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ON A LINEAR INVERSE POTENTIAL PROBLEM WITH APPROXIMATE DATA ON THE POTENTIAL FIELD ON AN APPROXIMATELY GIVEN SURFACE

E.B. Laneev, E.Yu. Ponomarenko

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Key words: ill-posed problem, inverse problem for the potential, Sretenskiy class of bodies, method of Tikhonov regularization.

AMS Mathematics Subject Classification: 35R25, 35R30.

Abstract. An approximate solution of the linear inverse problem for the Newtonian potential for bodies of constant thickness is constructed. The solution is stable with respect to the error in the data on the potential field given on an inaccurately known surface. The problem is reduced to an integral equation of the first kind, the proof of the stability of the solution is based on the Tikhonov regularization method.

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1 Introduction

The problem considered here is a linear version of the inverse potential problem, considered in [8]. The paper provides a solution to the problem of restoring the shape of the Newtonian potential density carrier for bodies of constant thickness belonging to the Sretensky class, defined in [9], which ensures the uniqueness of the solution of the inverse potential problem. In [9], the uniqueness of the inverse potential problem is proved for bounded homogeneous bodies having a common secant plane, such that every line perpendicular to it intersects the body at no more than two points lying on different sides of this plane. The problem is formulated in the framework of the odd-periodic model [4], which allows us to obtain a solution in the form of a Fourier series, which is essential for the application of numerical methods for solving the problem. The error of the periodic model with respect to the non-periodic one is studied in [5]. In the problem considered in this paper, information about the potential is given in the form of a potential field on a surface of a general form. Both the field and the surface are given approximately. The idea of the method in [6] is the basis for constructing a solution to the problem. The problem in this case, including for bounded bodies of constant thickness with variable density, is reduced to a linear integral equation of the first kind, the approximate solution of which, stable with respect to the error in data on the potential and the surface, is constructed on the basis of the Tikhonov regularization method [10], [11]. As an approximate solution, we consider the extremal of the Tikhonov functional, obtained as a solution of the Euler equation for this functional. The approximate solution is obtained in the form of a Fourier series with a regularizing factor. The convergence theorem of the approximate solution to the exact one is proved. The linear problem of reconstructing the distribution density function of sources with an infinitely thin carrier in the model of a heat-conducting body with convective heat exchange at the boundary, solved in [1], is closely related to the problem considered here.

2 Problem statement

In an infinite cylinder of rectangular cross-section

$$D^\infty = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < \infty\} \subset \mathbb{R}^3 \tag{2.1}$$

we consider the following model for the Newtonian potential

$$\begin{aligned} &\Delta v(M) = -4\pi\rho(M), \quad M \in D^\infty, \\ &v|_{x=0,l_x} = 0,\ v|_{y=0,l_y} = 0, \\ &v \to 0 \ \text{ when } \ z \to \pm\infty. \end{aligned} \tag{2.2}$$

We assume that the support of the density ρ is located in the domain $z > H > 0$ in the cylinder D^∞.

Let $\varphi(M,P)$ be the source function of problem (2.2) in the domain D^∞ of form (2.1). The function $\varphi(M,P)$ can be obtained as a series

$$\varphi(M,P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_P - z_M|}}{k_{nm}} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}, \tag{2.3}$$

where

$$k_{nm} = \sqrt{\left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2}.$$

If $z_P > H$, series (2.3) converges uniformly with respect to the variable M in the domain

$$D(-\infty, H-\varepsilon) = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < H - \varepsilon\},\ \varepsilon > 0. \tag{2.4}$$

In the domain of $D(-\infty, H-\varepsilon)$ the solution of problem (2.2) can be represented as

$$v(M) = 4\pi \int\limits_{\mathrm{supp}\varphi} \rho(P)\varphi(M,P)dV_P = \frac{8\pi}{l_x l_y} \int\limits_{\mathrm{supp}\varphi} dV_P \rho(P) \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_P - z_M)}}{k_{nm}} \times \sin(\frac{\pi n x_P}{l_x}) \sin(\frac{\pi m y_P}{l_y}) \sin(\frac{\pi n x_M}{l_x}) \sin(\frac{\pi m y_M}{l_y}). \tag{2.5}$$

It can be shown [4] that such a potential corresponds to a Newtonian potential with an odd-periodic source distribution function ρ in $\mathbb{R}^3$.

In the domain of $D(-\infty, H-\varepsilon)$ the field of potential (2.5) has the form

$$\begin{aligned} \mathbf{E}(M) = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z = -\nabla v(M) = -\frac{8\pi}{l_x l_y} \int\limits_{\mathrm{supp}\varphi} dV_P \rho(P) \\ \times \sum_{n,m=1}^{\infty} e^{-k_{nm}(z_P - z_M)} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y} \Big(\mathbf{i}\frac{\pi n}{l_x k_{nm}} \cos\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \\ + \mathbf{j}\frac{\pi m}{l_y k_{nm}} \sin\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y} + \mathbf{k} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \Big). \end{aligned} \tag{2.6}$$

Thus, within the framework of model (2.2), if the density ρ is given, then the potential of density ρ and the potential field can be calculated using formulas (2.5) and (2.6), respectively.

Let us formulate the inverse problem. We assume that the source density ρ in problem (2.2) corresponds to a body of constant thickness h, located on the plane $z = H$:

$$\rho(x,y,z) = \sigma(x,y)\theta(z-H)\theta(H+h-z), \tag{2.7}$$

where $\theta(z)$ is the Heaviside function. According to (2.7), we consider the source distribution density functions as constants along the z axis and variables in the (x, y) plane inside the density carrier.

THE INVERSE PROBLEM. Let in the framework of model (2.2) on the surface

$$S = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = F(x, y) < H\}, \quad F \in C^2(\Pi), \tag{2.8}$$

$$\Pi = \{(x, y) : 0 < x < l_x, 0 < y < l_y, \}\,. \tag{2.9}$$

the field $\mathbf{E}$ of form (2.6) of potential (2.5) be given as a vector function $\mathbf{E}^0$:

$$\mathbf{E}|_S = \mathbf{E}^0, \tag{2.10}$$

and the density ρ of form (2.7) is unknown. Let us set the problem of restoring the function ρ of form (2.7) for the field $\mathbf{E}^0$ given on S. Assuming that the parameters H and h are known, in fact, the inverse problem consists in reconstructing the function $\sigma(x, y)$ in (2.7) for the known function $\mathbf{E}^0$ on the surface S.

