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AN INTRODUCTION TO COMPOSITION OPERATORS IN SOBOLEV SPACES

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We propose a survey of the results on the composition operators in classical Sobolev spaces, obtained between 1975 and 2020. A first version of these notes were the subject of a series of lectures, given in Padova University in January 2018.

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1 Introduction

The composition of two maps f and g is defined by $(f \circ g)(x) := f(g(x))$, if the range of g is contained in the definition set of f. We denote by T_f the composition operator $T_f(g) := f \circ g$.

Definition 1. Let E be a set of real valued functions, and let $f : \mathbb{R} \to \mathbb{R}$. We say that f acts on E by composition (or: superposition) if $T_f(E) \subseteq E$.

Here are some elementary examples :

- Let E be a vector space of functions, which means that $g_1 + g_2 \in E$ and $\lambda g_1 \in E$, for all $g_1, g_2 \in E$ and all $\lambda \in \mathbb{R}$. Then every linear function $f : \mathbb{R} \to \mathbb{R}$ acts on E.
- Let E be an algebra of functions, which means that E is a vector space as above, and that $g_1 g_2 \in E$ for all $g_1, g_2 \in E$. Then any polynomial f such that $f(0) = 0$ acts on E.

We have a list of natural problems concerning operators T_f.

In case E is a vector space of functions, a composition operator T_f is said *trivial* if the function f is linear. Then we have the following questions :

$\mathcal{Q}_1$: *Do nontrivial composition operators exist ?*

In case E is an algebra of functions, the answer is positive. We will see that it is negative for certain Sobolev spaces.

$\mathcal{Q}_2$: *Describe explicitly the set of functions which act on E.*

For instance, if E is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then a function f acts on E if and only if f is itself continuous.

In case E is endowed with a norm, then the following problems make sense :

$\mathcal{Q}_3$: *Determine the functions* f *for which* $T_f : E \to E$ *is bounded.*

$\mathcal{Q}_4$: *Determine the functions* f *for which* $T_f : E \to E$ *is continuous.*

We propose a wide survey on the answers to the above questions, in case E is the classical Sobolev space $W_p^m(\mathbb{R}^n)$. Some results are given together with their proofs. Some proofs are simpler than the original ones.

2 Notation

$\mathbb{N}$ denotes the set of all positive integers, including 0. $\mathbb{Z}$ denotes the set of all integers. For $x \in \mathbb{R}^n$, $|x|$ denotes its euclidean norm.

If E, F are topological spaces, then $E \hookrightarrow F$ means that $E \subseteq F$, as sets, and the natural mapping $E \to F$ is continuous. If B is a Lebesgue measurable subset of $\mathbb{R}^n$, we denote by $|B|$ its Lebesgue measure. We denote by χ_A the characteristic function of a set A.

A *multi-index* is n-tuple $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For such α, and for all $h := (h_1, \dots, h_n) \in \mathbb{R}^n$, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ (this differs from the euclidean norm), $\alpha! := \alpha_1! \cdots \alpha_n!$, $h^\alpha := h_1^{\alpha_1} \cdots h_n^{\alpha_n}$. If f is a function defined on an open subset of $\mathbb{R}^n$, and $\alpha \in \mathbb{N}^n$ as above, we denote by $f^{(\alpha)}$ the partial derivative

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $h \in \mathbb{R}^n$, the *translation operator* is defined by $(\tau_h f)(x) := f(x-h)$ for all function f on $\mathbb{R}^n$. The *finite difference operator* is defined by $\Delta_h f := \tau_{-h} f - f$. The m-th power of Δ_h satisfies the following formula :

$$(\Delta_h^m f)(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(x+kh) \tag{2.1}$$

(easy proof by induction).

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $L_{1,loc}(\Omega)$ the set of (equivalence classes of) locally integrable functions on Ω, endowed with its natural topology (mean convergence on compact subsets of Ω), and by $\mathcal{D}(\Omega)$ the set of all indefinitely many times differentiable compactly supported functions on Ω, endowed with its natural topology, see [1, 1.56].

Let $Q := [-1/2, 1/2]^n$. We fix some function $\rho \in \mathcal{D}(\mathbb{R}^n)$ such that $\rho(x) = 1$ on Q and $\operatorname{supp} \rho \subseteq 2Q$.

Let E be a subset of $L_{1,loc}(\mathbb{R}^n)$. We say that a function $f \in L_{1,loc}(\mathbb{R}^n)$ belongs *locally* to E if $\varphi f \in E$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$; in case E is endowed with a norm, we say that a function $f \in L_{1,loc}(\mathbb{R}^n)$ belongs *locally uniformly* to E if

$$\sup_{a \in \mathbb{R}^n} \|(\tau_a \varphi) f\|_E < +\infty\,,$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Through the paper, "ball" means "closed ball with nonzero radius" (we exclude balls reduced to one point).

3 Composition operators in Lebesgue spaces

Proposition 3.1. *Let $1 \leq p < +\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then f acts on $L_p(\mathbb{R}^n)$ if and only if there exists $c > 0$ such that*

$$|f(t)| \leq c|t|\,, \quad \text{for all } t \in \mathbb{R}\,. \tag{3.1}$$

Proof. 1. If estimate (3.1) holds, it is easily seen that $g \in L_p(\mathbb{R}^n)$ implies $f \circ g \in L_p(\mathbb{R}^n)$. Indeed, the following holds :

$$\|f \circ g\|_p \leq c\,\|g\|_p, \quad \text{for all } g \in L_p(\mathbb{R}^n)\,. \tag{3.2}$$

2. Assume that f acts on $L_p(\mathbb{R}^n)$. Since the null function belongs to $L_p(\mathbb{R}^n)$, the same holds for the constant function $f(0)$. By condition $p < \infty$ we deduce $f(0) = 0$. Arguing by contradiction, let us assume that estimate (3.1) does not hold. Then, for some sequence $(a_k)_{k\geq 1}$, we have $|f(a_k)| > k|a_k|$ for all $k \geq 1$. Consider a sequence $(B_k)_{k\geq 1}$ of disjoint measurable sets in $\mathbb{R}^n$ such that

$$|a_k|^p|B_k| = k^{-p-1}\,. \tag{3.3}$$

Let

$$g := \sum_{k\geq 1} a_k \chi_{B_k}\,.$$

By (3.3), it follows easily that $g \in L_p(\mathbb{R}^n)$. Since

$$f \circ g = \sum_{k\geq 1} f(a_k)\chi_{B_k}\,,$$

(3.3) implies again $f \circ g \notin L_p(\mathbb{R}^n)$, a contradiction. □

Remark 1. The above proof works as well in case of $L_p(A)$, for any measurable subset A of $\mathbb{R}^n$ such that $|A| = +\infty$. For the generalization of Proposition 3.1 to L_p spaces on abstract measure spaces, we refer to [3, Theorem 3.1].

