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A NOTE ON QUASILINEAR ELLIPTIC SYSTEMS WITH L^∞-DATA

F. Balaadich, E. Azroul

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Key words: quasilinear elliptic systems, weak energy solution, Young measure.

AMS Mathematics Subject Classification: 35J50, 35J67, 35D30, 46E35.

Abstract. We prove the existence of a weak energy solution for the boundary value problem

$$-\operatorname{div} a(x,u,Du) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded open domain in $\mathbb{R}^n$ $(n \geq 3)$ and $f \in L^\infty(\Omega;\mathbb{R}^m)$. The existence result is proved using the concept of Young measures.

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1 Introduction

This article is concerned with the existence of weak energy solutions of the boundary value problems for quasilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div} a(x,u,Du) &= f \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ $(n \geq 3)$ with a smooth boundary $\partial\Omega$ and f belongs to $L^\infty(\Omega;\mathbb{R}^m)$. Here $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}^*$, is a vector-valued function and Du is the Jacobian matrix of u given by

$$Du(x) = \big(D_1u(x), D_2u(x), ..., D_nu(x)\big) \quad \text{with} \quad D_i = \partial/\partial_i(x_i).$$

We denote by $\mathbb{M}^{m\times n}$ the real space of all $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$ for all $\xi, \eta \in \mathbb{M}^{m\times n}$.

We assume that the function $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto a(x,s,\xi)$ is measurable for every $(s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(s,\xi) \mapsto a(x,s,\xi)$ is continuous for almost every $x \in \Omega$ and satisfies the following conditions: $\xi \mapsto a(x,u,\xi)$ is continuously differentiable and such that for a convex and C^1-mapping $A : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, we have

$$a(x,u,\xi) = \frac{\partial}{\partial \xi} A(x,u,\xi) \tag{1.2}$$

and

$$A(x,u,0) = 0 \tag{1.3}$$

for almost every $x \in \Omega$ and all $u \in \mathbb{R}^m$. Moreover, we assume that

$$|a(x,s,\xi)| \leq d_1(x) + |s|^{p-1} + |\xi|^{p-1} \tag{1.4}$$

for almost every $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$, where $0 \leq d_1 \in L^{p'}(\Omega)$, with $1/p + 1/p' = 1$ and the exponent p is such that $2 \leq p < n$. In addition, the mapping $\xi \to a(x, s, \xi)$ is monotone, i.e.,

$$\big(a(x, s, \xi) - a(x, s, \eta)\big) : (\xi - \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{M}^{m\times n}. \tag{1.5}$$

Finally, the following inequality holds:

$$|\xi|^p \leq a(x, s, \xi) : \xi \leq pA(x, s, \xi). \tag{1.6}$$

The concept of Young measure was introduced in [15] to prove the existence of solutions for (1.1) when $p \in (1, 2 - \frac{1}{n}]$ and $f = \mu$ is a measure. The authors used weak monotonicity assumptions on the function a and the weak derivative Du is replaced by the approximate derivative $apDu$. Hungerbühler has studied, in [19], the existence of weak solutions for (1.1) when the right-hand side belongs to the dual of the Sobolev space $W_0^{1,p}(\Omega; \mathbb{R}^m)$. He used also mild monotonicity assumptions and Young measures to achieve the result. The uniqueness and maximal regularity for nonlinear elliptic systems (1.1) have been proved in [16] when $f = \mu$ a Radon measure. Zhou [28] introduced the sign condition:

$$a_i(x, u, \xi) \cdot \xi_i \geq 0 \quad \text{for } i = 1, ..., m,$$

instead of the angle condition:

$$a(x, u, \xi) : M\xi \geq 0$$

assumed in [15], to prove the existence and regularity of solutions to (1.1) with $f = \mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. For more results, we refer the reader to see [14, 20, 21, 22, 23, 24, 26, 27] and [1, 2, 3, 4, 5, 6, 7, 8] where we have used the theory of Young measures for various quasilinear systems.

In [2, 3] we have proved the existence of weak solutions for various kinds of quasilinear elliptic systems similar to (1.1), for $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$, under various kinds of monotonicity assumptions and based on the theory of Young measures. See also [10, 11, 12, 13] for more results and [25] for different theories and methods used in nonlinear analysis.

In this paper, the source term in (1.1) is assumed to be in $L^\infty(\Omega; \mathbb{R}^m)$ and a to satisfy conditions (1.2)-(1.6). The main objective is to prove the existence of a weak energy solution using the concept of Young measure and energy functionals. Moreover, a is assumed to be the derivative over the third argument of another function A. This assumption is necessary in order to associate with the problem an energy functional, and then to minimize this functional to obtain a weak solution. The main result of the paper consists in justification of sufficient assumptions for such minimization

A prototype example that is covered by our assumptions (1.2)-(1.6) is the following p-Laplacian problem: Consider

$$A(x, u, \xi) = \frac{1}{p}|\xi|^p, \quad a(x, u, \xi) = |\xi|^{p-2}\xi$$

where $p \geq 2$.