3 Reducing the inverse problem to an integral equation in the case of a flat surface S

Let us consider the $z-$component of a field (2.6) with a density (2.7) in the domain $D(-\infty, H-\varepsilon)$ of form (2.4). The value of ε is arbitrarily small and can be chosen so that the surface S of form (2.8) is located in the domain $D(-\infty, H-\varepsilon)$, that is, $\varepsilon < H - \max\limits_{(x,y)} F(x, y)$.

Given formula (2.7) for the density ρ, and also given that $z_M < z_P - \varepsilon$ if $M \in D(-\infty, H-\varepsilon)$, for the component E_z of field (2.6), we obtain

$$\begin{aligned} E_z(M) = &-\frac{8\pi}{l_x l_y} \int\limits_0^{l_x} \int\limits_0^{l_y} \sigma(x_P, y_P) \int\limits_H^{H+h} dz_P \sum_{n,m=1}^{\infty} e^{-k_{nm}(z_P - z_M)} \\ &\times \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} dx_P dy_P \\ = &\frac{16\pi}{l_x l_y} \int\limits_0^{l_x} \int\limits_0^{l_y} \sum_{n,m=1}^{\infty} e^{-k_{nm}(H + \frac{h}{2} - z_M)} \frac{\operatorname{sh} k_{nm}\frac{h}{2}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \\ &\times \sigma(x, y) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dx dy = \int\limits_0^{l_x} \int\limits_0^{l_y} K_z(x_M, y_M, z_M, x, y)\sigma(x, y) dx dy, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} K_z(x_M, y_M, z_M, x, y) = &\frac{16\pi}{l_x l_y} \sum_{n,m=1}^{\infty} e^{-k_{nm}(H + \frac{h}{2} - z_M)} \frac{\operatorname{sh} k_{nm}\frac{h}{2}}{k_{nm}} \\ &\times \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \quad k_{nm} = \sqrt{\left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2}. \end{aligned} \tag{3.2}$$

So, if the function σ in (2.7) is known, then we obtain the component of the field E_z in form (3.1).

If now, in accordance with the inverse problem, the field $\mathbf{E}$, or only its component E_z on a flat surface (2.8) when $F(x, y) \equiv a < H$, is known, i.e. according to (2.10)

$$E_z\Big|_{z=a} = E_z^0,$$

from (3.1) we obtain an integral equation of the first kind, linear with respect to the desired function σ:

$$\int_0^{l_x}\int_0^{l_y} K(x_M,y_M,x,y)\sigma(x,y)dxdy = E_z^0(x_M,y_M), \quad (x_M,y_M)\in\Pi, \tag{3.3}$$

where the kernel of the integral operator according to representation (3.2) has the form

$$K(x_M,y_M,x,y) = K_z(x_M,y_M,a,x,y)$$
$$= \frac{16\pi}{l_x l_y}\sum_{n,m=1}^{\infty} e^{-k_{nm}(H+\frac{h}{2}-a)}\frac{\operatorname{sh} k_{nm}\frac{h}{2}}{k_{nm}}\sin\frac{\pi n x_M}{l_x}\sin\frac{\pi m y_M}{l_y}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}. \tag{3.4}$$

We shall now obtain an equation similar to (3.3) in the case when the surface S has form (2.8) with the function F of general form.

4 Reducing the inverse problem to an integral equation in the case of a surface S of general form

We note that the z-component, like every component of field (2.6) of potential (2.5), is a harmonic function in the domain $D(-\infty,H)$. It also follows from (2.6) that the component E_z satisfies the conditions

$$E_z|_{x=0,l_x} = 0 \quad E_z|_{y=0,l_y} = 0,$$
$$E_z \to 0 \text{ when } z\to-\infty.$$

Taking into account condition (2.10) of the inverse problem for E_z of form (2.6), we obtain the problem

$$\begin{aligned}&\Delta E_z(M) = 0, \quad M\in D(-\infty,H),\\ &E_z|_S = E_z^0,\\ &E_z|_{x=0,l_x} = 0 \quad E_z|_{y=0,l_y} = 0,\\ &E_z\to 0 \text{ when } z\to-\infty.\end{aligned} \tag{4.1}$$

If E_z^0 is z-component of field (2.6) on the surface S of form (2.8), then problem (4.1) is the Dirichlet problem in the domain

$$D(-\infty,F) = \{(x,y,z): 0<x<l_x, 0<y<l_y, -\infty<z<F(x,y)\} \tag{4.2}$$

which has an unique solution, represented with formula (2.6).

From condition (2.10) of the inverse problem for field (2.6), an additional condition for the normal derivative on the surface S can be obtained. Indeed, field (2.6) is potential, and in the domain $D(-\infty,H)$ satisfies the equations

$$\operatorname{rot}\mathbf{E}(M) = 0, \quad M\in D(-\infty,H),$$
$$\operatorname{div}\mathbf{E}(M) = 0.$$

For the normal derivative of the component E_z on the surface S of form (2.8), given by the equation $z=F(x,y)<H$, we obtain

$$n_1\frac{\partial E_z}{\partial n}\Big|_S = (\mathbf{n}_1,\nabla E_z)\big|_S = (\frac{\partial E_z}{\partial x}F'_x + \frac{\partial E_z}{\partial y}F'_y - \frac{\partial E_z}{\partial z})\Big|_S,$$

where $\mathbf{n}_1 = (F'_x, F'_y, -1)$ is the inner normal with respect to the domain $D(-\infty, F)$ of form (4.2). Then, extracting from the equation $div\,\mathbf{E} = 0$, valid at the points of the surface $S \subset D(-\infty, H)$, the derivative with respect to the variable z, we obtain

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\mathbf{n}_1, \nabla E_z)\big|_S = (\frac{\partial E_z}{\partial x} F'_x + \frac{\partial E_z}{\partial y} F'_y + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y})\Big|_S. \tag{4.3}$$