In case of linear operators on normed spaces, it is well known that boundedness is equivalent to continuity. Of course that does not hold for nonlinear ones. In particular, composition operators can be bounded but not continuous.

Proposition 3.2. *Assume $1 \leq p \leq +\infty$. Let (X, μ) be a measure space. Assume that (X, μ) is non trivial, i.e. there exists a measurable set A in X such that $0 < \mu(A) < +\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be such that T_f takes $L_p(X, \mu)$ to itself. If T_f is continuous from $L_p(X, \mu)$ to itself, then f is continuous.*

Proof. Assume that T_f is continuous from $L_p(X, \mu)$ to itself. Without loss of generality, assume $f(0) = 0$. Let A be as in the above statement. For all real numbers u, v,

$$f \circ u\chi_A - f \circ v\chi_A = (f(u) - f(v))\chi_A\,,$$

hence

$$\|f \circ u\chi_A - f \circ v\chi_A\|_p = |f(u) - f(v)|\,\mu(A)^{1/p}\,. \tag{3.4}$$

Clearly

$$\lim_{v\to u} v\chi_A = u\chi_A \quad \text{in } L_p\,.$$

By continuity of T_f, and by (3.4), we obtain the continuity of f. □

By Propositions 3.1 and 3.2, it follows that, in case of $L_p(\mathbb{R}^n)$, there exist bounded composition operators which are not continuous. Proposition 3.2 admits a converse statement :

Proposition 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for some constant $c > 0$, $|f(t)| \leq c\,|t|$, for all $t \in \mathbb{R}$. Let (X, μ) be a measure space and let $1 \leq p < +\infty$. Then T_f is continuous from $L_p(X, \mu)$ to itself.*

Proof. It suffices to prove the following : for all sequence (g_j) converging to g in $L_p(X, \mu)$, there exists a subsequence (g_{j_k}) such that $(f \circ g_{j_k})$ converges to $f \circ g$ in $L_p(X, \mu)$. By the classical measure theoretic result (see, for instance, the proof of Theorem 3.11 in [15]), there exists a subsequence (g_{j_k}) and a function $h \in L_p(X, \mu)$ such that

$$g_{j_k} \to g \quad a.e.\,, \quad |g_{j_k}| \leq h\,.$$

By the continuity of f, it holds $f \circ g_{j_k} \to f \circ g$ a.e.. By the assumption on f,

$$|f \circ g_{j_k} - f \circ g| \leq 2c\,h\,.$$

By the Lebesgue dominated convergence Theorem, we conclude that $\|f \circ g_{j_k} - f \circ g\|_p$ tends to 0. □

Remark 2. If $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, T_f is easily seen to be continuous from $L_\infty(X, \mu)$ to itself. The details are left to the reader.

4 Automatic boundedness

Definition 2. Let E be a normed space. A mapping $T : E \to E$ is said *bounded* if, for all bounded set A of E, the set $T(A)$ is bounded.

For instance, according to estimate (3.2), any composition operator, which sends $L_p(\mathbb{R}^n)$ to itself, is bounded on $L_p(\mathbb{R}^n)$. More generally, for all "reasonable" function space, a weak form of boundedness is satisfied by composition operators. Thus we have a kind of automatic boundedness for a large class of function spaces.

Proposition 4.1. *Let E, F be vector subspaces of $L_{1,loc}(\Omega)$. Assume that*

- *E and F are endowed with complete norms such that the embeddings of E and F into $L_{1,loc}(\Omega)$ are continuous.*
- *$\mathcal{D}(\Omega)$ is embedded into E.*
- *$\varphi g \in F$, for all $\varphi \in \mathcal{D}(\Omega)$ and $g \in F$.*

For all $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $T_f(E) \subseteq F$, there exist a closed ball $B \subset \Omega$ and two numbers $c_1, c_2 > 0$ such that, for all $g \in E$,

$$\|g\|_E \leq c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq B \qquad \Rightarrow \qquad \|f \circ g\|_F \leq c_2\,. \tag{4.1}$$

Proof. By contradiction, assume that, for all B, c_1, c_2 there exists $g \in E$ such that

$$\|g\|_E \leq c_1\,, \quad \operatorname{supp} g \subseteq B\,, \quad \|f \circ g\|_F > c_2\,. \tag{4.2}$$

Consider a sequence $(B_j)_{j\geq 1}$ of disjoint closed balls in Ω. Take functions $\varphi_j \in \mathcal{D}(\Omega)$ such that $\varphi_j(x) = 1$ on $\frac{1}{2}B_j$ (the ball of the same center and half radius than B_j) and $\varphi_j(x) = 0$ out of B_j. It is easily seen (Closed Graph Theorem, see [17, Chapter II, §6, Theorem 1] or [14, Theorem 2.15]) that, for $\varphi \in \mathcal{D}(\Omega)$, the linear multiplication operator $g \mapsto \varphi g$ is bounded on F. Thus we can consider

$$M_j := \sup\{\|\varphi_j g\|_F \,:\, \|g\|_F \leq 1\}\,.$$

According to (4.2), there exist functions g_j such that

$$\|g_j\|_E \leq 2^{-j}\,, \quad \operatorname{supp} g_j \subseteq \frac{1}{2}B_j\,, \quad \|f \circ g_j\|_F > jM_j\,.$$

Let $g := \sum_j g_j$. Clearly $g \in E$ and, by the embedding $E \hookrightarrow L_{1,loc}(\Omega)$,

$$g(x) = \sum_{j \geq 0} g_j(x) \quad \text{a.e.}\,.$$

By considering supports, $\varphi_j(f \circ g) = f \circ g_j$, hence

$$jM_j \leq \|\varphi_j(f \circ g)\|_F \leq M_j \|f \circ g\|_F$$

for any $j \geq 1$, a contradiction. □

Remark 3. If $\Omega = \mathbb{R}^n$, and if E is translation and dilation invariant, the conclusion of Proposition 4.1 can be improved : indeed *for all* balls or cubes B, there exist $c_1, c_2 > 0$ such that (4.1) holds for all $g \in E$.

