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CALDERÓN – LOZANOVSKIĬ CONSTRUCTION FOR A COUPLE OF GLOBAL MORREY SPACES

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Communicated by V.S. Guliyev

Key words: Calderón-Lozanovskiĭ construction, global Morrey spaces, Banach ideal spaces, interpolation theorems in global Morrey spaces.

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Abstract. It is shown that under certain conditions on the parameters included in the definition of global Morrey spaces, the Calderón – Lozanovskiĭ construction on a pair of global Morrey spaces gives a space of the same type. Calculation of the Calderón – Lozanovskiĭ construction for a pair of global Morrey spaces allowed us to obtain new interpolation theorems even for classical global Morrey spaces.

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1 Introduction

In 1938, due to the applications in elliptic partial differential equations, Morrey [28] introduced a class of function spaces, nowadays named after him. In recent years, there is an increasing interest in applications of Morrey spaces in various areas of analysis, such as partial differential equations, potential theory and harmonic analysis; we refer, for example, to [1], [11], [12], [21], [25], [33], [35] and their references.

We begin with some basic notation from the theory of Morrey spaces.

Let μ be Lebesgue measure in $\mathbb{R}^n$, let $S(\mu, \mathbb{R}^n) = S(\mu)$ be the space of all Lebesgue measurable functions $x : \mathbb{R}^n \to \mathbb{R}$ and let $\chi(D)$ stand for the characteristic function of a set $D \subset \mathbb{R}^n$. Along with the Lebesgue spaces $L^p \equiv L^p(\mathbb{R}^n)$, $p \in [1, \infty]$ ideal spaces X are often used in harmonic analysis. Recall their definition (see, for example, [20], [24]).

A Banach space X of measurable functions on Ω is said to be ideal if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $|y(t)| \leq |x(t)|$ for almost all $t \in \Omega$ that $y \in X$ and $\|y|X\| \leq \|x|X\|$ (the symbol $\|x|X\|$ denotes the norm of an element x in the space X). Let $v \in S(\mu)$, $v > 0$ almost everywhere (v is a weight). We denote by the symbol X_v a new ideal space in which the norm is given by the equation $\|x|X_v\| = \|x \cdot v|X\|$. When $X = L^p$, our definition of weighted space differs somewhat from the often used one: when the weight is included in the measure.

Along with function spaces we need ideal spaces of sequences. Let $e^i = \{..., 0, 1, 0, ...\}$, ($i \in \mathbb{Z}$, the unit stands in the i-th place) be the standard basis in the space of two-side sequences. We denote by the symbol l an ideal space of sequences $x = \sum_{i=-\infty}^{\infty} x_i e^i$ ($x_i \in \mathbb{R}$) with the norm $\|x|l\|$. All the properties listed above for function spaces are preserved for sequence spaces. For details concerning the theory of sequence spaces, see [23].

The classical Morrey space M_{λ,L^p}, $(\lambda \in \mathbb{R})$ (see [28]), consists of all functions $x \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|x|M_{\lambda,L^p}\| = \sup_{t\in\mathbb{R}^n}\sup_{r>0} r^{-\lambda}\|x(t+.)\chi(B(0,r))|L^p\|.$$

We note that if $\lambda = 0$, then $M_{\lambda,L^p} = L^p$, if $\lambda = \frac{n}{p}$, then $M_{\lambda,L^p} = L^\infty$, if $\lambda < 0$ or $\lambda > \frac{n}{p}$, then M_{λ,L^p} consists only of functions equivalent to zero.

As a natural generalization of Lebesgue spaces, the interpolation properties of Morrey spaces became an interesting question. The first result on this problem is due to Stampacchia [34] and, independently, Campanato and Murthy [17]. They obtained an interpolation property for linear operators from Lebesgue spaces to Morrey spaces on $\mathbb{R}^n$ and showed that, if a linear operator T is bounded from $L^{q_i}(\mathbb{R}^n)$ to Morrey spaces $M_{\lambda_i,L^{p_i}}(\mathbb{R}^n)$ with the operator norm M_i, $i \in \{0,1\}$, then T is also bounded from $L^{q_\theta}(\mathbb{R}^n)$ to $M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R}^n)$ when

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \lambda_\theta = (1-\theta)\lambda_0 + \theta\lambda_1 \tag{1.1}$$

for some $\theta \in (0,1)$ with the operator norm not more than a positive constant multiple of $M_0^{1-\theta}M_1^\theta$. In 1969, Peetre [31] found that the previous conclusion still holds true when $(L^{q_0}(\mathbb{R}^n),\ L^{q_1}(\mathbb{R}^n))$ and $L^{q_\theta}(\mathbb{R}^n)$ are replaced, respectively, by a certain abstract pair (A_0, A_1) and an interpolation space A constructed from (A_0, A_1).

However, the converse result in general is not true. In 1995, Ruiz and Vega [32] proved that, when $n \geq 2$, $u \in (0,n)$, $\theta \in (0,1)$, $1 \leq p_2 < p_3 < \frac{n-1}{u} < p_1 < \infty$ and $\lambda_1 = \frac{1}{p_1} - \frac{1}{u}$, $\lambda_2 = \frac{1}{p_2} - \frac{1}{u}$, $\lambda_3 = \frac{1}{p_3} - \frac{1}{u}$ for any given $C \in (0,\infty)$, there exists a positive continuous linear operator $T : M_{\lambda_i,L^{p_i}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $i \in \{1,2,3\}$, with the operator norm satisfying $\|T|M_{\lambda_i,L^{p_i}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)\| \leq K_i$, $i \in \{1,2\}$, but $\|T|M_{\lambda,L^{p_3}}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)\| \geq CK_0^{1-\theta}K_1^\theta$ for $\frac{1}{p_3} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This implies the lack of convexity of operators on Morrey spaces.