Using the equations $rot\,\mathbf{E} = 0$ at the points of the surface $S \subset D(-\infty, H)$, namely

$$\frac{\partial E_z}{\partial x}\Big|_S = \frac{\partial E_x}{\partial z}\Big|_S, \quad \frac{\partial E_z}{\partial y}\Big|_S = \frac{\partial E_y}{\partial z}\Big|_S,$$

from (4.3) we obtain

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\frac{\partial E_x}{\partial z} F'_x + \frac{\partial E_y}{\partial z} F'_y + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y})\Big|_S. \tag{4.4}$$

We shall consider the field $\mathbf{E}^0$ in (2.10), given on S, as a function of the variables x and y on the rectangle Π of form (2.9). Differentiating the components of the field $\mathbf{E}^0$ by the arguments x and y, we obtain

$$\frac{\partial}{\partial x} E^0_x = \frac{\partial}{\partial x} E_x(x, y, F(x, y)) = (\frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial z} F'_x)\Big|_S,$$

$$\frac{\partial}{\partial y} E^0_y = \frac{\partial}{\partial y} E_y(x, y, F(x, y)) = (\frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial z} F'_y)\Big|_S.$$

Substituting these derivatives in (4.4), we obtain the expression for the normal derivative in terms of the derivatives of the components of the vector $\mathbf{E}^0$:

$$n_1 \frac{\partial E_z}{\partial n}\Big|_S = (\frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial z} F'_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial z} F'_y)\Big|_S = \frac{\partial}{\partial x} E^0_x + \frac{\partial}{\partial y} E^0_y. \tag{4.5}$$

If we add condition (4.5) to (4.1), then the component E_z of field (2.6) in the domain $D(-\infty, F) \subset D(-\infty, H)$ of form (4.2) is a solution of the problem

$$\begin{aligned}
&\Delta E_z(M) = 0, \quad M \in D(-\infty, F),\\
&E_z|_S = E^0_z,\\
&\frac{\partial E_z}{\partial n}|_S = \frac{1}{n_1}(\frac{\partial E^0_x}{\partial x} + \frac{\partial E^0_y}{\partial y}), \quad \mathbf{n}_1 = (F_x, F_y, -1),\\
&E_z|_{x=0,l_x} = 0, \quad E_z|_{y=0,l_y} = 0,\\
&E_z \to 0 \text{ при } z \to -\infty,
\end{aligned} \tag{4.6}$$

where the vector $\mathbf{E}^0 = (E^0_x, E^0_y, E^0_z)$ is field (2.10) in the formulation of the inverse problem.

We shall show now that, following the scheme in [6], the inverse problem can be reduced to an integral equation.

The source function $\varphi(M, P)$ of problem (2.2) can be represented as the sum of the fundamental solution and the function $W(M, P)$, harmonic in P:

$$\varphi(M, P) = \frac{1}{4\pi r_{MP}} + W(M, P), \tag{4.7}$$

where r_{MP} is the distance between points M and P. Let us put the point M in the domain

$$D(R, F) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, R < z < F(x, y), R = Const < 0\}$$

and apply Green formula in the domain $D(R,F)$ to the solution of problem (4.6) $E_z(P)$ and to functions $(4\pi r_{MP})^{-1}$ and $W(M,P)$. Then we obtain

$$E_z(M) = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)\frac{1}{4\pi r_{MP}} - E_z(P)\frac{\partial}{\partial n_P}\frac{1}{4\pi r_{MP}}(M,P)\Big]d\sigma_P,\ M\in D(R,F) \tag{4.8}$$

and

$$0 = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)W(M,P) - E_z(P)\frac{\partial W}{\partial n_P}(M,P)\Big]d\sigma_P, \quad M\in D(R,F) \tag{4.9}$$

Here the normal is external to the domain $D(R,F)$. Summing (4.8) and (4.9) and taking into account (4.7), we obtain

$$E_z(M) = \int\limits_{\partial D(R,F)} \Big[\frac{\partial E_z}{\partial n}(P)\varphi(M,P) - E_z(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big]d\sigma_P, \quad M\in D(R,F).$$

Given the boundary conditions for E_z and φ in problems (4.6) and (2.2), as well as replacing the external normal with the internal one, we obtain the representation of the component of the field E_z as the sum of the surface integrals

$$\begin{aligned} E_z(M) = &\int\limits_S \Big[-\frac{1}{n_1}\Big(\frac{\partial E_x^0}{\partial x}(P) + \frac{\partial E_y^0}{\partial y}(P)\Big)\varphi(M,P) + E_z^0(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P \\ &- \int\limits_{\Pi(R)} \Big[\frac{\partial E_z}{\partial n_P}(P)\varphi(M,P) - E_z(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P, \quad M\in D(R,F), \end{aligned} \tag{4.10}$$

where the rectangle $\Pi(R)$ has the form

$$\Pi(R) = \{(x,y,z) : 0<x<l_x, 0<y<l_y, z=R\}, \quad R < \min_{(x,y)} F(x,y). \tag{4.11}$$

The integral over the rectangle $\Pi(R)$, due to the representation of field (2.6) and the representation of the source function for a fixed point $z_M > z_P = R$ in accordance with (2.3)

$$\varphi(M,P) = \frac{2}{\pi l_x l_y}\sum_{n,m=1}^{\infty}\frac{e^{-k_{nm}(z_M-R)}}{k_{nm}}\sin\frac{\pi n x_M}{l_x}\sin\frac{\pi m y_M}{l_y}\sin\frac{\pi n x_P}{l_x}\sin\frac{\pi m y_P}{l_y},$$

converges to zero when $R\to-\infty$.