As an example of use of Proposition 4.1, we give the following variant of Proposition 3.1 :

Proposition 4.2. *Let $1 \leq p < +\infty$, let Ω be an open subset of $\mathbb{R}^n$ such that $|\Omega| < +\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then f acts on $L_p(\Omega)$ if and only if there exist $\alpha, \beta > 0$ such that*

$$|f(t)| \leq \alpha |t| + \beta\,, \quad \text{for all } t \in \mathbb{R}\,. \tag{4.3}$$

Proof. Since the sufficiency of condition (4.3) is clear, we deal only with necessity. Assume that f acts on $L_p(\Omega)$. Without loss of generality, we can assume that $f(0) = 0$. By Proposition 4.1, there exist a cube $Q' \subset \Omega$ and two numbers $c_1, c_2 > 0$ such that, for all $g \in L_p(\Omega)$,

$$\|g\|_p \leq c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq Q' \qquad \Rightarrow \qquad \|f \circ g\|_p \leq c_2\,. \tag{4.4}$$

Let $b \in \Omega$ and $r > 0$ be such that $Q' = b + 2rQ$. For any $a \in \mathbb{R}$, and $0 < \varepsilon \leq 1$, let

$$g_{a,\varepsilon}(x) := a\rho\left(\frac{x-b}{r\varepsilon}\right)\,.$$

Then the support of $g_{a,\varepsilon}$ is contained in Q'. We choose ϵ depending on a in the following way.

In the case of large a, more precisely if $|a| \geq R := r^{-n/p} c_1 \|\rho\|_p^{-1}$, we choose ε such that

$$|a|\, r^{n/p} \|\rho\|_p\, \varepsilon^{n/p} = c_1\,. \tag{4.5}$$

If $|a| < R$, we take $\varepsilon = 1$. In both cases, we obtain $\|g_{a,\varepsilon}\|_p \leq c_1$, hence $\|f \circ g_{a,\varepsilon}\|_p \leq c_2$. Since

$$\rho\left(\frac{x-b}{r\varepsilon}\right) = 1$$

on the cube $b + \varepsilon r Q$, this implies

$$\int_{b+\varepsilon r Q} |f(a)|^p \,\mathrm{d}x \leq c_2^p\,,$$

hence $|f(a)|^p \varepsilon^n \leq c_3$, for some constant c_3.

If $|a| \geq R$, by using (4.5), we obtain $|f(a)| \leq c_4 |a|$, for some constant c_4. If $|a| < R$, we obtain $|f(a)| \leq c_3^{1/p}$. □

Remark 4. The above proof can be viewed as a prototype of a number of results on composition operators, as we will see further.

5 Definition and main properties of Sobolev spaces

Definition 3. Let $f \in L_{1,loc}(\mathbb{R}^n)$, and $\alpha \in \mathbb{N}^n$. We say that f has a weak derivative of order α if there exists $g \in L_{1,loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\varphi^{(\alpha)}(x)\,\mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

If such g exists, it is easily seen to be unique, up to equality almost everywhere; then we denote it by $f^{(\alpha)}$ and we call it the *weak derivative* of f of order α.

Definition 4. *Let $m \in \mathbb{N}$ and $1 \le p \le +\infty$. The Sobolev space $W_p^m(\mathbb{R}^n)$ is the set of functions $f \in L_{1,loc}(\mathbb{R}^n)$ such that, for all $|\alpha| \le m$, $f^{(\alpha)}$ exists in the weak sense, and $f^{(\alpha)} \in L_p(\mathbb{R}^n)$.*

$W_p^m(\mathbb{R}^n)$ is a vector subspace of $L_p(\mathbb{R}^n)$. It will be endowed with the following norm :

$$\|f\|_{W_p^m(\mathbb{R}^n)} := \sum_{|\alpha| \le m} \|f^{(\alpha)}\|_p \,. \tag{5.1}$$

We give here some useful properties of Sobolev spaces.

First of all, $W_p^m(\mathbb{R}^n)$ is a function space which satisfies the assumptions of Proposition 4.1, see [1, Theorem 3.3].

The behavior of (5.1) with respect to dilations is described in the following assertion, with a simple proof :

Proposition 5.1. *It holds*

$$\|f(\lambda(.))\|_{W_p^m(\mathbb{R}^n)} \le \lambda^{m-(n/p)} \|f\|_{W_p^m(\mathbb{R}^n)}\,,$$

for all $\lambda \ge 1$.

Then we have the so-called *Sobolev embedding theorems*, see [1, Theorem 4.12] :

Proposition 5.2. *If*

$$m_1 - m_2 \ge \frac{n}{p_1} - \frac{n}{p_2} > 0,$$

then $W_{p_1}^{m_1}(\mathbb{R}^n) \hookrightarrow W_{p_2}^{m_2}(\mathbb{R}^n)$.

In particular $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ if $m > n/p$. In fact, we have a more precise statement, where $C_b(\mathbb{R}^n)$ denotes the space of bounded continuous functions on $\mathbb{R}^n$:

Proposition 5.3. *If $m > n/p$, or $p = 1$ and $m = n$, then $W_p^m(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.*

The assumptions on the parameters are sharp : $W_p^m(\mathbb{R}^n)$ is not embedded in $L_\infty(\mathbb{R}^n)$ in the case $m < n/p$, or $m = n/p$ and $p > 1$.

Remark 5. The elements of $W_p^m(\mathbb{R}^n)$ are equivalence classes of functions with respect to the equality almost everywhere. Thus the precise meaning of Proposition 5.3 is the following : all $f \in W_p^m(\mathbb{R}^n)$ contains a (necessarily unique) bounded continuous representative such that $\|f\|_\infty \le c\|f\|_{W_p^m}$, for some constant $c > 0$ depending only on m, p, n.

Proposition 5.4. *If $m > n/p$, or $m = n$ and $p = 1$, the Sobolev space $W_p^m(\mathbb{R}^n)$ is a subalgebra of $C_b(\mathbb{R}^n)$.*

See [1, Theorem 4.39] for the proof.

6 Necessity of Lipschitz continuity

In case $m \geq 1$, any function which acts on W_p^m by composition is necessarily Lipschitz continuous, at least locally. This is a major distinction with the case of L_p. To prove this property, we need some preliminary results.

Lemma 6.1. *Assume that $W_p^m(\mathbb{R}^n)$ is not embedded into $L_\infty(\mathbb{R}^n)$. There exists a sequence $(\theta_j)_{j\geq 1}$ in $\mathcal{D}(\mathbb{R}^n)$ such that*

$$\theta_j(x) = 1 \quad \text{on} \quad 2^{-j}Q\,, \quad \operatorname{supp}\theta_j \subseteq Q\,, \quad \lim_{j\to+\infty} \|\theta_j\|_{W_p^m(\mathbb{R}^n)} = 0\,.$$

Proof. In case $m < n/p$, we take $\theta_j(x) = \rho(2^j x)$. By Proposition 5.1,

$$\|\theta_j\|_{W_p^m(\mathbb{R}^n)} \leq 2^{j(m-(n/p))}\|\rho\|_{W_p^m(\mathbb{R}^n)}\,.$$

Thus the sequence (θ_j) has the desired properties.