In the case $n = 1$, Blasco, Ruiz and Vega [9] in 1999 proved that, for a particular u, if $1 < p_0 < p_1 < u < \infty$ and $\lambda_1 = \frac{1}{p_1} - \frac{1}{u}$, $\lambda_2 = \frac{1}{p_2} - \frac{1}{u}$, then there exist $q_0,\ q_1 \in (1,\infty)$ and a positive continuous linear operator T which is bounded from $M_{\lambda_i,L^{p_i}}(\mathbb{R})$ to $L^{q_i}(\mathbb{R})$, $i \in \{0,1\}$, but not bounded from $M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R})$ to $L^{q_\theta}(\mathbb{R})$ when conditions (1.1) are satisfied. These counterexamples show that Morrey spaces have no interpolation property in general.

Nevertheless, under some restriction, Morrey spaces also have some interpolation properties. Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ and p_θ, λ_θ be defined by (1.1). Recently, Lemarie-Rieusset [21], [22] showed that for $p_0,\ p_1,\ \lambda_0,\ \lambda_1,\ \theta,\ p_\theta$ and λ_θ as above,

$$[M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n)]_\theta = M_{\lambda_\theta,L^{p_\theta}}(\mathbb{R}^n) \tag{1.2}$$

if and only if

$$p_0\lambda_0 = p_1\lambda_1, \tag{1.3}$$

holds, which gives a necessary and sufficient condition ensuring the interpolation property of Morrey spaces on $\mathbb{R}^n$. Here, $[M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n)]_\theta$ denotes the space obtained using the first of Calderón interpolation methods [16] for a pair of Morrey spaces $(M_{\lambda_0,L^{p_0}}(\mathbb{R}^n), M_{\lambda_1,L^{p_1}}(\mathbb{R}^n))$.

Note that the situation changes radically for pairs of local Morrey spaces [13], [14], [15], [2], [3], [6]. For example, if in (1.2) global Morrey spaces are replaced by local Morrey spaces, then equality (1.2) will hold without restriction on the indices (1.3).

In this paper, we give a generalization of equality (1.2) to general Morrey spaces. Namely, for any functions $\varphi_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ each of which is concave, positively homogeneous of degree one, nondecreasing and continuous in each variable and such that $\varphi_i(0,0) = 0$, (i=0,1,2), the triple of spaces

$$\{\varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}),\quad \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1});\quad \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})\}.$$

has interpolation properties. Here, $\overline{\varphi}(t,s) \equiv \varphi_2(\varphi_0(t,s), \varphi_1(t,s))$, $(t,s \geq 0)$, $(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$ is a pair of general Morrey spaces, and $\varphi(X_0, X_1)$ denotes the space constructed from the pair of ideal spaces (X_0, X_1) using the construction of Calderón – Lozanovskiĭ. In particular, we show that for any concave function φ, the triple of spaces

$$\{M_{\lambda_0,L^{p_0}}(\mathbb{R}^n),\quad M_{\lambda_1,L^{p_1}}(\mathbb{R}^n),\quad \widetilde{\varphi}(M_{\lambda,L^1}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))\} \tag{1.4}$$

has interpolation properties, when condition (1.3) is met. Here, $\widetilde{\varphi}(t,s) \equiv \varphi(t^{\theta_0}s^{1-\theta_0}, t^{\theta_1}s^{1-\theta_1})$, $(\theta_0 = 1/p_0,\ \theta_1 = 1/p_1, \lambda = \lambda_0/\theta_0;\ t,s \geq 0)$.

Note that if instead of the triplet of global Morrey spaces (1.4) we consider the corresponding triplet of local Morrey spaces, then the triplet of local Morrey spaces will have the interpolation property not only when (1.3) is satisfied, but also in a much more general case [3], [6].

2 Basis constructions

We now replace the Lebesgue space L^p in the definition of the classical Morrey space by an ideal space X, the outer sup-norm by the norm in an any ideal space L and replace the balls $B(0,r)$ by homothetic sets $U(0,r) \subset \mathbb{R}^n$. Below, we always assume that $0 \in U(0,1)$ and $\mu(U(0,1))) \in (0,\infty)$. Moreover, we often assume that $U(0,1)$ is star-shaped with respect to the point 0, that is, if $t \in U(0,1)$, then $\gamma t \in U(0,1)$ for all $\gamma \in (0,1)$. In general, the star-shapedness assumption is not necessary, but sometimes is useful.

We also need local Morrey spaces constructed from a family of sets $\{U(0,r_i)\}$ with discretely varying parameter.

We denote by Υ the set of non-negative number sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions

$$\forall i: \qquad \tau_i < \tau_{i+1}, \qquad \bigcup_i (\tau_i, \tau_{i+1}] = R_+.$$

When $\tau_{i+1} = \infty$, we assume that $(\tau_i, \infty] = (\tau_i, \infty)$.

Definition 1. [2]. Let an ideal space X on $\mathbb{R}^n$, an ideal space l of two-sided sequences with the standard basis $\{e^i\}$ and a sequence $\tau \in \Upsilon$ be given. By Morrey space $M^\tau_{l,X}$ we mean the set of all functions $x \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|x|M^\tau_{l,X}\| = \sup_{t\in\mathbb{R}^n} \|\sum_{i=-\infty}^{\infty} e^i \|x(t+.)\chi(U(0,\tau_i))|X\||l\|.$$

The spaces introduced in Definition 1 are called global discrete Morrey spaces.

Discrete spaces are more convenient to consider at least for the following reasons. Firstly, all classical Morrey spaces can be realized as discrete Morrey spaces (see the example below), and secondly, one does not need to think about the measurability of the function $\|x(t+.)\chi(B(0,r))|X\|$.

