{j_{k-1}-1}\left(\varphi_s \sum_{i=1}^{s} f_i\right)^q\right) \leq \frac{1}{2^q - 1}\left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=1}^{s} f_i\right)^q\right)$$

$$\leq \frac{1}{2^q - 1}\left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i\right)^q + \sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i\right)^q\right).$$

Hence,

$$2^{(k-1)} \leq \frac{2^{\frac{1}{q}-1}}{(2^q - 1)^q}\left(\left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i\right)^q\right)^{\frac{1}{q}} + \left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i\right)^q\right)^{\frac{1}{q}}\right). \tag{5.7}$$

For the left-hand side $I(f)$ of inequality (1.1) we have

$$I(f) = \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=1}^{n}\left(\varphi_s \sum_{i=1}^{s} f_i\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq 4\left(\sum_k 2^{\theta(k-1)} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta\right)^{\frac{1}{\theta}}. \tag{5.8}$$

Combining (5.7) with (5.8), we have

$$I(f) \ll \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=1}^{j_{k-1}} f_i\right)^q\right)^{\frac{1}{q}}\right.\right.$$

$$\left.\left.+\left(\sum_{s=j_{k-1}}^{j_k}\left(\varphi_s \sum_{i=j_{k-1}}^{s} f_i\right)^q\right)^{\frac{1}{q}}\right)^\theta\right)^{\frac{1}{\theta}}.$$

In both cases $\theta > 1$ and $0 < \theta \leq 1$, we get that

$$I(f) \ll \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q\right)^{\frac{\theta}{q}} \left(\sum_{i=1}^{j_{k-1}} f_i\right)^\theta\right)^{\frac{1}{\theta}}$$

$$+\left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \left(\sum_{i=j_{k-1}}^{s} f_i\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} = I_1 + I_2. \tag{5.9}$$

Let us estimate I_1

$$I_1 = \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{j} f_i\right)^\theta \mu(j)\right)^{\frac{1}{\theta}}, \tag{5.10}$$

where

$$\mu(j) = \sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \delta(j - j_{k-1})$$

and $\delta(\cdot)$ is the Dirac delta-function. By Theorem A from (5.10) we have

$$I_1 \le \left\{ \sup_{r\ge1} \left(\sum_{j=r}^{\infty} \mu(j) \right)^{\frac{1}{\theta}} u_r^{-1} \right\} \|f\|_{p,u}. \tag{5.11}$$

Since

$$\sum_{j=r}^{\infty} \mu(j) = \sum_{j_{k-1}\ge r} \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \le \sum_{n=r}^{\infty} w_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}},$$

we have

$$\sup_{r\ge1} \left(\sum_{n=r}^{\infty} w_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_r^{-1} \ll B_5. \tag{5.12}$$

From (5.11) and (5.12) we obtain

$$I_1 \le B_5 \|f\|_{p,u}. \tag{5.13}$$

Let us estimate I_2:

$$I_2 \le \left(\sum_k \sum_{n=j_k}^{j_{k+1}-1} w_n^\theta \left(\sum_{s=j_{k-1}}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{k-1}}^{j_k} f_i \right)^{\theta} \right)^{\frac{1}{\theta}}$$

$$\le \left(\sum_k \left(\sum_{i=j_{k-1}}^{j_k} f_i u_i u_i^{-1} \right)^{p\frac{\theta}{p}} \sum_{n=j_k}^{\infty} w_n^\theta \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}.$$

Using the condition (2.3), we get

$$I_2 \ll \left(\sum_k \left(\sum_{i=j_{k-1}}^{j_k} |f_i u_i|^p \right)^{\frac{\theta}{p}} \sup_{i\le j_k} u_i^{-\theta} \sum_{n=j_k}^{\infty} w_n^\theta \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\le \left(\sum_k \left(\sum_{i=j_{k-1}}^{j_k} |f_i u_i|^p \right)^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} \sup_k \left(\sum_{n=j_k}^{\infty} w_n^\theta \right)^{\frac{1}{\theta}} \left(\sum_{s=1}^{j_k} \varphi_s^q \right)^{\frac{1}{q}} \sup_{i\le j_k} u_i^{-1}.$$

Therefore, by applying (2.4) with $\alpha = \frac{\theta}{p}$, we obtain that

$$I_2 \ll \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{1}{p}} \sup_{r\ge1} \left(\sum_{n=r}^{\infty} w_n^\theta \right)^{\frac{1}{\theta}} \left(\sum_{s=1}^{r} \varphi_s^q \right)^{\frac{1}{q}} \sup_{i\le r} u_i^{-1},$$

so that

$$I_2 \le B_6 \|f\|_{p,u}. \tag{5.14}$$

From (5.9), (5.13) and (5.14) we have

$$I(f) \ll \max\{B_5, B_6\}\|f\|_{p,u}. \tag{5.15}$$

Therefore, from inequality (5.15), we get $C \ll B$. The latter together with (5.6) gives that $C \approx B$, where C is the best constant in (1.1). □

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