Now assume that $m = n/p$ and $1 < p < +\infty$. Let

$$\theta_j(x) := \frac{1}{j}\sum_{k=1}^{j} \rho(2^k x)\,.$$

If $|\alpha| = m$, then the function $x \mapsto \rho^{(\alpha)}(2^k x)$ has support in the set $S_k := 2^{-k+1}Q \setminus 2^{-k}Q$. Thus, for all $1 \leq k \leq j$ and $x \in S_k$,

$$\left|\theta_j^{(\alpha)}(x)\right| = \frac{1}{j}2^{mk}\left|\rho^{(\alpha)}(2^k x)\right| \leq cj^{-1}2^{mk}\,.$$

Hence

$$\|\theta_j^{(\alpha)}\|_p^p = \sum_{k=1}^{j}\int_{S_k}\left|\theta_j^{(\alpha)}(x)\right|^p \mathrm{d}x \leq cj^{-p}\sum_{k=1}^{j} 2^{kmp}2^{-nk} = cj^{1-p}\,.$$

Thus the sequence $(\|\theta_j^{(\alpha)}\|_p)$ tends to 0 for all $|\alpha| = m$. The same holds, with a simple proof, for $|\alpha| < m$. □

Lemma 6.2. *Define the sequence of functions $(B_m)_{m\geq 1}$ in $L_1(\mathbb{R})$ by $B_1 := \mathbf{1}_{[0,1]}$ and $B_{m+1} := B_m * B_1$ for all m. Then*

$$\Delta_h^m f(x) = \int_{-\infty}^{+\infty} B_m(t)\left(\sum_{|\alpha|=m}\frac{m!}{\alpha!}f^{(\alpha)}(x+th)\,h^\alpha\right)\mathrm{d}t\,, \tag{6.1}$$

for almost all $x \in \mathbb{R}^n$, all $h \in \mathbb{R}^n$, all $m \geq 1$ and all $f \in W_p^m(\mathbb{R}^n)$.

Proof. We consider the case of an m times continuously differentiable function f. An approximation procedure will complete the proof in general case.

Step 1 : case $n = 1$. In this case, formula (6.1) reduces to

$$\Delta_h^m f(x) = \int_{-\infty}^{+\infty} B_m(t) f^{(m)}(x+th)\,h^m \mathrm{d}t\,. \tag{6.2}$$

We prove it by induction. The case $m = 1$ is well known. Assuming that (6.2) holds, we obtain

$$\Delta_h^{m+1} f(x) = h^m \int_{-\infty}^{+\infty} B_m(t)\Delta_h f^{(m)}(x+th)\,\mathrm{d}t$$

$$= h^{m+1}\int_{-\infty}^{+\infty} B_m(t)\left(\int_{-\infty}^{+\infty} B_1(s) f^{(m+1)}(x+(t+s)h)\,\mathrm{d}s\right)\mathrm{d}t\,.$$

By Fubini, a change of variable, and the definition of B_m, we obtain formula (6.2) at rank $m+1$.

Step 2 : general case. We fix x, h in $\mathbb{R}^n$, and set $g(t) := f(x + t(h/|h|)))$ for all $t \in \mathbb{R}$. Then $\Delta_h^m f(x) = \Delta_{|h|}^m g(0)$. Applying Step 1 to the function g, we obtain (6.1). The details are left to the reader. □

Lemma 6.3. *For all $m \geq 1$, $1 \leq p \leq \infty$, there exists $c > 0$ such that*

$$\left(\int_{\mathbb{R}^n} |\Delta_h^m f(x)|^p\,\mathrm{d}x\right)^{1/p} \leq c|h|^m \|f\|_{W_p^m(\mathbb{R}^n)}$$

for all $h \in \mathbb{R}^n$ and all $f \in W_p^m(\mathbb{R}^n)$.

Proof. By definition of B_m, $B_m \geq 0$ and $\int_{-\infty}^{+\infty} B_m(t)\,\mathrm{d}t = 1$. Applying (6.1), we obtain

$$\|\Delta_h^m f\|_p \leq |h|^m \sum_{|\alpha|=m} \frac{m!}{\alpha!}\|f^{(\alpha)}\|_p\,.$$

□

Theorem 6.1. *Assume that $m \geq 1$ and that $W_p^m(\mathbb{R}^n)$ is not embedded into $L_\infty(\mathbb{R}^n)$. Then any function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W_p^m(\mathbb{R}^n)$ to itself, is Lipschitz continuous on $\mathbb{R}$.*

Proof. Throughout the proof, $\|.\|$ will denote the norm in $W_p^m(\mathbb{R}^n)$.

Step 1 : construction of the comb-shaped function. This construction was first introduced by S. Igari [11]. Let $A_N := \mathbb{Z}^n \cap [-N, N]^n$, for every positive integer N. We fix a real number s such that

$$0 < s < \frac{1}{2m+1}\,. \tag{6.3}$$

Let b, b' be real numbers. Then we consider integers $N, j \geq 1$, and a real number $r > 0$, whose values will be fixed depending on b, b'. Our test function will be defined by

$$g(x) := \sum_{\mu\in A_N} \rho\left(\frac{1}{s}\left(\frac{x}{r} - \mu\right)\right)(b' - b) + \theta_j(x)\,b\,. \tag{6.4}$$

The first condition on parameters will be

$$3rN \leq 2^{-j}\,. \tag{6.5}$$

By inequality $s < 1/2$ and by condition (6.5), we deduce that the cubes $r(2sQ + \mu)$ are disjoint, and that $r(2sQ + \mu) \subset r(Q + \mu) \subset 2^{-j}Q$, if $\mu \in A_N$. Hence

$$g(x) = b'\,,\quad \text{if } x \in r(sQ + \mu) \text{ for some } \mu \in A_N\,, \tag{6.6}$$

$$g(x) = b\,,\quad \text{if } x \in 2^{-j}Q \setminus \bigcup_{\mu\in A_N} r(2sQ + \mu)\,. \tag{6.7}$$

By (6.5), we have $r \leq 1$. Then Proposition 5.1 gives us

$$\Big\|\sum_{\mu\in A_N} \rho\Big(\frac{1}{s}\Big(\frac{\cdot}{r} - \mu\Big)\Big)\Big\| \leq c_1 r^{(n/p)-m} N^{n/p}\,, \tag{6.8}$$

for some constant c_1.