Note that all discrete Morrey spaces are ideal.

The following example shows that most recently investigated Morrey spaces can be implemented as discrete Morrey spaces.

Example 1. Let $U(0,1)$ be a star-shaped set of a positive measure, $\lambda > 0$, $p \in [1,\infty]$, the ideal space X and the space $M_{\lambda,p;X}$, the norm in which is given by the equality

$$\|x|M_{\lambda,p;X}\| = \begin{cases} \sup_{t\in\mathbb{R}^n}\left(\int_0^\infty (r^{-\lambda}\|x(t+.)\chi(U(0,r))|X\|)^p \frac{dr}{r}\right)^{1/p}, & \text{for } p \in [1,\infty); \\ \sup_{t\in\mathbb{R}^n}\sup_r\{r^{-\lambda}\|x(t+.)\chi(U(0,r))|X\|\}, & \text{for } p = \infty \end{cases}$$

be given.

If $p \in [1,\infty)$, then for each function $x \in M_{\lambda,p;X}$ the following inequalities hold:

$$\sup_{t\in\mathbb{R}^n} 2^{-\lambda}(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p} \le \|x|M_{\lambda,p;X}\|$$

$$\le \sup_{t\in\mathbb{R}^n} 2^{\lambda}\cdot(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p}.$$

Thus, for $p \in [1,\infty)$ on the space $M_{\lambda,p;X}$ we can introduce an equivalent norm

$$\|x|M_{\lambda,p;X}\|_b = \sup_{t\in\mathbb{R}^n}(\sum_i (2^{-\lambda i}\|x(t+.)\chi(U(0,2^i))|X\|)^p)^{1/p}.$$

If $p = \infty$, then for each $x \in M_{\lambda,\infty;X}$ the following inequalities hold:

$$\sup_{t\in\mathbb{R}^n} 2^{-\lambda}\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\| \le \|x|M_{\lambda,\infty;X}\|$$

$$\le \sup_{t\in\mathbb{R}^n} 2^{\lambda}\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|.$$

So on the space $M_{\lambda,\infty;X}$

$$\|x|M_{\lambda,\infty;X}\|_b = \sup_{t\in\mathbb{R}^n}\{\sup_i 2^{-i\lambda}\|x(t+.)\chi(U(0,2^i))|X\|\}$$

is an equivalent norm.

Put $\tau_i = 2^i$, $(i \in \mathbb{Z})$, for the sequence of points $\{\tau_i\}_{-\infty}^{\infty}$ consider the corresponding partition τ for R_+ and define a weight sequence by setting $\omega_\lambda(i) = 2^{-\lambda i}$, $(i \in \mathbb{Z})$. Then we get that for all $p \in [1,\infty]$ up to equivalence of the norms:

$$M^{\tau}_{l^p_{\omega_\lambda},X} = M_{\lambda,p;X}.$$

Let C_{cv} denote the set of all functions $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$ concave, positively homogeneous of degree one, nondecreasing and continuous in each variable and such that $\varphi(0,0) = 0$.

The class C_{cv} is a cone with respect to the operations of addition and multiplication by a non-negative number.

We recall the definition of the construction of Calderón – Lozanovskiĭ.

Definition 2. Let a couple of ideal spaces (X_0, X_1) on Ω and $\varphi \in C_{cv}$ be given. The space $\varphi(X_0, X_1)$ consists of all measurable functions x, for which there is a pair of functions $x_0 \in X_0$, $x_1 \in X_1$ such that almost everywhere holds the inequality

$$|x(t)| \le \varphi(x_0(t), x_1(t)).$$

On the space $\varphi(X_0, X_1)$ the norm is introduced by the equality

$$\|x|\varphi(X_0,X_1)\|$$

$$= \inf\{\lambda > 0 : |x(t)| \le \lambda\varphi(x_0(t), x_1(t)) \text{ (for a. e. } t \in \Omega),$$
$$x_i \in X_i, \|x_i|X_i\| \le 1; (i = 0,1)\}. \qquad (2.1)$$

The space $\varphi(X_0, X_1)$ is an ideal Banach space equipped with this norm.

If $\varphi_\theta(t,s) = t^\theta \cdot s^{1-\theta}$ then the definition of the space $\varphi_\theta(X_0, X_1)$, which is usually denoted by $X_0^\theta \cdot X_1^{1-\theta}$, was proposed by A.P. Calderón [16]; for an arbitrary $\varphi \in C_{cv}$ the space $\varphi(X_0, X_1)$ was defined by G.Ya. Lozanovskiĭ [26].

The equality proposed below is well known

$$\|x|\varphi_\theta(X_0, X_1)\| = \inf\{\|x_0|X_0\|^\theta \cdot \|x_1|X_1\|^{1-\theta} : |x(t)| \le x_0^\theta(t) \cdot x_1^{1-\theta}(t) \quad \text{a.e. on } \Omega\}.$$

The Calderón – Lozanovskiĭ construction of $\varphi(X_0, X_1)$ has found many applications in the theory of ideal spaces [27], in the theory of interpolation of linear operators [10], [19], [30], in the geometric theory of Banach spaces [8].

In cases in which exact estimates of constants are important, we can introduce on the space $\varphi(X_0, X_1)$ norms different from (2.1) as follows. Let $\psi(a_1, a_2) : \mathbb{R}^2 \to \mathbb{R}_+$ be a norm on $\mathbb{R}^2$. Then on $\varphi(X_0, X_1)$ the norm is defined by the equality

$$\|x|\{\varphi(X_0, X_1), \psi\}\| = \inf\{\psi(a_1, a_2) :$$

$$|x(t)| \le \varphi(x_0(t), x_1(t)), \quad \text{a.e. on } \Omega, \quad x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1)\}. \tag{2.2}$$

The space $\varphi(X_0, X_1)$ is an ideal Banach space equipped with the norm $\| \,.\, |\{\varphi(X_0, X_1), \psi\}\|$.