The remaining part of this paper is organized as follows: a brief review on Young measures is presented in Section 2, while Section 3 is devoted to state the existence result and its proof.

2 A brief review on Young measures

By $C_0(\mathbb{R}^m)$ we denote the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$, by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

Lemma 2.1 (See p. 19 in [17]). *Let $\{z_j\}_{j\geq 1}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x\in\Omega$, such that for almost each $\varphi\in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \text{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda)d\nu_x(\lambda)$ for a.e. $x\in\Omega$.

Definition 1. *We call $\{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{z_k\}_k$.*

Remark 1.

- *In [9], it is shown that for any Carathéodory function $\varphi : \Omega\times\mathbb{R}^m \to \mathbb{R}$ and $\{z_k\}_k$ a sequence that generates the Young measure ν_x, we then have*

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x, .)\rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda)d\nu_x(\lambda)$$

weakly in $L^1(\Omega')$ for all measurable $\Omega'\subset\Omega$, provided that the negative part $\varphi^-(x, z_k)$ is equiintegrable.

- *Ball shows also in [9], that if z_k generates the Young measure ν_x, then for $\varphi\in L^1(\Omega; C_0(\mathbb{R}^m))$*

$$\lim_{k\to\infty}\int_\Omega g(x, z_k(x))dx = \int_\Omega \langle \nu_x, g(x, .)\rangle dx.$$

Lemma 2.2 ([18]). *If $|\Omega| < \infty$ then*

$$z_k \to z \text{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x\in\Omega.$$

Lemma 2.3 ([1]). *If $\{Dz_k\}_k$ is bounded in $L^p(\Omega;\mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dz_k has the following properties:*

(i) ν_x is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} := \int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) = 1$ for almost every $x\in\Omega$.

(ii) The weak L^1-limit of Dz_k is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(iii) ν_x satisfies $\langle \nu_x, id\rangle = Dz(x)$ for almost every $x\in\Omega$.

We conclude this section by recalling the following Fatou-type inequality.

Lemma 2.4 ([15]). *Let $\varphi : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $z_k : \Omega\to\mathbb{R}^m$ a sequence of measurable functions such that $z_k \to z$ in measure and such that Dz_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x\in\Omega$. Then*

$$\liminf_{k\to\infty}\int_\Omega \varphi(x, z_k, Dz_k)dx \geq \int_\Omega\int_{\mathbb{M}^{m\times n}} \varphi(x, z, \lambda)d\nu_x(\lambda)dx$$

provided that the negative part $\varphi^-(x, z_k, Dz_k)$ is equiintegrable.

For more results and details about Young measures, we refer the reader not familiar with this concept to see for example [9, 17, 18, 25].

3 Existence of weak energy solution

Before we state the main result of this paper, let us introduce the following definition of weak energy solutions of (1.1).

Definition 2. *A weak energy solution of* (1.1) *is a function* $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

$$\int_\Omega (a(x,u,Du) : D\varphi)dx = \int_\Omega f(x)\varphi dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m).$$

The main result is given in the following.

Theorem 3.1. *Assume* $f \in L^\infty(\Omega;\mathbb{R}^m)$ *and* (1.2)-(1.6) *hold. Then there exists a weak energy solution of* (1.1).

Proof of the main result. Let us define the energy functional $J : W_0^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ by

$$J(u) = \int_\Omega A(x,u,Du)dx - \int_\Omega fudx.$$

Proposition 3.1. *The functional* J *is well-deffined on* $W_0^{1,p}(\Omega;\mathbb{R}^m)$ *and* $J \in C^1(W_0^{1,p}(\Omega;\mathbb{R}^m),\mathbb{R})$ *with the derivative given by*

$$\langle J'(u),\varphi\rangle = \int_\Omega (a(x,u,Du) : D\varphi)dx - \int_\Omega f\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Proof. For any $x \in \Omega$, $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\xi \in \mathbb{M}^{m\times n}$, we have

$$A(x,u,\xi) = \int_0^1 \frac{d}{dt}A(x,u,t\xi)dt = \int_0^1 a(x,u,t\xi) : \xi dt.$$