Step 2 : adjustment of parameters. Now we assume that f acts on $W^m_p(\mathbb{R}^n)$ by composition. By Proposition 4.1, we can find constants δ_1, δ_2 such that $\|f \circ u\| \leq \delta_2$ for every function u such that $\|u\| \leq \delta_1$, and u has support in Q. In order to apply this property to $u = g$, we need the following inequalities :

$$|b|\,\|\theta_j\| \leq \frac{\delta_1}{2}\,, \tag{6.9}$$

$$\frac{\delta_1}{3c_1|b-b'|} \leq r^{(n/p)-m} N^{n/p} \leq \frac{\delta_1}{2c_1|b-b'|}\,. \tag{6.10}$$

Now we discuss the choice of j, N, r with respect to b, b', such that conditions (6.5), (6.9) and (6.10) hold. First, we choose $j = j(b) \geq 1$ such that (6.9) holds. This is possible by Lemma 6.1. In the case $m < n/p$, we define

$$r := \left(\frac{\delta_1}{2c_1|b-b'|} N^{-n/p}\right)^{\frac{p}{n-mp}}\,,$$

which ensures condition (6.10) ; since

$$rN = \left(\frac{\delta_1}{2c_1|b-b'|}\right)^{\frac{p}{n-mp}} N^{\frac{mp}{pm-n}}\,,$$

condition (6.5) holds for all sufficiently large N, depending on $|b-b'|$.

In the case $m = n/p$, we take N such that (6.10) holds. Such a choice is possible if $|b - b'| \leq c_2$, where $c_2 > 0$ depends only on p, n, δ_1. Then we put $r := 2^{-j}/3N$.

Step 3 : end of the proof. By combining inequalities (6.8), (6.9) and (6.10), we deduce $\|g\| \leq \delta_1$. Using Lemma 6.3, we obtain

$$\|\Delta^m_h(f\circ g)\|_p \leq \delta_3 |h|^m\,,$$

for all $h \in \mathbb{R}^n$, where δ_3 depends only on δ_2, m, n, p. Let $Q^+ :=]0, 1/2]^n$ and $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^n$. By condition (6.3) we have

$$x + \ell r s e_1 \in r(Q+\mu) \subset 2^{-j}Q \quad (\ell = 0, \ldots, m)\,,$$

$$x + \ell r s e_1 \notin \bigcup_{\mu' \in A_N} r(2sQ + \mu')\,, \quad (\ell = 1, \ldots, m)\,,$$

for all $x \in r(sQ^+ + \mu)$; for such x, equalities (6.6) and (6.7), and formula (2.1), imply that

$$\left|\Delta^m_{rse_1}(f\circ g)(x)\right| = |f(b') - f(b)|\,.$$

Hence

$$\begin{aligned}\delta_3 &\geq c_3 r^{-m}\left(\sum_{\mu\in A_N}\int_{r(sQ^+ + \mu)} \left|\Delta^m_{rse_1}(f\circ g)(x)\right|^p \mathrm{d}x\right)^{1/p}\\ &\geq c_4 |f(b') - f(b)|\, N^{n/p} r^{(n/p)-m}\,.\end{aligned}$$

By (6.10) we obtain the existence of a constant δ_4 such that $|f(b') - f(b)| \leq \delta_4 |b - b'|$ for all $b, b' \in \mathbb{R}$ satisfying $|b' - b| \leq c_2$. Thus f is uniformly Lipschitz continuous. □

Theorem 6.2. *Assume that $m \geq 1$. Then any function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W^m_p(\mathbb{R}^n)$ to itself, is locally Lipschitz continuous on $\mathbb{R}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which acts on $W^m_p(\mathbb{R}^n)$. Let $a \in \mathbb{R}$. We introduce a localized version of T_f with the help of the following statement :

Lemma 6.4. *There exists a nonlinear operator U_a which sends $W^m_p(\mathbb{R}^n)$ to itself, such that, for all $g \in W^m_p(\mathbb{R}^n)$,*

$$U_a g(x) = f(a + g(x)) - f(a)\,, \quad \text{for all} \quad x \in Q\,,$$

$$\|g\|_{W^m_p(\mathbb{R}^n)} \le \delta_1 \quad \text{and} \quad \operatorname{supp} g \subseteq Q \qquad \Rightarrow \qquad \|U_a g\|_{W^m_p(\mathbb{R}^n)} \le \delta_2\,.$$

The proof is essentially the same as that of Proposition 4.1, see [6, Lemma 1] for details.

Returning to the proof of Theorem 6.2, we argue in the same way as in the proof of Theorem 6.1, just replacing T_f by U_a. We define g by (6.4), with $\theta_j(x)$ replaced by $\rho(2x)$, $s = 1/4$ and $r = 1/6N$. Inequality (6.9) becomes $|b| \le \delta_3$, for some constant δ_3 depending only on δ_1. The double inequality (6.10) reduces to

$$\frac{\delta_4}{|b - b'|} \le N^m \le \frac{\delta_5}{|b - b'|}\,, \tag{6.11}$$

where δ_4, δ_5 depend on δ_1 and c_1. If $|b - b'| \le \delta_4$, we can choose N satisfying (6.11). We obtain a constant δ_6 such that

$$|f(a + b) - f(a + b')| \le \delta_6 |b - b'|\,,$$

for b, b' satisfying $|b| \le \delta_3$ and $|b - b'| \le \delta_4$. Thus f is Lipschitz continuous in a neighborhood of a. □

An easy modification of the above proof gives us the following statement :

Proposition 6.1. *Let us assume $m \ge 3$, and define p_1 by :*

$$2 - \frac{n}{p_1} := m - \frac{n}{p}\,. \tag{6.12}$$

Then every function $f : \mathbb{R} \to \mathbb{R}$, such that T_f sends $W^m_p(\mathbb{R}^n)$ to $W^2_{p_1}(\mathbb{R}^n)$, is locally Hölder continuous of order $2/m$.

7 A case of degeneracy: Dahlberg Theorem

As announced in Introduction, Sobolev spaces provide simple examples of spaces for which the answer to question $\mathcal{Q}_1$ is negative.

Theorem 7.1. *Assume that m is an integer satisfying*

$$1 + \frac{1}{p} < m < \frac{n}{p}\,. \tag{7.1}$$

Then, for each function $f : \mathbb{R} \to \mathbb{R}$ which acts on $W^m_p(\mathbb{R}^n)$ by composition, there exists $c \in \mathbb{R}$ such that $f(t) = ct$ for all $t \in \mathbb{R}$.