Of course all the norms on $\varphi(X_0, X_1)$, defined by equation (2.2), are equivalent. If we put $\psi_\infty(a_1, a_2) = \; max\{|a_1|, |a_2|\}$, then the norm on the space $\{\varphi(X_0, X_1), \psi_\infty\}$ coincide with the norm defined in (2.1). For example (see [4]), using the introduced norms one can to define the exact dual space $\{\varphi(X_0, X_1), \psi\}'$ and exact dual norm on the space $\{\varphi(X_0, X_1), \psi\}$.

For each $\varphi \in C_{cv}$ for all $a, b, c, d > 0$ the following inequality holds

$$\varphi(a + b, c + d) = (c + d)\varphi(\frac{a + b}{c + d}, 1) = (c + d)\varphi(\frac{a}{c}\frac{c}{c + d} + \frac{b}{d}\frac{d}{c + d}, 1) \ge$$

$$(c + d)\{\frac{c}{c + d}\varphi(\frac{a}{c}, 1) + \frac{d}{c + d}\varphi(\frac{b}{d}, 1)\} = \varphi(a, c) + \varphi(b, d). \tag{2.3}$$

Now we will show that condition (1.3) is equivalent to the fact that the corresponding Morrey spaces are obtained using Calderón's constructions $\varphi_{\theta_0}(.,.)$, $\varphi_{\theta_1}(.,.)$ for one special pair of spaces.

Lemma 2.1. *Let the space $M_{l,X}^\tau$ be constructed from the spaces X, l, the sequence $\tau \in \Upsilon$ and the set $U(0,1)$. Let $\theta \in (0,1)$. Then*

$$(M_{l,X}^\tau)^\theta (L^\infty)^{1-\theta} = M_{l^\theta, X^\theta}^\tau$$

and the norms on these spaces coincide.

Proof. Let $x \in (M_{l,X}^\tau)^\theta (L^\infty)^{1-\theta}$. This means that there exists $x_0 \in M_{l,X}^\tau$ with $\|x_0|M_{l,X}^\tau\| = 1$ such that the equality $|x(t)| = \lambda x_0^\theta(t) \cdot 1^{1-\theta}$, $(t \in R^n)$ holds and $\lambda = \|x|(M_{l,X}^\tau)^\theta\|$. Then the following relations follow

$$\||\frac{x}{\lambda}|^{1/\theta}|(M_{l,X}^\tau)^\theta (L^\infty)^{1-\theta}\| = 1 \quad \Leftrightarrow \quad \sup_t \|\Sigma_{-\infty}^\infty \|x^{1/\theta}(t + .)\chi(U(0, r_i))|X\|e^i|l\|^\theta = \lambda$$

$$\Leftrightarrow \quad \sup_t \|\Sigma_{-\infty}^\infty \left((\|x(t + .)\chi(U(0, r_i))|X\|)^\theta\right)^{1/\theta} e^i|l\|^\theta = \lambda \Leftrightarrow \quad \|x|M_{l^\theta, X^\theta}^\tau\| = \lambda.$$

Let us prove the reverse inequality. Let $x \in M_{l^\theta, X^\theta}^\tau$, $x \ge 0$ and $\|x|M_{l^\theta, X^\theta}^\tau\| = 1$. This means that the equality

$$\sup_t \|\Sigma_{-\infty}^\infty \left((\|x^{1/\theta}(t + .)\chi(B(0, r_i))|X\|)^\theta\right)^{1/\theta} e^i|l\|^\theta = 1.$$

Put $x_0(t) = x^{1/\theta}(t)$. Then obvious equality $x(t) = x_0^\theta(t) \cdot (1)^{1-\theta}$, $t \in \Omega$ holds. Let us check that the equality $\|x_0|M^\tau_{l,X}\| = 1$ holds. Indeed,

$$\|x_0|M^\tau_{l,X}\| = \sup_t \|\Sigma^\infty_{-\infty}\|x_0(t+.)\chi(U(0,r_i))|X\|e^i|l\|$$

$$= \sup_t \|\Sigma^\infty_{-\infty}\|x^{1/\theta}(t+.)\chi(U(0,r_i))|X\|e^i|l\|$$

$$= \sup_t \|\Sigma^\infty_{-\infty}\ ((\|x^{1/\theta}(t+.)\chi(U(0,r_i))|X\|)^\theta)\ ^{1/\theta}e^i|l\| = 1.$$

□

Corollary 2.1. *Let $0 < \lambda < \frac{n}{p}$ and $\theta \in (0,1)$ be given. We define the numbers γ and q by the equalities*

$$\nu = \theta\lambda, \quad q = \frac{p}{\theta}.$$

Then the space $(M^\tau_{\lambda,L^p})^\theta(L^\infty)^{1-\theta}$ and the space M^τ_{γ,L^q} coincide and the norms in these spaces are equal.