The integral over the surface S in (4.10) is reduced to the integral with respect to the variables x_P and y_P, given that $\dfrac{\partial\varphi}{\partial n}(M,P) = (\mathbf{n}, \nabla_P\varphi(M,P))$, $\mathbf{n} = \dfrac{\mathbf{n_1}}{n_1}$, $\mathbf{n_1} = (F'_x, F'_y, -1)$, and $d\sigma_P = n_1 dx_P dy_P$,

$$\begin{aligned} E_z(M) = \int\limits_0^{l_x}\int\limits_0^{l_y} \Big[&-\Big(\frac{\partial E_x^0}{\partial x_P}(x_P,y_P) + \frac{\partial E_y^0}{\partial y_P}(x_P,y_P)\Big)\varphi(M,P) \\ &+ E_z^0(x_P,y_P)(\mathbf{n_1}, \nabla_P\varphi(M,P))\Big]_{P\in S} dx_P dy_P. \end{aligned}$$

Integrating by parts, taking into account the boundary conditions for φ, we obtain

$$E_z(M) = \int_0^{l_x}\int_0^{l_y} \Big[E_x^0(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)\big|_{P\in S} + E_y^0(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)\big|_{P\in S} + E_z^0(x_P,y_P)(\mathbf{n_1},\nabla_P\varphi(M,P))\big|_{P\in S}\Big]dx_Pdy_P. \quad (4.12)$$

Let us introduce the notation

$$\Phi(x_M,y_M) = E_z(M)\big|_{z_M=a}, \quad a < \min_{(x,y)} F(x,y), \quad (4.13)$$

where E_z is the function of form (4.12). Since the field $\mathbf{E}^0$ is given, Φ is a known function, and the source function $\varphi(M,P)$ for $M\in\Pi(a)$ of form (4.11) where $z=a$ and $P\in S$ of form (2.8) can be represented as an uniformly convergent series (2.3).

On the other hand, since E_z of form (4.12) is a component of field (2.6) of the potential, integral representation (3.1) is valid for E_z. Then, from integral representation (3.1) in order to determine the unknown density of σ, we obtain the Fredholm integral equation of the first kind with respect to the desired function σ, similar to (3.3)

$$\int_0^{l_x}\int_0^{l_y} K(x_M,y_M,x,y)\sigma(x,y)dxdy = \Phi(x_M,y_M), \quad (x_M,y_M)\in\Pi. \quad (4.14)$$

where the kernel of the integral operator has form (3.4) and the rectangle Π has form (2.9).

5 Exact solution of the inverse problem

When solving the inverse potential problem, we assume that the field $\mathbf{E}^0$ in (2.10) is field (2.6) on surface (2.8), so the solution of equation (4.14) exists in $L_2(\Pi)$. Since the system of eigenfunctions of the Dirichlet problem for the Laplace equation in the rectangle Π

$$\Big\{\sin\frac{\pi nx}{l_x}\Big\}\cdot\Big\{\sin\frac{\pi my}{l_y}\Big\}\Big|_{n,m=1}^{n,m=\infty}$$

is complete, the kernel of integral equation (4.14) is closed and the equation has an unique solution.

The solution of integral equation (4.14) can be obtained as a Fourier series

$$\sigma(x,y) = \sum_{n,m=1}^{\infty}\tilde\sigma_{nm}\sin\frac{\pi nx}{l_x}\sin\frac{\pi my}{l_y} = \sum_{n,m=1}^{\infty}\tilde\Phi_{nm}K_{nm}\sin\frac{\pi nx}{l_x}\sin\frac{\pi my}{l_y}, \quad (5.1)$$

where $\tilde\Phi_{nm}$ are the Fourier coefficients

$$\tilde\Phi_{nm} = \frac{4}{l_xl_y}\int_0^{l_x}\int_0^{l_y}\Phi(x,y)\sin\frac{\pi nx}{l_x}\sin\frac{\pi my}{l_y}dxdy \quad (5.2)$$

of the function Φ of form (4.13), and

$$K_{nm} = e^{k_{nm}(H+\frac{h}{2}-a)}\frac{k_{nm}}{4\pi\,\mathrm{sh}\,k_{nm}\frac{h}{2}}, \quad k_{nm} = \sqrt{\Big(\frac{\pi n}{l_x}\Big)^2+\Big(\frac{\pi m}{l_y}\Big)^2}. \quad (5.3)$$

Since, when solving equation (4.14), we consider that the function Φ of form (4.13) corresponds to the density σ of form (2.7), the coefficients $\tilde{\Phi}_{nm}(a) = \sigma_{nm}/K_{nm}$ decrease faster than the value $e^{k_{nm}(H-a)}k_{nm}$ increases and series (5.1) converges to σ in $L_2(\Pi)$.

In the case when $\sigma(M) = \sigma_0\chi_D(M)$, where $\chi_D(M)$ is the characteristic function of some domain $D \subset \Pi$ and σ_0 is a known constant, the solution of the inverse problem is reduced to finding the support D of the source density function. To do this, we can use the formula

$$D = \{(x,y) \in \Pi : \frac{1}{\sigma_0}\sigma(x,y) > \lambda = Const,\ 0 < \lambda < 1\}. \tag{5.4}$$

As it is known [10, 11], the Fredholm equation of the first kind is an ill-posed problem. Its approximate solution is unstable with respect to the error of the right part and requires the use of regularizing algorithms. Let us construct an approximate right-hand side of the integral equation in the case of an inaccurate data on the field $\mathbf{E}^0$ and the surface S and estimate its error.

6 Approximate calculation of the normal to an inaccurately defined surface

As follows from (4.13), (4.12), when forming the right-hand side of integral equation (4.14), it is necessary to calculate the vector function of the normal $\mathbf{n}_1$ to the surface S of form (2.8), which is the gradient of the function $F(x,y) - z$,

$$\mathbf{n}_1 = grad\,(F(x,y) - z) = \nabla_{xy}F - \mathbf{k}. \tag{6.1}$$

Let the surface S is given with an error, namely, instead of the exact function F in (2.8), the function F^μ is known, given on a rectangle Π of form (2.9), such that

$$\|F^\mu - F\|_{L_2(\Pi)} \leqslant \mu. \tag{6.2}$$

For the approximate calculation of integral (4.12), it is necessary to calculate the normal to the surface given approximately, which is also an ill-posed problem, since the calculation of the normal $\mathbf{n}_1$ is associated with the calculation of the derivatives of the function F.

To obtain a stable solution to this problem, we use the approach of [7], that is, we consider the problem of calculating the gradient of a function as the problem of calculating values of an unbounded operator [2].