This theorem was first proved by B. Dahlberg [9] for smooth functions f. Indeed, we have a slightly stronger property :

Proposition 7.1. *Under condition (7.1), let us define p_1 by condition (6.12). Then, for each function $f : \mathbb{R} \to \mathbb{R}$ such that T_f sends $W^m_p(\mathbb{R}^n)$ to $W^2_{p_1}(\mathbb{R}^n)$, there exists $c \in \mathbb{R}$ such that $f(t) = ct$ for all $t \in \mathbb{R}$.*

By Proposition 5.2, we have $W^m_p(\mathbb{R}^n) \hookrightarrow W^2_{p_1}(\mathbb{R}^n)$. Thus Theorem 7.1 follows by Proposition 7.1.

Proof. Step 1. We assume first that f is of class C^2. Since $W^m_p(\mathbb{R}^n)$ does not contain nonzero constant functions, we have $f(0) = 0$. By Proposition 4.1, there exist two numbers $c_1, c_2 > 0$ such that, for all $g \in W^m_p(\mathbb{R}^n)$,

$$\|g\|_{W^m_p(\mathbb{R}^n)} \leq c_1 \quad \text{and} \quad \operatorname{supp} g \subseteq 2Q \qquad \Rightarrow \qquad \|f \circ g\|_{W^2_{p_1}(\mathbb{R}^n)} \leq c_2 \,. \tag{7.2}$$

Define the function $u \in \mathcal{D}(\mathbb{R}^n)$ by

$$u(x) := x_1 \rho(x) \,, \tag{7.3}$$

where x_1 denotes the first coordinate of $x \in \mathbb{R}^n$. Let $a > 0$, and $0 < \varepsilon \leq 1$ (a number to be determined with respect to a). Let us define $g_a \in \mathcal{D}(\mathbb{R}^n)$ by

$$g_a(x) := a u\left(\frac{x}{\varepsilon}\right) .$$

Then $\operatorname{supp} g_a \subset 2Q$, and $\|g_a\|_{W^m_p(\mathbb{R}^n)} \leq c_1$ if

$$a\, \varepsilon^{(n/p)-m} \|u\|_{W^m_p(\mathbb{R}^n)} = c_1 \,. \tag{7.4}$$

Due to the condition $m < n/p$, the above equality determines ε as a function of a, if a is sufficiently large. Hence it holds $\|f \circ g_a\|_{W^2_{p_1}(\mathbb{R}^n)} \leq c_2$ for all large numbers a. Since

$$(f \circ g_a)(x) = f\left(\frac{a}{\varepsilon} x_1\right) , \quad x \in \varepsilon Q \,,$$

we deduce that

$$\left(\frac{a}{\varepsilon}\right)^{2p_1} \int_{\varepsilon Q} \left| f''\left(\frac{a}{\varepsilon} x_1\right)\right|^{p_1} \mathrm{d}x \leq c_2^{p_1} \,.$$

By using (7.4) and a change of variable, we obtain a constant $c_3 > 0$ such that

$$a^{p_1 - 1} \int_{-a/2}^{+a/2} |f''(t)|^{p_1} \, \mathrm{d}t \leq c_3 \,, \tag{7.5}$$

for all large numbers a. By the assumption $m > 1 + (1/p)$, we have $p_1 > 1$. If we take a to $+\infty$, we deduce that

$$\int_{-\infty}^{+\infty} |f''(t)|^{p_1} \, \mathrm{d}t = 0 \,.$$

Hence $f''(t) = 0$ almost everywhere. Since f'' is continuous, we conclude that $f(t) = ct$, for some constant c.

Step 2. We turn now to the general case. By Theorem 6.2 and Proposition 6.1, we know that f is continuous. Let $\omega \in \mathcal{D}(\mathbb{R})$, with support in $[-1, +1]$, even, such that $\int \omega(t)\, \mathrm{d}t = 1$. Let us set $\omega_j(t) := j\omega(jt)$ for all positive integers j. The convolution $\omega_j * f$ is defined, and it is a smooth function. Let us define

$$f_j(t) := (\omega_j * f)(t) - (\omega_j * f)(0) \,.$$

For all function g with support in Q,

$$(f_j \circ g)(x) = \rho(x) \int_{\mathbb{R}} (f((g(x) + t)\rho(x)) - f(t\rho(x)))\, \omega_j(t) \, \mathrm{d}t$$

for all $x \in \mathbb{R}^n$. In other words :

$$\operatorname{supp} g \subseteq Q \quad \Rightarrow \quad f_j \circ g = \rho \int_{\mathbb{R}} (f \circ ((g + t)\rho) - f \circ (t\rho))\, \omega_j(t) \, \mathrm{d}t \,. \tag{7.6}$$

Let $M := \sup\{\|\rho\, h\|_{W_p^m(\mathbb{R}^n)} \,:\, \|h\|_{W_p^m(\mathbb{R}^n)} \leq 1\}$. Let j_0 be the first integer such that

$$j_0 \geq 2c_1^{-1}\|\rho\|_{W_p^m(\mathbb{R}^n)}\,.$$

Let g be such that $\operatorname{supp} g \subseteq Q$ and

$$\|g\|_{W_p^m(\mathbb{R}^n)} \leq \frac{c_1}{2M}\,.$$

Then, for all $j \geq j_0$, and all $|t| \leq 1/j$, it holds

$$\|(g+t)\rho\|_{W_p^m(\mathbb{R}^n)} \leq c_1\,.$$

By (7.2), we obtain

$$\|f_j \circ g\|_{W_{p_1}^2(\mathbb{R}^n)} \leq 2Mc_2$$

for all $j \geq j_0$. All together, we have obtained constants $c_3, c_4 > 0$ such that

$$\|g\|_{W_p^m(\mathbb{R}^n)} \leq c_3 \quad \text{and} \quad \operatorname{supp} g \subseteq Q \qquad \Rightarrow \qquad \|f_j \circ g\|_{W_{p_1}^2(\mathbb{R}^n)} \leq c_4\,, \tag{7.7}$$

for all $j \geq j_0$. Reasoning as in Step 1, we conclude that, for some constants a_j, $j \geq j_0$, we have $f_j(t) = a_j t$ for all $t \in \mathbb{R}$. Thus we obtain

$$(\omega_j * f)(t) = (\omega_j * f)(0) + a_j t$$

for all $t \in \mathbb{R}$. Since f is continuous, we know that $\lim_{j\to+\infty}(\omega_j * f)(t) = f(t)$ for all $t \in \mathbb{R}$. Taking $t = 1$, we obtain $\lim_{j\to+\infty} a_j = f(1)$. We conclude that $f(t) = f(1)t$ for all $t \in \mathbb{R}$. □