Proof. If the inequalities $|x(t)| \le \gamma|x_0(t)|^\theta$ and $\|x_0|M^\tau_{\lambda,L^p}\| \le 1$ are satisfied for all t, then the following relations are valid

$$\sup_t\{\sup_{r>0} r^{-\lambda}\|\chi(U(0,r))(\frac{|x(t+.)|}{\gamma})^{1/\theta}|L^p\|\} \le 1$$

$$\Leftrightarrow \sup_t\{\sup_{r>0} r^{-\lambda\theta}\|\chi(U(0,r))|x(t+.)|^{1/\theta}|L^p\|^\theta\} \le \gamma$$

$$\Leftrightarrow \sup_t\{\sup_{r>0} r^{-\nu}\||\chi(U(0,r))x(t+.)||L^q\|\} \le \gamma.$$

□

Corollary 2.2. *Let a couple of Morrey spaces $(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})$ be given. Condition (1.3) is satisfied if and only if there are numbers λ, p and $\theta_0, \theta_1 \in (0,1)$ for which the following equalities are satisfied*

$$M_{\lambda_0,L^{p_0}} = (M_{\lambda,L^1})^{\theta_0}(L^\infty)^{1-\theta_0}, \quad M_{\lambda_1,L^{p_1}} = (M_{\lambda,L^1})^{\theta_1}(L^\infty)^{1-\theta_1}$$

Proof. Define the parameters θ_0, θ_1 by the equalities $\theta_0 = \frac{1}{p_0}, \theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0} \equiv \frac{\lambda_1}{\theta_1}$. Then the following equalities are valid

$$M_{\lambda_0,L^{p_0}} = (M_{\lambda,L^1})^{\theta_0}(L^\infty)^{1-\theta_0}, \quad M_{\lambda_1,L^{p_1}} = (M_{\lambda,L^1})^{\theta_1}(L^\infty)^{1-\theta_1}$$

and it suffices to apply Corollary 2.1. □

Let us note the connection between the Calderón – Lozanovskiĭ construction and the generalized Orlicz - Morrey space.

First we recall the definition of Young functions. A function $N : \mathbb{R}_+ \to \mathbb{R}_+$ is called a Young function if N is convex, left-continuous, strictly increases and $\lim_{t\to 0} N(t) = N(0) = 0$, $\lim_{t\to\infty} N(t) = \infty$.

Let a Young function N be given, by which the Orlicz space $L^N(\mathbb{R}^n)$ is constructed. A natural generalization of the Lebesgue-Morrey space is the Orlicz-Morrey space, the norm in which is given by the equality

$$\|x|M^\tau_{l,L^N}\| = \sup_t\{\|\sum e^i\|\chi(U(0,r_i))x(t+.)|L^N\|\|l\|\}$$

$$= \sup_t \{ \| \sum e^i \{ \inf\{ \lambda_i > 0 : \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda_i}) | L^1 \| \leq 1 \} \} \| |l \|.$$

If we put $l = l^\infty_\omega$, then the formula for the norm in the space $M^\tau_{l^\infty_\omega, L^N}$ has the form

$$\|x|M^\tau_{l^\infty_\omega, L^N}\| = \sup_t \{ \| \sum e^i \| \chi(U(0,r_i)) x(t+.) | L^N \| |l \| \}$$

$$= \sup_t \{ \sup_i \omega(i) \{ \inf\{ \lambda_i > 0 : \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda_i}) | L^1 \| \leq 1 \} \}. \tag{2.4}$$

Let $\theta \in (0,1)$, $p = \frac{1}{\theta}$, $N_\theta(t) = t^{\frac{1}{\theta}}$, $(t \in [0,\infty))$. Then from (2.4) follows the equality

$$M^\tau_{l^\infty_\omega, L^{N_\theta}} = M^\tau_{l^\infty_\omega, L^p}$$

and the norms in these spaces coincide.

Another natural generalization of the Lebesgue-Morrey space is the Orlicz-Morrey space $\varphi_N(M^\tau_{l,L^1}, L^\infty)$, constructed by the Calderón – Lozanovskiĭ construction.

Let a Young function N be given. We define the function $\varphi_N(.,1)$ by the equality $\varphi_N(s,1) = N^{-1}(s)$, $(s \in [0,\infty))$, and put $\varphi_N(s,t) = t\varphi_N(s/t,1)$. Then $\varphi_N \in C_{cv}$.

Lemma 2.2. *Let N be a Young function, and the function $\varphi_N \in C_{cv}$ is constructed.*
Then the following equality is true

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| = \sup_t \{ \inf\{ \lambda > 0 : \| \sum e^i \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda}) | L^1 \| |l \| \leq 1 \} \}.$$

Proof. If $|x(t)| \leq \gamma\varphi_N(x_0(t),1)$ and $\|x_0|M^\tau_{l,L^1}\| = 1$, then

$$|x(t)| \leq \gamma\varphi_N(x_0(t),1) \Leftrightarrow \frac{|x(t)|}{\gamma} \leq \varphi_N(x_0(t),1) \Leftrightarrow \|N(\frac{|x(t)|}{\gamma})|M^\tau_{l,L^1}\| = \|x_0|M^\tau_{l,L^1}\|$$

$$\Leftrightarrow \sup_t \{ \| \sum e^i \| \chi(U(0,2^i)) N(\frac{|x(t+.)|}{\gamma}) | L^1 \| |l \| \} = \|x_0|M^\tau_{l,L^1}\|$$

$$\Rightarrow \sup_t \{ \inf\{ \lambda > 0 : \| \sum e^i \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda}) | L^1 \| |l \| \leq 1 \} \} \leq \gamma.$$

From here it follows that

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| \geq \sup_t \{ \inf\{ \lambda > 0 : \| \sum e^i \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda}) | L^1 \| |l \| \leq 1 \} \}.$$

On the other hand, if

$$\sup_t \{ \inf\{ \lambda > 0 : \| \sum e^i \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{\lambda}) | L^1 \| |l \| \leq 1 \} \} < 1,$$

then

$$\sup_t \{ \| \sum e^i \| \chi(U(0,r_i)) N(\frac{|x(t+.)|}{1}) | L^1 \| |l \| \} \leq 1.$$