Using (1.4), we get

$$\begin{aligned} A(x,u,\xi) &\le \int_0^1 \big(d_1(x) + |u|^{p-1} + t^{p-1}|\xi|^{p-1}\big)|\xi|dt \\ &\le d_1(x)|\xi| + |u|^{p-1}|\xi| + \frac{1}{p}|\xi|^p. \end{aligned} \tag{3.1}$$

This and the Hölder inequality imply that

$$0 \le \int_\Omega |A(x,u,Du)|dx \le \|d_1\|_{p'}\|Du\|_p + \|u\|_p^{p-1}\|Du\|_p + \frac{1}{p}\|Du\|_p^p$$

and

$$\int_\Omega |fu|dx \le \|f\|_{q'}\|u\|_q, \quad \text{where } 1 < q < p.$$

Next we deduce that J is well-defined on $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Let us fix $x \in \Omega$ and $0 < |r| < 1$. According to the mean value theorem, there exists $\theta \in [0,1]$ such that

$$\begin{aligned} &\big|a(x,u,Du+\theta D\varphi)\big||D\varphi| \\ &= \frac{\big|A(x,u,Du+rD\varphi) - A(x,u,Du)\big|}{|r|} \\ &\le \big(d_1(x) + |u|^{p-1} + |Du+\theta rD\varphi|^{p-1}\big)|D\varphi| \\ &\le \Big(d_1(x) + |u|^{p-1} + 2^{p-2}\big(|Du|^{p-1} + (\theta r)^{p-1}|D\varphi|^{p-1}\big)\Big)|D\varphi|. \end{aligned}$$

Hölder's inequality gives that

$$\int_\Omega d_1(x)|D\varphi|dx \leq \|d_1\|_{p'}\|D\varphi\|_p,$$

$$\int_\Omega |Du|^{p-1}|D\varphi|dx \leq \|Du\|_p^{p-1}\|D\varphi\|_p$$

and

$$\int_\Omega |D\varphi|^{p-1}|D\varphi|dx = \|D\varphi\|_p^p.$$

From these inequalities, we deduce that

$$\Big(d_1(x) + |u|^{p-1} + 2^{p-2}\big(|Du|^{p-1} + (\theta r)^{p-1}|D\varphi|^{p-1}\big)\Big)|D\varphi| \in L^1(\Omega).$$

Thanks to the Lebesgue theorem, it follows that

$$\langle J'(u), \varphi\rangle = \int_\Omega a(x,u,Du) : D\varphi dx - \int_\Omega f\varphi dx.$$

Assume now that $u_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Then $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. According to Lemma 2.1 there is a Young measure ν_x generated by Du_k in $L^p(\Omega;\mathbb{M}^{m\times n})$ and satisfying the properties of Lemma 2.3. Using (1.5) and [2, Lemma 5.3], we get that

$$\begin{aligned}0 &\leq \big(a(x,u,\lambda) - a(x,u,Du+\tau\xi)\big) : (\lambda - Du - \tau\xi)\\ &= a(x,u,Du) : (\lambda - Du) - a(x,u,\lambda) : \tau\xi\\ &\quad - a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi),\end{aligned}$$

which gives

$$-a(x,u,\lambda) : \tau\xi \geq -a(x,u,Du) : (\lambda - Du) + a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi),$$

for every $\lambda, \xi \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$. We have $\xi \mapsto a(x,u,\xi)$ is continuously differentiable, hence we can write

$$\begin{aligned}&a(x,u,Du+\tau\xi) : (\lambda - Du - \tau\xi)\\ &= a(x,u,Du+\tau\xi) : (\lambda - Du) - a(x,u,Du+\tau\xi) : \tau\xi\\ &= a(x,u,Du) : (\lambda - Du)\\ &+ \tau\Big(\big(\nabla a(x,u,Du)\xi\big) : (\lambda - Du) - a(x,u,Du) : \xi\Big) + o(\tau),\end{aligned}$$

where ∇ is the derivative of a with respect to its third variable. Therefore,

$$-a(x,u,\lambda) : \tau\xi \geq \tau\Big(\big(\nabla a(x,u,Du)\xi\big) : (\lambda - Du) - a(x,u,Du) : \xi\Big) + o(\tau)$$

which gives, since τ is arbitrary in $\mathbb{R}$, that

$$a(x,u,\lambda) : \xi = a(x,u,Du) : \xi + \big(\nabla a(x,u,Du)\xi\big) : (Du - \lambda) \tag{3.2}$$

on the support of ν_x. Since $(a(x,u_k,Du_k))_k$ is equiintegrable by (1.4) and $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, it follows that its weak L^1-limit $\overline{a}$ is given by

$$\begin{aligned}\overline{a}(x) &:= \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda)d\nu_x(\lambda)\\ &\overset{(3.2)}{=} a(x,u,Du)\underbrace{\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda)}_{:=1} + \big(\nabla a(x,u,Du)\big)^t \underbrace{\int_{\operatorname{supp}\nu_x}(Du-\lambda)d\nu_x(\lambda)}_{:=0}\\ &= a(x,u,Du).\end{aligned}$$