8 Composition operators on W_p^1

First of all, we recall a classical result :

Theorem 8.1. *For all $f : \mathbb{R} \to \mathbb{R}$, the following properties are equivalent :*

(1) f is Lipschitz continuous,

(2) f has a weak derivative in $L_\infty(\mathbb{R})$,

(3) There exists $g \in L_\infty(\mathbb{R})$ and a constant $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} \quad f(x) = \int_0^x g(t)\,\mathrm{d}t + c\,.$$

Proof. The implication (3) ⇒ (1) is straightforward. The equivalence (2) ⇔ (3) is easy to prove. Concerning (1) ⇒ (3), we refer to [10, Theorem 7.18] (Alternatively, we can observe that any Lipschitz continuous function is absolutely continuous, then apply [15, Theorem 8.17]). □

Theorem 8.2. *Let $f : \mathbb{R} \to \mathbb{R}$, such that $f(0) = 0$. Then f acts on $W_p^1(\mathbb{R}^n)$ if and only if*

- *f is Lipschitz continuous, if $W_p^1(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R})$,*
- *f is locally Lipschitz continuous, if $W_p^1(\mathbb{R}^n) \subset L_\infty(\mathbb{R})$.*

This theorem is due to Marcus and Mizel [12]. Roughly speaking, sufficiency result relies upon the formula $\partial_j(f \circ g) = (f' \circ g)\partial_j g$. In the case $W_p^1(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, we just need that f' belongs to L_∞ on the range of g. The necessity of the Lipschitz conditions follows by Theorems 6.1 and 6.2.

9 Full description of acting functions in higher order Sobolev spaces

Let us give a sufficient condition for composition :

Theorem 9.1. *Assume that* $m \geq \max(2, n/p)$*, or* $m = 2$*,* $p = 1$*. If a function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $f(0) = 0$ *and* $f' \in W_p^{m-1}(\mathbb{R})$*, then* f *acts on* $W_p^m(\mathbb{R}^n)$*.*

Proof. A preliminary remark : under the assumptions of Theorem 9.1, it holds

$$W_p^{m-1}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R}) .$$

That follows by Proposition 5.3.

Here we restrict ourselves to the case $m = 2$. The method that we use is typical of the general case. Also we assume that f is of class C^m, with bounded derivatives up to order m, and that g is smooth, with derivatives tending to 0 at infinity; see [4, 5] and [16, 5.2.4, Theorem 2] for the approximation procedure to cover the general case.

Let $g \in W_p^2(\mathbb{R}^n)$. We have to prove that the second order derivatives of $f \circ g$ belongs to $L_p(\mathbb{R}^n)$. It holds

$$\partial_j\partial_k(f \circ g) = (f'' \circ g)(\partial_j g)(\partial_k g) + (f' \circ g)\partial_j\partial_k g . \tag{9.1}$$

The second term belongs to L_p, because $f' \in L_\infty$. Thus we can concentrate on the first one. By the applying Cauchy-Schwarz inequality, we obtain

$$\|(f'' \circ g)\partial_j g\, \partial_k g\|_p \leq U_j^{1/2p} U_k^{1/2p} , \tag{9.2}$$

where

$$U_j := \int_{\mathbb{R}^n} |(f'' \circ g)(x)|^p |\partial_j g(x)|^{2p}\, \mathrm{d}x .$$

Let us introduce

$$h(x) := \int_x^{+\infty} |f''(t)|^p\, \mathrm{d}t .$$

Then $-U_j$ is equal to

$$\int_{\mathbb{R}^n} (h' \circ g)(x)\partial_j g(x)\partial_j g(x)|\partial_j g(x)|^{2p-2}\, \mathrm{d}x = \int_{\mathbb{R}^n} \partial_j(h \circ g)(x)\, \partial_j g(x)|\partial_j g(x)|^{2p-2}\, \mathrm{d}x .$$

An integration by parts gives

$$U_j = (2p-1) \int_{\mathbb{R}^n} (h \circ g)(x)\, \partial_j^2 g(x)|\partial_j g(x)|^{2p-2}\, \mathrm{d}x .$$

Hence

$$U_j \leq (2p-1)\|f''\|_p^p \int_{\mathbb{R}^n} |\partial_j^2 g(x)|\, |\partial_j g(x)|^{2p-2}\, \mathrm{d}x . \tag{9.3}$$

In case $p = 1$, the above inequality becomes $U_j \leq \|f''\|_1\, \|\partial_j^2 g\|_1$. That completes the proof of Theorem in the case $m = 2$, $p = 1$.

In case $p > 1$, we use the Hölder inequality to derive

$$U_j \leq (2p-1)\|f''\|_p^p\, \|\partial_j^2 g\|_p \left(\int_{\mathbb{R}^n} |\partial_j g(x)|^{2p}\, \mathrm{d}x \right)^{1-(1/p)} .$$

By Proposition 5.2 and condition $2 \geq n/p$, $W_p^2(\mathbb{R}^n) \hookrightarrow W_{2p}^1(\mathbb{R}^n)$. That completes the proof of Theorem 9.1. □

Remark 6. The above proof shows also that the composition operator is bounded under assumptions of Theorem 9.1. More precisely, there exist a constant $c = c(p, n) > 0$ such that

$$\|f \circ g\|_{W_p^2(\mathbb{R}^n)} \leq c\|f''\|_p \left(\|g\|_{W_p^2(\mathbb{R}^n)} + \|g\|_{W_p^2(\mathbb{R}^n)}^{2-(1/p)}\right) . \tag{9.4}$$

We turn now to the complete description of composition operators. Due to Theorems 7.1 and 8.2, we will consider only the case $m \geq 2$, together with the three following subcases :

- $m > n/p$, or $m = n$ and $p = 1$.
- $m = n/p$ and $p > 1$.
- $m = 2$, $p = 1$ and $n \geq 3$.

Theorem 9.2. *Let $m \geq 2$, $1 \leq p < +\infty$. If $m > n/p$, or if $m = n$ and $p = 1$, then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W_p^m(\mathbb{R}^n)$ if and only if $f(0) = 0$ and f belongs locally to $W_p^m(\mathbb{R})$.*

Proof. 1- Assume that f belongs locally to $W_p^m(\mathbb{R})$, and that $g \in W_p^m(\mathbb{R}^n)$. By Proposition 5.3, g is bounded. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(t) = 1$ on the range of g. Then $f \circ g = (\varphi\, f) \circ g$. Since $\varphi\, f \in W_p^m(\mathbb{R})$, we can apply Theorem 9.1, and conclude that $f \circ g \in W_p^m(\mathbb{R}^n)$.