Therefore $N(|x(.)|) \in M^\tau_{l,L^1}$, $\|N(|x(.)|)|M^\tau_{l,L^1}\| \leq 1$ and $|x(t)| \equiv \varphi_N(N(|x(t)|),1)$. From here it follows that

$$\|x|\varphi_N(M^\tau_{l,L^1}, L^\infty)\| \leq 1.$$

□

If we put $l = l^\infty_\omega$, then the formula for the norm in the space $\varphi_N(M^\tau_{l^\infty_\omega, L^1}, L^\infty)$ has the form

$$\|x|\varphi_N(M^\tau_{l^\infty_\omega, L^1}, L^\infty)\| = \sup_t\{\inf\{\lambda > 0 : \sup_i\{\omega(i)\|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda})|L^1\| \leq 1\}\}\}$$

$$= \sup_t\{\sup_i\{\inf\{\lambda_i > 0 : \|\chi(U(0,r_i))N(\frac{|x(t+.)|}{\lambda_i})|L^1\| \leq \frac{1}{\omega_i}\}\}\}. \tag{2.5}$$

Let $\theta \in (0,1)$. We define a Young function by the equality $N_\theta(t) = t^{\frac{1}{\theta}}$, $(t \in [0,\infty))$ and put $p = \frac{1}{\theta}$, $\omega_\theta(i) = (\omega(i))^{\frac{1}{\theta}}$, $(i \in Z)$.

Then it follows from (2.5) that

$$\varphi_{N_\theta}(M^\tau_{l^\infty_\omega, L^1}, L^\infty) = M^\tau_{l^\infty_{\omega_\theta}, L^p}$$

and the norms in these spaces coincide.

Note that from (2.5) it turns out that the space $\varphi_n(M^\tau_{l^\infty_\omega, L^1}, L^\infty)$ coincides with the Orlicz-Morrey space introduced by E. Nakai [29]. It is for these spaces that interpolation theorems are formulated below.

The following theorem is a basis for obtaining interpolation theorems for global Morrey spaces.

Theorem 2.1. *Let X_0, X_1 be two ideal spaces, φ, φ_0, $\varphi_1 \in C_{cv}$ and*

$$\overline{\varphi}(t,s) = \varphi(\varphi_0(t,s), \varphi_1(t,s)), \quad t, s \in \mathbb{R}_+. \tag{2.6}$$

Then $\overline{\varphi} \in C_{cv}$, the following equality is true

$$\overline{\varphi}(X_0, X_1) = \varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1)),$$

and for each $x \in \overline{\varphi}(X_0,X_1)$ the inequalities

$$\|x|\varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1))\|$$

$$\leq \|x|\overline{\varphi}(X_0,X_1)\| \leq 2\|x|\varphi(\varphi_0(X_0,X_1), \varphi_1(X_0,X_1))\| \tag{2.7}$$

hold.

Proof. Let us prove first that $\overline{\varphi} \in C_{cv}$.

The positive homogeneity of first degree of the function $\overline{\varphi}$ is obvious. Let us check the concavity. Indeed, using inequality (2.3), we obtain

$$\overline{\varphi}(\frac{t_0+t_1}{2}, 1) = \varphi(\varphi_0(\frac{t_0+t_1}{2}, 1), \varphi_1(\frac{t_0+t_1}{2}, 1))$$

$$\geq \varphi(\frac{1}{2}(\varphi_0(t_0,1) + \varphi_0(t_1,1)), \frac{1}{2}(\varphi_1(t_0,1) + \varphi_1(t_1,1)))$$

$$= \frac{1}{2}\varphi(\varphi_0(t_0,1) + \varphi_0(t_1,1), \varphi_1(t_0,1) + \varphi_1(t_1,1))$$

$$\geq \frac{1}{2}\{\varphi(\varphi_0(t_0,1), \varphi_1(t_0,1)) + \varphi(\varphi_0(t_1,1), \varphi_1(t_1,1))\} = \frac{1}{2}\{\overline{\varphi}(t_0,1) + \overline{\varphi}(t_1,1)\}.$$

Let us prove the left inequality in (2.7).

Let $x \in \overline{\varphi}(X_0,X_1)$ and $\|x|\overline{\varphi}(X_0,X_1)\| < 1$. Then there are $x_0 \in X_0$, $x_1 \in X_1$ such that

$$|x(t)| \leq \overline{\varphi}(x_0(t), x_1(t)), \quad (t \in \Omega); \quad \|x_0|X_0\| < 1; \quad \|x_1|X_1\| < 1.$$

Let us define new functions z_0, z_1 by the equalities:

$$z_0(t) = \varphi_0(x_0(t), x_1(t)), \quad z_1(t) = \varphi_1(x_0(t), x_1(t)); \quad (t \in \Omega).$$

Then

$$z_0 \in \varphi_0(X_0, X_1), \quad \|z_0|\varphi_0(X_0, X_1)\| < 1, \quad z_1 \in \varphi_1(X_0, X_1), \quad \|z_1|\varphi_1(X_0, X_1)\| < 1$$

and

$$\varphi(z_0(t), z_1(t)) = \overline{\varphi}(\varphi_0(x_0(t), x_1(t)), \varphi_1(x_0(t), x_1(t))), \quad (t \in \Omega).$$

Therefore

$$x \in \varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1)), \quad \|x|\varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1))\| < 1.$$

These relations prove the left inequality in (2.7).

Let us prove the right inequality in (2.7).

Let

$$x \in \varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1)), \quad \|x|\varphi(\varphi_0(X_0, X_1), \varphi_1(X_0, X_1))\| < 1.$$

Then there are $x_0, x_1 \in X_0$, $y_0, y_1 \in X_1$ such that

$$\|x_0|X_0\| < 1; \quad \|x_1|X_0\| < 1; \quad \|y_0|X_1\| < 1, \quad \|y_1|X_1\| < 1$$

and

$$|x(t)| \leq \varphi(\varphi_0(x_0(t), y_0(t)), \varphi_1(x_1(t), y_1(t))), \quad (t \in \Omega).$$

Let us define new functions by the equalities: $z_0(t) = \max\{x_0(t), x_1(t)\}$, $z_1(t) = \max\{y_0(t), y_1(t)\}$.