As an approximation to the function $\nabla_{xy}F$ in (6.1) calculated from the known function F^μ, associated with the function F by condition (6.2), we consider the gradient of the extremal of the functional

$$N^\beta[W] = \left\|W - F^\mu\right\|^2_{L_2(\Pi)} + \beta\left\|\nabla W\right\|^2_{L_2(\Pi)}, \quad \beta > 0. \tag{6.3}$$

For simplicity of calculating the extremal, we consider such surfaces S, for which

$$F|_{x=0,l_x} = 0, \quad F|_{y=0,l_y} = 0.$$

This condition, in particular, occurs in the case when S can be considered as a perturbation of the plane $z = 0$. Then the extremal of functional (6.3) is the solution of the following problem for the Euler equation

$$-\beta\Delta W + W = F^\mu,$$
$$W|_{x=0,l_x} = 0,\ W|_{y=0,l_y} = 0.$$

The solution of this problem is

$$W^{\mu}_{\beta}(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde{F}^{\mu}_{nm}}{1+\beta k^2_{nm}} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{6.4}$$

where the Fourier coefficients $\tilde{F}^{\mu}_{nm}$ are calculated by formulas of form (5.2) and k_{nm} has form (5.3). It is easy to see that series (6.4) converges uniformly on Π.

As an approximate value of the gradient of the function F^{μ}, we consider the vector function

$$\begin{aligned}\nabla_{xy} W^{\mu}_{\beta}(x,y) = \sum_{n,m=1}^{\infty} & \frac{\tilde{F}^{\mu}_{nm}}{1+\beta k^2_{nm}} \\ & \times \left(\mathbf{i}\frac{\pi n}{l_x} \cos\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} + \mathbf{j}\frac{\pi m}{l_y} \cos\frac{\pi m y}{l_y} \sin\frac{\pi n x}{l_x} \right).\end{aligned} \tag{6.5}$$

Series (6.5) converges in $L_2(\Pi)$.

Let F^- be an odd-periodic continuation of the function F, given on the rectangle Π of form (2.9), with a period of $2l_x$ for the variable x and with a period of $2l_y$ for the variable y, i.e.

$$\begin{aligned}
&F^-(x,y) = F(x,y), \quad (x,y)\in\Pi,\\
&F^-(-x,y) = -F(x,y), \quad (x,y)\in\Pi,\\
&F^-(x,-y) = -F(x,y), \quad (x,y)\in\Pi,\\
&F^-(-x,-y) = F(x,y), \quad (x,y)\in\Pi,\\
&F^-(x+2l_x n, y+2l_y m) = F^-(x,y), \quad (x,y)\in\mathbb{R}^2, \quad n,m=\pm1,\pm2,\ldots.
\end{aligned}$$

Theorem 6.1. [7] *Let* $F^- \in C^2(\mathbb{R}^2)$, $\beta = \beta(\mu) > 0$, $\beta(\mu) \to 0$ *and* $\mu/\sqrt{\beta(\mu)} \to 0$ *when* $\mu \to 0$. *Then*

$$\left\| \nabla_{xy} W^{\mu}_{\beta(\mu)} - \nabla_{xy} F \right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2} \left\| \Delta F \right\|_{L_2(\Pi)} \to 0 \text{ when } \mu \to 0.$$

Based on the theorem, we can use formula (6.5) to approximate the normal to the surface using formula (6.1):

$$\mathbf{n}^{\mu}_{1,\beta} = \nabla_{xy} W^{\mu}_{\beta} - \mathbf{k}. \tag{6.6}$$

With a known estimate

$$\left\| \Delta F \right\|_{L_2(\Pi)} \leqslant M,$$

it follows from the statement of the theorem that

$$\left\| \mathbf{n}^{\mu}_{1,\beta} - \mathbf{n}_1 \right\|_{L_2(\Pi)} = \left\| \nabla_{xy} W^{\mu}_{\beta} - \nabla_{xy} F \right\|_{L_2(\Pi)} \leqslant \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2} M.$$

The maximum for the β expression on the right is achieved when

$$\beta(\mu) = \frac{\mu}{M} \tag{6.7}$$

and, thus denoting in accordance with (6.6) and (6.7)

$$\mathbf{n}^{\mu}_1 = \mathbf{n}^{\mu}_{1,\beta(\mu)} = \nabla_{xy} W^{\mu}_{\beta(\mu)} - \mathbf{k}, \tag{6.8}$$

we shall obtain:

$$\left\| \mathbf{n}^{\mu}_1 - \mathbf{n}_1 \right\|_{L_2(\Pi)} \leqslant \sqrt{M\mu} \xrightarrow[\mu\to 0]{} 0. \tag{6.9}$$

It is also not difficult to obtain the estimate

$$\left\| W^{\mu}_{\beta(\mu)} - F \right\|_{L_2(\Pi)} \leqslant 2\mu. \tag{6.10}$$

The surface defined by the equation $z = W^{\mu}_{\beta(\mu)}(x,y)$, we denote as

$$S^{\mu} = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = W^{\mu}_{\beta(\mu)}(x,y)\}. \tag{6.11}$$

7 Solution of the inverse problem in the case of an approximately given field $\mathbf{E}^0$ on an approximately given surface

Let instead of the exact vector function $\mathbf{E}^0$ in condition (2.10) of the inverse problem, the function $\mathbf{E}^{0,\delta} = (E_x^{0,\delta}, E_y^{0,\delta}, E_z^{0,\delta})$ is known, given as a function on the rectangle Π of form (2.9), such that

$$\|\mathbf{E}^{0,\delta} - \mathbf{E}^0\|_{L_2(\Pi)} \leqslant \delta. \tag{7.1}$$

In this case, we assume that the surface S of form (2.8) is given approximately by condition (6.2).

We assume that we also know that

$$a_1 < F(x,y) < a_2. \tag{7.2}$$

In this case using the results of the previous paragraph, the right part $\Phi(M)$ of form (4.13) in integral equation (4.14) will be calculated approximately on a rectangle

$$\Pi(a) = \left\{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = a\right\}, \quad a < \min_{(x,y)} W^{\mu}_{\beta(\mu)}(x,y), \quad a < a_1 \tag{7.3}$$

in accordance with formula (4.12) and (4.13) as a function

$$E_z^{\delta,\mu}(M) = \int_0^{l_x}\int_0^{l_y} [E_x^{0,\delta}(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} + E_y^{0,\delta}(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu}$$
$$+ E_z^{0,\delta}(x_P,y_P)(\mathbf{n}_1^\mu, \nabla_P\varphi(M,P))|_{P\in S^\mu}]dx_P dy_P, \quad M \in \Pi(a), \tag{7.4}$$

where the surface S^μ has form (6.11), the approximate normal $\mathbf{n}_1^\mu$ is calculated by formula (6.8) and the function

$$\varphi(M,P) = \frac{2}{l_x l_y}\sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_P-a)}}{k_{nm}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}$$

is source function (2.3) of problem (2.2).