2- Assume that T_f sends $W_p^m(\mathbb{R}^n)$ to itself. By considering $f \circ g$, where $g \in \mathcal{D}(\mathbb{R}^n)$ satisfies $g(x) = x_1$ on an arbitrary ball of $\mathbb{R}^n$, we conclude that f, together with all its derivatives up to order m, belongs to L^p on each bounded interval of $\mathbb{R}$. □

Theorem 9.3. *Let $m = n/p \geq 2$ and $p > 1$. Then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W_p^m(\mathbb{R}^n)$ if and only if $f(0) = 0$ and f' belongs locally uniformly to $W_p^{m-1}(\mathbb{R})$.*

Proof. The sufficiency of the condition on f follows by a modification of the proof of Theorem 9.1, see [4, 5] or [16, 5.2.4, Theorem 2].

To prove the necessity, we use the same ideas as in the proof of Theorem 6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which acts on $W_p^m(\mathbb{R}^n)$. We introduce constants δ_1, δ_2 as in the proof of Theorem 6.1. Let b be a real number. Let $j = j(b) \geq 1$ such that (6.9) holds. Let us consider the function

$$g_b(x) := \lambda u(2^j x) + \theta_j(x)\, b\,,$$

where u is the function introduced in (7.3), and λ is a constant, to be fixed below. By the assumption $m = n/p$, it holds $\|u(2^j(.))\| \leq \|u\|$. Thus, the choice of $\lambda := \delta_1/2\|u\|$ implies $\|g_b\| \leq \delta_1$. Hence we have

$$\|f \circ g_b\| \leq \delta_2\,. \tag{9.5}$$

On the cube $2^{-j}Q$, it holds $(f \circ g_b)(x) = f(\lambda 2^j x_1 + b)$, hence

$$\partial_1^m (f \circ g_b)(x) = \lambda^m 2^{jm} f^{(m)}(\lambda 2^j x_1 + b)\,.$$

Then using (9.5), a change of variable, and condition $m = n/p$, we find a constant $\delta_3 > 0$ such that

$$\int_{b-(\lambda/2)}^{b+(\lambda/2)} |f^{(m)}(y)|^p \,\mathrm{d}y \leq \delta_3\,,$$

for every $b \in \mathbb{R}$. Thus we have proved that $f^{(m)}$ belongs to $L_p(\mathbb{R})$ locally uniformly. Since we know yet that $f' \in L_\infty$, it follows easily that f' belongs to $W_p^{m-1}(\mathbb{R})$ locally uniformly. □

Theorem 9.4. *If $n \geq 3$, then a function $f : \mathbb{R} \to \mathbb{R}$ acts on $W_1^2(\mathbb{R}^n)$ if and only if $f(0) = 0$ and $f'' \in L_1(\mathbb{R})$.*

Proof. Sufficiency of $f'' \in L_1(\mathbb{R})$ follows by Theorem 9.1. To prove necessity, we proceed as in the proof of Theorem 7.1. Then the estimate (7.5) becomes

$$\int_{-a/2}^{+a/2} |f''(t)|\,\mathrm{d}t \le c_3\,,$$

for all large a. By taking $a \to +\infty$, we obtain $f'' \in L_1(\mathbb{R})$. □

10 Continuity of composition on Sobolev spaces

The more precise versions of Theorems 9.1, 9.2, 9.3 show that all the composition operators which send $W_p^m(\mathbb{R}^n)$ to itself are bounded. They are also continuous, according to the following :

Theorem 10.1. *Let m be an integer ≥ 1, $1 \le p < \infty$, and let $f : \mathbb{R} \to \mathbb{R}$. If f acts by composition on $W_p^m(\mathbb{R}^n)$, then the composition operator T_f is continuous from $W_p^m(\mathbb{R}^n)$ to itself.*

This theorem was proved step by step between 1976 and 2019 :

- for $m = 1$ and $p = 2$, by Ancona [2],
- for $m = 1$ and any p, by Marcus and Mizel [13],
- for $m > n/p$ and $1 < p < \infty$, by Lanza de Cristoforis and the author [6],
- in the general case by Moussai and the author [7], who proved also this "automatic" continuity on the so-called Adams-Frazier spaces $W_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, where $\dot{W}$ denotes the homogeneous Sobolev space, and on the spaces $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, conveniently realized.

References

[1] R. Adams, J. Fournier, *Sobolev spaces.* Elsevier (2003).

[2] A. Ancona, *Continuité des contractions dans les espaces de Dirichlet.* C.R.A.S. 282 (1976), 871–873, and Springer LNM 563 (1976), 1–26.

[3] J. Appell, P. Zabrejko, *Nonlinear superposition operators.* Cambridge U.P. (1990).

[4] G. Bourdaud, *Le calcul fonctionnel dans les espaces de Sobolev.* Invent. math. 104 (1991), 435–446.

[5] G. Bourdaud, *Superposition in homogeneous and vector valued Sobolev spaces.* Trans. Amer. Math. Soc. 362 (2010), 6105–6130.

[6] G. Bourdaud, M. Lanza de Cristoforis, *Regularity of the symbolic calculus in Besov algebras.* Studia Math. 184 (2008), 271–298.

[7] G. Bourdaud, M. Moussai, *Continuity of composition operators in Sobolev spaces.* Ann. I. H. Poincaré - AN 36 (2019), 2053–2063.

[8] G. Bourdaud, W. Sickel, *Composition operators on function spaces with fractional order of smoothness.* RIMS Kokyuroku Bessatsu B26 (2011), 93–132.

[9] B.E.J. Dahlberg, *A note on Sobolev spaces.* Proc. Symp. Pure Math. 35 (1979), no. 1, 183–185.

[10] J. Foran, *Fundamentals of real analysis.* Marcel Dekker (1991).

[11] S. Igari, *Sur les fonctions qui opèrent sur l'espace* $\widehat{A}^2$. Ann. Inst. Fourier 15 (1965), 525–536.

[12] M. Marcus, V.J. Mizel, *Complete characterization of functions which act via superposition on Sobolev spaces.* Trans. Amer. Math. Soc. 251 (1979), 187–218.

[13] M. Marcus, V.J. Mizel, *Every superposition operator mapping one Sobolev space into another is continuous.* J. Funct. Anal. 33 (1979), 217–229.

[14] W. Rudin, *Functional analysis.* McGraw-Hill (1973).

[15] W. Rudin, *Real and complex analysis.* 3rd ed., McGraw-Hill (1987).

[16] T. Runst, W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* de Gruyter, Berlin, 1996.

[17] K. Yosida. *Functional analysis.* 6th edition, Springer (1980).

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