Then

$$\varphi_0(x_0(t), y_0(t)) \leq \varphi_0(z_0(t), z_1(t)), \quad (t \in \Omega); \quad \varphi_1(x_1(t), y_1(t)) \leq \varphi_1(z_0(t), z_1(t)), \quad (t \in \Omega);$$

$$\|z_0|X_0\| < 2; \quad \|z_1|X_1\| < 2.$$

For all $t \in \Omega$ holds the inequality

$$|x(t)| \leq \varphi(\varphi_0(z_0(t), z_1(t)), \varphi_1(z_0(t), z_1(t))) = \overline{\varphi}(z_0(t), z_1(t)), \quad (t \in \Omega).$$

Therefore $\|x|\overline{\varphi}(X_0, X_1)\| < 2$. These relations prove the right inequality in (2.7). □

Corollary 2.3. *Let a couple ideal space X_i on $\mathbb{R}^n$, a couple ideal space of sequence l_i, $(i = 0, 1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

Let φ, φ_0, $\varphi_1 \in C_{cv}$ be fixed, and the function $\overline{\varphi} \in C_{cv}$ is constructed by equality (2.6).

Then

$$\overline{\varphi}(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}) = \varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))$$

and for all $x \in \overline{\varphi}(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})$ the following inequalities are valid

$$\|x|\varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))\| \leq \|x|\overline{\varphi}(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})\|$$

$$\leq 2\, \|x|\varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))\|.$$

Corollary 2.4. *Let $0 \le \theta_0, \theta_1 \le 1$, $\varphi \in C_{cv}$ be fixed, and the function $\varphi_{\theta_0,\theta_1} \in C_{cv}$ is constructed by the equality*

$$\varphi_{\theta_0,\theta_1}(t,s) = \varphi(t^{\theta_0} s^{1-\theta_0}, t^{\theta_1} s^{1-\theta_1}). \tag{2.8}$$

Then

$$\varphi_{\theta_0,\theta_1}(M^\tau_{l,X}, L^\infty) = \varphi((M^\tau_{l,X})^{\theta_0}(L^\infty)^{1-\theta_0}, (M^\tau_{l,X})^{\theta_1}(L^\infty)^{1-\theta_1})$$

and the following inequalities are valid

$$\|x|\varphi((M^\tau_{l,X})^{\theta_0}(L^\infty)^{1-\theta_0}, (M^\tau_{l,X})^{\theta_1}(L^\infty)^{1-\theta_1})\| \le \|x|\varphi_{\theta_0,\theta_1}(M^\tau_{l,X}, L^\infty)\|$$

$$\le 2\, \|x|\varphi((M^\tau_{l,X})^{\theta_0}(L^\infty)^{1-\theta_0}, (M^\tau_{l,X})^{\theta_1}(L^\infty)^{1-\theta_1})\|.$$

Corollary 2.5. *Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ and $\varphi \in C_{cv}$ be given and condition (1.3) be satisfied. Let $\theta_0 = \frac{1}{p_0}$, $\theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0}$, and the function $\varphi_{\theta_0,\theta_1}$ is defined by (2.8).*
Then

$$\varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^\infty) = \varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})$$

and the following inequalities are valid

$$\|x|\varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})\| \le \|x|\varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^\infty)\| \le 2\, \|x|\varphi(M_{\lambda_0,L^{p_0}}, M_{\lambda_1,L^{p_1}})\|.$$

To obtain interpolation theorems, we need one geometric property of an ideal space.

Definition 3. Say (see, for example, [20], [24]) that an ideal space $X \subset S(\mu,\Omega)$ has the Fatou property if from $0 \le x_n \uparrow x$; $x_n \in X$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

It is well known that the Lebesgue spaces L^p_ω, (l^p_ω) for $p \in [1,\infty]$ have the Fatou property, and the space c^0 has not the Fatou property.

The following theorem is not a very general fact for the Calderón – Lozanovskiĭ construction on a couple of ideal spaces. The question of when the space $\varphi(X_0,X_1)$ has the Fatou property depends on the properties of the couple of ideal spaces (X_0,X_1) and the function φ is discussed in more detail in [5].

Theorem 2.2. *[5]. Let $\varphi \in C_{cv}$ and an interpolation couple of ideal spaces (X_0,X_1) be given. If X_0 and X_1 have the Fatou property, then the space $\varphi(X_0,X_1)$ has the Fatou property too.*

The next theorem shows that if parameters of the global Morrey space have the Fatou property, then the global Morrey space also has the Fatou property.

Theorem 2.3. *[7]. Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the space $M^\tau_{l,X}$ be constructed from the spaces X, l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

If both ideal spaces l and X have the Fatou property, then the space $M^\tau_{l,X}$ has the Fatou property too.

We apply Theorems 2.1 - 2.3 to obtain interpolation theorems. Namely, we write out conditions for the coincidence of the Calderón – Lozanovskiĭ construction on a couple of Morrey spaces with the value of the Gustavsson – Peetre – Ovchinnikov interpolation functor on a couple of Morrey spaces.

We recall [10] that a couple of normed spaces (A_0,A_1) is referred to as an interpolation couple if both spaces are embedded in a separable topological linear space V.