Let us estimate the error in calculating the function $E_z^{\delta,\mu}$ of form (7.4) with respect to the function E_z of form (4.12) on the rectangle $\Pi(a)$ – the right-hand side of integral equation (4.14), i.e. we estimate the difference

$$\left| E_z^{\delta,\mu}(M) - E_z(M) \right| \leqslant \left| E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M) \right| + \left| E_z^{\delta,\mu,1}(M) - E_z^{\delta}(M) \right|$$
$$+ \left| E_z^{\delta}(M) - E_z(M) \right|, \quad M \in \Pi(a). \tag{7.5}$$

where $\Pi(a)$ has form (7.3). In this estimate the function $E_z^{\delta,\mu,1}$ of form (7.4) is introduced, where formally the approximate normal $\mathbf{n}_1^\mu$ is replaced by the exact normal $\mathbf{n}_1$ (note that $\mathbf{n}_1(x_P,y_P)|_{P\in S^\mu} = \mathbf{n}_1(x_P,y_P)|_{P\in S}$):

$$E_z^{\delta,\mu,1}(M) = \int_0^{l_x}\int_0^{l_y} [E_x^{0,\delta}(x_P,y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} + E_y^{0,\delta}(x_P,y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu}$$
$$+ E_z^{0,\delta}(x_P,y_P)(\mathbf{n}_1, \nabla_P\varphi(M,P))|_{P\in S^\mu}]dx_P dy_P, \quad \mathbf{n}_1 = (F_x', F_y', -1), \tag{7.6}$$

and is also introduced the function E_z^δ of form (7.4), which is calculated on an exactly specified surface

$$E_z^\delta(M) = \int_0^{l_x}\int_0^{l_y} [E_x^{0,\delta}(x_P, y_P)\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S} + E_y^{0,\delta}(x_P, y_P)\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S} + E_z^{0,\delta}(x_P, y_P)(\mathbf{n}_1, \nabla_P\varphi(M,P))|_{P\in S}]dx_P dy_P, \quad \mathbf{n}_1 = (F_x', F_y', -1). \tag{7.7}$$

Let us estimate the difference between functions (7.4) and (7.6) in the right-hand side of inequality (7.5):

$$\Big| E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M)\Big|_{M\in\Pi(a)} = \Big| \int_0^{l_x}\int_0^{l_y} E_z^{0,\delta}(x_P, y_P)((\mathbf{n}_1^\mu - \mathbf{n}_1), \nabla_P\varphi(M,P))|_{P\in S^\mu} dx_P dy_P\Big|$$
$$\leqslant \int_0^{l_x}\int_0^{l_y}\Big[\big|E_z^{0,\delta}(x_P, y_P)\big|\cdot\big|\mathbf{n}_1^\mu(P) - \mathbf{n}_1(P)\big|\cdot\big|\nabla_P\varphi(M,P)\big|\Big]_{P\in S^\mu} dx_P dy_P$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}} \big|\nabla_P\varphi(M,P)\big| \int_0^{l_x}\int_0^{l_y} \big|E_z^{0,\delta}(x_P, y_P)\big|\cdot\big|\mathbf{n}_1^\mu(P) - \mathbf{n}_1(P)\big|_{P\in S^\mu} dx_P dy_P.$$

Using the Cauchy-Bunyakovsky inequality, estimate (6.9) and estimate $\|E_z^{0,\delta}\| \leqslant \|\mathbf{E}^0\| + \delta$, we obtain

$$\Big| E_z^{\delta,\mu}(M) - E_z^{\delta,\mu,1}(M)\Big|_{M\in\Pi(a)} = \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}} \big|\nabla_P\varphi(M,P)\big|\,\|E_z^{0,\delta}\|\cdot\|\mathbf{n}_1^\mu - \mathbf{n}_1\|$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P\in S^\mu}} \big|\nabla_P\varphi(M,P)\big|\,(\|\mathbf{E}^0\| + \delta)\cdot\sqrt{M\mu} \leqslant C_1\sqrt{\mu}. \tag{7.8}$$

Let us estimate the difference between functions (7.6) and (7.7) in the right-hand side of inequality (7.5) using the Lagrange formula

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^{\delta}(M)\Big|_{M\in\Pi(a)}$$
$$= \Big| \int_0^{l_x}\int_0^{l_y} \Big[E_x^{0,\delta}(x_P, y_P)\Big(\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S^\mu} - \frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ E_y^{0,\delta}(x_P, y_P)\Big(\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S^\mu} - \frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ E_z^{0,\delta}(x_P, y_P)\big(\mathbf{n}_1, \nabla_P\varphi(M,P)|_{P\in S^\mu} - \nabla_P\varphi(M,P)|_{P\in S}\big)\Big]dx_P dy_P\Big|$$
$$= \Big| \int_0^{l_x}\int_0^{l_y} \Big[E_x^{0,\delta}(x_P, y_P)\Big(\frac{\partial^2}{\partial x_P z_P}\varphi(M,P_1)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big)$$
$$+ E_y^{0,\delta}(x_P, y_P)\Big(\frac{\partial^2}{\partial y_P z_P}\varphi(M,P_2)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big)$$
$$+ E_z^{0,\delta}(x_P, y_P)\big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P_3)\big)(z_P\big|_{P\in S^\mu} - z_P\big|_{P\in S})\Big]dx_P dy_P\Big|, \quad M\in\Pi(a).$$