As $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ is reflexive, it follows that $(a(x,u_k,Du_k))_k$ converges in $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L^{p'}$-limit is also $\overline{a}(x)=a(x,u,Du)$. This and the Hölder inequality imply

$$\big|\langle J'(u_k)-J'(u),\varphi\rangle\big|\le\int_\Omega\big|a(x,u_k,Du_k)-a(x,u,Du)\big||D\varphi|dx$$

and so

$$\|J'(u_k)-J'(u)\|\le\|a(x,u_k,Du_k)-a(x,u,Du)\|_{p'}\longrightarrow 0$$

as $k\to\infty$. □

Lemma 3.1. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By (3.1) and Hölder's inequality, it is obvious that J is bounded from below. Using (1.6), we have

$$\begin{aligned}J(u)&=\int_\Omega A(x,u,Du)dx-\int_\Omega fudx\\&\ge\frac{1}{p}\int_\Omega|Du|^pdx-\|f\|_{q'}\|u\|_q,\quad(\text{with }1<q<p)\\&\ge\frac{1}{p}\int_\Omega|Du|^pdx-c\|u\|_{1,p}\longrightarrow+\infty\end{aligned}$$

as $\|u\|_{1,p}\to\infty$, since $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuously embedded in $L^q(\Omega;\mathbb{R}^m)$. Then J is coercive. Let $(u_k)\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence which converges weakly to u in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Hence $u_k\to u$ in $L^p(\Omega;\mathbb{R}^m)$ and in measure on Ω (for a subsequence still indexed by k), by the compact embedding of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ in $L^p(\Omega;\mathbb{R}^m)$. Since $\nu_x=\delta_{Du(x)}$ for a.e. $x\in\Omega$ by Lemma 2.3, then Lemma 2.2 implies $Du_k\to Du$ in measure. We have $(A(x,u_k,Du_k))_k$ is equiintegrable by (3.1), it follows then by Lemma 2.4 that

$$\int_\Omega\int_{\mathbb{M}^{m\times n}}A(x,u,\lambda)d\nu_x(\lambda)dx\le\liminf_{k\to\infty}\int_\Omega A(x,u_k,Du_k)dx.\tag{3.3}$$

On the other hand, assumption (1.5) and the relation $a(x,u,\xi)=\frac{\partial}{\partial\xi}A(x,u,\xi)$ imply, in particular, that $\xi\mapsto A(x,u,\xi)$ is convex, i.e.,

$$\underbrace{A(x,u,\lambda)}_{=:F(\lambda)}\ge\underbrace{A(x,u,Du)+a(x,u,Du):(\lambda-Du)}_{=:G(\lambda)},\quad\forall\lambda\in\mathbb{M}^{m\times n}.$$

Since $\lambda\mapsto F(\lambda)$ is a C^1-function by Proposition 3.1, then for $\tau\in\mathbb{R}$

$$\frac{F(\lambda+\tau\xi)-F(\lambda)}{\tau}\le\frac{G(\lambda+\tau\xi)-G(\lambda)}{\tau}\quad\text{for }\tau<0$$

and

$$\frac{F(\lambda+\tau\xi)-F(\lambda)}{\tau}\ge\frac{G(\lambda+\tau\xi)-G(\lambda)}{\tau}\quad\text{for }\tau>0.$$

Hence $\nabla F=\nabla G$, i.e.,

$$A(x,u,\lambda)=A(x,u,Du)\quad\text{for all }\lambda\in\operatorname{supp}\nu_x.\tag{3.4}$$

Going back to (3.3), it follows by (3.4) that

$$\begin{aligned}\int_\Omega\int_{\mathbb{M}^{m\times n}}A(x,u,\lambda)d\nu_x(\lambda)&=\int_\Omega\int_{\operatorname{supp}\nu_x}A(x,u,Du)d\nu_x(\lambda)dx\\&=\int_\Omega A(x,u,Du)dx\\&\le\liminf_{k\to\infty}\int_\Omega A(x,u_k,Du_k)dx.\end{aligned}$$

This fact implies that

$$J(u) \leq \liminf_{k\to\infty} J(u_k).$$

Hence, J is weakly lower semi-continuous and the proof is complete. □

Since J is proper, weakly semi-continuous and coercive, then J has a minimizer which is in fact a weak energy solution of (1.1). The proof of the main result is complete.

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