Let $\varphi \in C_{cv}$ and a interpolation couple (A_0, A_1) be given. Denote by $(A_0, A_1)_\varphi$ the Gustavsson – Peetre – Ovchinnikov interpolation functor [10], [19], [30] calculated for the couple (A_0, A_1):

$$a \in (A_0, A_1)_\varphi \Leftrightarrow a = \sum_{-\infty}^{\infty} a_i; a_i \in A_0 \bigcap A_1, \text{the series converges in the space} A_0 + A_1;$$

$$\|a|(A_0, A_1)_\varphi\|$$

$$= \inf\{\max\{\sup_n\{\sup_{\varepsilon_i=\pm 1} \|\sum_{-n}^{n} \varepsilon_i \frac{a_i}{\varphi(1, 2^i)}|A_0\|, \quad \sup_{\varepsilon_i=\pm 1} \|\sum_{-n}^{n} \varepsilon_i \frac{a_i}{\varphi(2^i, 1)}|A_1\|\}\} : a = \sum_{-\infty}^{\infty} a_i\} < \infty.$$

Theorem 2.4. *[10], [19], [30]. Let $\varphi \in C_{cv}$ and an interpolation couple of ideal spaces (X_0, X_1) on Ω be given. If X_0 and X_1 have the Fatou property, then*

$$\{\varphi(X_0, X_1), \psi\} = (X_0, X_1)_\varphi,$$

the norms in these spaces are equivalent, and the equivalence constant does not depend on X_0, X_1 and the function φ.

Thus, the triple of spaces $\{X_0,\ X_1;\ \ \varphi(X_0, X_1)\}$ is an interpolation triple.
From Theorems 2.1 – 2.4 we obtain the following interpolation theorem.

Theorem 2.5. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequence l_i, $(i = 0, 1)$, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let all spaces X_0, X_1, l_0, l_1 have the Fatou property. Let the spaces $M^\tau_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0, 1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi, \varphi_0, \varphi_1 \in C_{cv}$ be given. Define the function $\overline{\varphi}$ by equality (2.6). We form the triple of spaces*

$$\{\varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}), \quad \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}); \quad \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})\}.$$

Let an interpolation couple (A_0, A_1) be given.
1) *If a linear operator S is bounded as an operator*

$$S : A_i \to \varphi_i(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}), \quad (i = 0, 1),$$

then

$$S : (A_0, A_1)_\varphi \to \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$$

and is bounded.
2) *If a linear operator P is bounded as an operator*

$$P : \varphi_i(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}) \to A_i, \quad (i = 0, 1),$$

then

$$P : \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}) \to (A_0, A_1)_\varphi$$

and is bounded.

Proof. From Theorems 2.2 and 2.3 it follows that all spaces

$$M^\tau_{l_0,X_0},\ M^\tau_{l_1,X_1}; \quad \varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}), \quad \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1});$$

$$\varphi(\varphi_0(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1}), \varphi_1(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})); \quad \overline{\varphi}(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$$

have the Fatou property. Therefore, it follows from Theorem 2.4, that

$$\begin{aligned}&\varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))\\ &= (\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))_{\varphi} \end{aligned} \tag{2.9}$$

and the norms in these spaces are equivalent.

From Theorem 2.1 it follows that

$$\overline{\varphi}(M^{\tau}_{l_0,X_)}, M^{\tau}_{l_1,X_1}) = \varphi(\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1})) \tag{2.10}$$

and the norms in these spaces are equivalent.

From (2.9) – (2.10) it follows that

$$\overline{\varphi}(M^{\tau}_{l_0,X_)}, M^{\tau}_{l_1,X_1}) = (\varphi_0(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}), \varphi_1(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}))_{\varphi}$$

and the norms in these spaces are equivalent.

From the latter relation we obtain statements 1) and 2). □

Corollary 2.6. *(An interpolation theorem for classical Morrey spaces.)*

Let $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $\theta \in (0,1)$ *and* $\varphi \in C_{cv}$ *be given and condition (1.3) be satisfied. Let* $\theta_0 = \frac{1}{p_0}$, $\theta_1 = \frac{1}{p_1}$, $\lambda = \frac{\lambda_0}{\theta_0}$, *and the function* $\varphi_{\theta_0,\theta_1}$ *is defined by (2.8).*

Then statements 1) and 2) in Theorem 2.5 hold for the triple of spaces

$$\{M_{\lambda_0,L^{p_0}}, \quad M_{\lambda_1,L^{p_1}}; \quad \varphi_{\theta_0,\theta_1}(M_{\lambda,L^1}, L^{\infty})\}.$$

Corollary 2.7. *(An interpolation theorem for generalized Orlicz – Morrey spaces.)*

Let two Young functions N_0, N_1 *be given, and the functions* $\varphi_{N_i}(s,1) = N_i^{-1}(s)$ *and* $\varphi_{N_i}(s,t) = t\varphi_{N_i}(s/t,1)$ $(i = 0,1)$ *are constructed.*

Let $\varphi \in C_{cv}$ *be fixed, and the function* $\varphi_{N_0,N_1} \in C_{cv}$ *be defined by the formula*

$$\varphi_{N_0,N_1}(t,s) = \varphi(tN_0^{-1}(\frac{s}{t}),\ tN_1^{-1}(\frac{s}{t})); \quad t,s > 0.$$

Then statements 1) and 2) in Theorem 2.5 hold for the triple of spaces

$$\{\varphi_{N_0}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty}),\ \varphi_{N_1}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty}); \varphi_{N_0,N_1}(M^{\tau}_{l^{\infty}_{\omega},L^1}, L^{\infty})\}.$$

Remark 1. In the article we considered the Morrey space defined on $\mathbb{R}^n$. If we consider the Morrey space defined on a subset $\Omega \subset \mathbb{R}^n$, $(0 \in \Omega)$ then in Definitions 1 it is necessary to replace $U(0,\tau)$ by $U(0,\tau) \cap \Omega$. All results will remain true.

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