Since according to (6.11) $z_P\big|_{P\in S^\mu} = W^\mu_{\beta(\mu)}(x_P,y_P)$ and $z_P\big|_{P\in S} = F(x_P,y_P)$, we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)}$$
$$= \Big| \int_0^{l_x}\int_0^{l_y} \Big[E_x^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial x_P z_P}\varphi(M,P_1)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big)$$
$$+ E_y^{0,\delta}(x_P,y_P)\Big(\frac{\partial^2}{\partial y_P z_P}\varphi(M,P_2)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big)$$
$$+ E_z^{0,\delta}(x_P,y_P)\big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P_3)\big)(W^\mu_{\beta(\mu)}(x_P,y_P) - F(x_P,y_P))\Big] dx_P dy_P \Big|. \quad (7.9)$$

We introduce the following notation using (7.2)

$$\begin{aligned} z_1(x_P,y_P) &= \min\{W^\mu_{\beta(\mu)}(x_P,y_P), a_1\}, \\ z_2(x_P,y_P) &= \max\{W^\mu_{\beta(\mu)}(x_P,y_P), a_2\}. \end{aligned} \quad (7.10)$$

Now from (7.9) using (7.10) we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)}$$
$$\leqslant \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \frac{\partial^2}{\partial x_P z_P}\varphi(M,P)\Big| \int_0^{l_x}\int_0^{l_y} \big|E_x^{0,\delta}(x,y)\big| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \frac{\partial^2}{\partial y_P z_P}\varphi(M,P)\Big| \int_0^{l_x}\int_0^{l_y} |E_y^{0,\delta}(x,y)| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P)\big)\Big| \int_0^{l_x}\int_0^{l_y} |E_z^{0,\delta}(x,y)| \cdot \Big| W^\mu_{\beta(\mu)}(x,y) - F(x,y)\Big| dxdy.$$

Applying the Cauchy-Bunyakovsky inequality, assuming that $\delta < \delta_0$, and using the estimate (6.10), we obtain

$$\Big| E_z^{\delta,\mu,1}(M) - E_z^\delta(M) \Big|_{M\in\Pi(a)} = \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \frac{\partial^2}{\partial x_P z_P}\varphi(M,P)\Big| \, \|E_x^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \frac{\partial^2}{\partial y_P z_P}\varphi(M,P)\Big| \, \|E_y^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$+ \max_{\substack{M\in\Pi(a)\\ P:z_1<z_P<z_2}} \Big| \big(\mathbf{n}_1, \frac{\partial}{\partial z_P}\nabla_P\varphi(M,P)\big)\Big| \, \|E_z^{0,\delta}\| \cdot \|W^\mu_{\beta(\mu)} - F\|$$
$$\leqslant C\|\mathbf{E}^{0,\delta}\|\mu \leqslant C(\|\mathbf{E}^0\|) + \delta)\mu \leqslant C_2\mu. \quad (7.11)$$

Let us estimate the difference between functions (7.7) and (4.12) in the right-hand side of inequality (7.5):

$$\Big| E_z^{\delta}(M) - E_z(M) \Big|_{M\in\Pi(a)} = \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} \Big[\big(E_x^{0,\delta}(x_P,y_P) - E_x^0(x_P,y_P)\big)\Big(\frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ \big(E_y^{0,\delta}(x_P,y_P) - E_y^0(x_P,y_P)\big)\Big(\frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S}\Big)$$
$$+ \big(E_z^{0,\delta}(x_P,y_P) - E_z^0(x_P,y_P)\big)\big(\mathbf{n}_1, \nabla_P\varphi(M,P)\big)_{P\in S}\Big] dx_P dy_P \Big| .$$

Using the Cauchy-Bunyakovsky inequality, as well as (7.1), we obtain from here

$$\Big| E_z^{\delta}(M) - E_z(M) \Big| = \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| \frac{\partial}{\partial x_P}\varphi(M,P)|_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_x^{0,\delta}(x,y) - E_x^0(x,y)| dx dy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| \frac{\partial}{\partial y_P}\varphi(M,P)|_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_y^{0,\delta}(x,y) - E_y^0(x,y)| dx dy$$
$$+ \max_{\substack{M\in\Pi(a)\\ P\in S}} \Big| (\mathbf{n}_1, \nabla_P\varphi(M,P))_{P\in S} \Big| \int\limits_0^{l_x}\int\limits_0^{l_y} |E_z^{0,\delta}(x,y) - E_z^0(x,y)| dx dy$$
$$\leqslant C_3\|\mathbf{E}^{0,\delta} - \mathbf{E}^0\| \leqslant C_3\delta, \quad M\in\Pi(a). \tag{7.12}$$

Collecting estimates (7.8), (7.11), (7.12) and assuming that $\mu < \mu_0$, from (7.5) we obtain

$$\big| E_z^{\delta,\mu}(M) - E_z(M) \big|_{M\in\Pi(a)} \leqslant C_1\sqrt{\mu} + C_2\mu + C_3\delta \leqslant C_4\sqrt{\mu} + C_3\delta. \tag{7.13}$$

Denoting, similarly to (4.13), the approximate right-hand side of integral equation (4.14)

$$\Phi^{\delta,\mu}(x_M,y_M) = E_z^{\delta,\mu}(M)\big|_{M\in\Pi(a)}, \tag{7.14}$$

from (7.13) we obtain an estimate in L_2 of the error of the approximate right-hand side of integral equation (4.14)

$$\big\|\Phi^{\delta,\mu} - \Phi\big\|_{L_2(\Pi)} \leqslant \overline{C}_4\sqrt{\mu} + \overline{C}_3\delta = \gamma(\mu,\delta) \xrightarrow[\substack{\mu\to 0\\ \delta\to 0}]{} 0, \tag{7.15}$$

where $\overline{C}_4, \overline{C}_3$ are some constants.

Let us now construct an approximate solution of integral equation (4.14) with right-hand side (7.14) by the Tikhonov regularization method [10, 11]. As an approximate solution, we consider the extremal of the Tikhonov functional

$$M[w] = \big\|Kw - \Phi^{\delta,\mu}\big\|^2_{L_2(\Pi)} + \alpha\,\|w\|^2_{L_2(\Pi)}, \quad \alpha > 0, \tag{7.16}$$

where K is the integral operator in (4.14). The extremal $\sigma_\alpha^{\delta,\mu}$ can be obtained as a solution of the Euler equation

$$K^*Kw + \alpha w = K^*\Phi^{\delta,\mu}$$

for functional (7.16) and has the form

$$\sigma_\alpha^{\delta,\mu}(x,y) = \sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^{\delta,\mu} K_{nm}}{1+\alpha K_{nm}^2} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \quad \alpha > 0. \tag{7.17}$$

Here $\tilde{\Phi}^{\delta,\mu}_{nm}$ are the Fourier coefficients

$$\tilde{\Phi}^{\delta,\mu}_{nm} = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} \Phi^{\delta,\mu}(x,y) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dxdy \tag{7.18}$$

of the function $\Phi^{\delta,\mu}$ of form (7.14). The value K_{nm} in formula (7.17) has form (5.3).

Let us note that for $\delta = 0$, $\mu = 0$ and $\alpha = 0$, formula (7.17) turns into an explicit representation of exact solution (5.1). When $\delta > 0$, $\mu > 0$ and $\alpha = 0$, (7.17), generally speaking, may diverge in accordance with the fact that the inverse problem is ill-posed. For $\delta > 0$, $\mu > 0$ and $\alpha > 0$, the convergence is provided by the regularizing factor $(1+\alpha K^2_{nm})^{-1}$.

The following theorem proves the convergence of approximate solution (7.17) in $L_2(\Pi)$ to the exact solution of the integral equation.

Theorem 7.1. *For any $\alpha = \alpha(\gamma) > 0$ such that $\alpha(\gamma) \to 0$, $\gamma/\sqrt{\alpha(\gamma)} \to 0$ when $\gamma \to 0$, the function $\sigma^{\delta,\mu}_{\alpha(\gamma)}$ of form (7.17), where according to (7.15) $\gamma = \gamma(\mu,\delta) = \overline{C}_4\sqrt{\mu} + \overline{C}_3\delta$, converges to the exact solution of integral equation (4.14) in $L_2(\Pi)$ when $\delta \to 0$, $\mu \to 0$.*

Proof. Following the general scheme [2] of estimating an approximate solution of a linear integral equation, introducing a function σ_α of form (7.17) when $\delta = 0, \mu = 0$, we obtain

$$\|\sigma^{\delta,\mu}_\alpha - \sigma\|_{L_2} \leqslant \|\sigma^{\delta,\mu}_\alpha - \sigma_\alpha\|_{L_2} + \|\sigma_\alpha - \sigma\|_{L_2}. \tag{7.19}$$

To estimate the first difference in the right-hand side of inequality (7.19), we use estimate (7.15)

$$\|\sigma^{\delta,\mu}_\alpha - \sigma_\alpha\|_{L_2} \leqslant \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \Big(\frac{K_{nm}}{1+\alpha K^2_{nm}}\Big)^2 |\tilde{\Phi}^{\delta,\mu}_{nm} - \tilde{\Phi}_{nm}|^2\Big]^{1/2} \leqslant \max_x \Big(\frac{x}{1+\alpha x^2}\Big) \|\Phi^{\delta,\mu} - \Phi\|_{L_2} \leqslant \frac{\gamma}{2\sqrt{\alpha(\gamma)}}. \tag{7.20}$$

We estimate the second difference in the right-hand side of inequality (7.19). We note that according to (5.1) $\tilde{\Phi}_{nm} K_{nm} = \tilde{\sigma}_{nm}$, so we obtain

$$\|\sigma_\alpha - \sigma\|_{L_2} \leqslant \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \Big(\frac{\alpha K^2_{nm}}{1+\alpha K^2_{nm}}\Big)^2 |\tilde{\Phi}_{nm} K_{nm}|^2\Big]^{1/2} = \Big[\frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \Big(\frac{\alpha K^2_{nm}}{1+\alpha K^2_{nm}}\Big)^2 \tilde{\sigma}^2_{nm}\Big]^{1/2}.$$

Since the series depending on the parameter α is majorized by a converging numerical series with coefficients $\tilde{\sigma}^2_{nm}$ it is possible to pass to the limit in α, and hence

$$\|\sigma_\alpha - \sigma\|_{L_2} \to 0, \quad \text{when} \quad \alpha \to 0. \tag{7.21}$$

It follows from (7.19), (7.20), (7.21), and the assumptions of the theorem that

$$\|\sigma^{\delta,\mu}_{\alpha(\gamma)} - \sigma\|_{L_2} \to 0, \quad \text{when} \quad \delta \to 0, \mu \to 0.$$

□

In the case when $\sigma(M) = \sigma_0\chi_D(M)$, where $\chi_D(M)$ is the characteristic function of the domain D in accordance with (5.4), we construct an approximation $D_\lambda^{\delta,\mu}$ to the support D of the density σ based on the approximate density function of sources (7.17)

$$D_\lambda^{\delta,\mu} = \{(x,y) \in \Pi : \frac{1}{\sigma_0}\sigma_{\alpha(\gamma)}^{\delta,\mu}(x,y) > \lambda = Const,\ 0 < \lambda < 1\}. \tag{7.22}$$

A criterion for the quality of the approximation can be the measure of the symmetric difference between domain (7.22) and the domain D of form (5.4).

Theorem 7.2. *Under the conditions of Theorem* 7.1 *the measure of the symmetric difference* $\boldsymbol{mes}(D_\lambda^{\delta,\mu}\Delta D) \to 0$ *when* $\delta \to 0$, $\mu \to 0$.

Proof. It follows from theorem 7.1,

$$\|\frac{1}{\sigma_0}\sigma_\alpha^{\delta,\mu} - \chi_D\|_{L_2(\Pi)} \to 0 \quad \text{when} \quad \delta \to 0,\ \mu \to 0.$$

From the convergence of $\frac{1}{\sigma_0}\sigma_\alpha^\delta$ to χ_D in L_2, the convergence in measure follows (see [3]). Further, the proof repeats verbatim the proof of the theorem in [1]. □

Formulas (7.4), (7.14), (7.17), (7.18), (7.22) give the solution to the inverse problem.

8 Conclusions

The inverse problem of the Newtonian potential for bodies of constant thickness is posed and solved in the case when the potential field on the surface of the general form is known. In this case, the density function of the distribution of potential sources is found as an approximate regularized solution of the linear integral Fredholm equation of the first kind, which is stable both with respect to the error in setting the potential and to the error in the surface.

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