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# ON THE LAGRANGE MULTIPLIER RULE FOR MINIMIZING SEQUENCES

**A.V. Arutyunov, S.E. Zhukovskiy**

Communicated by V.I. Burenkov

**Key words:** constraint optimization, Lagrange multiplier rule, optimality condition, minimizing sequence, Caristi-like condition.

**AMS Mathematics Subject Classification:** 49K27.

**Abstract.** In the paper, an optimization problem with equality-type constraints is studied. It is assumed that the minimizing function and the functions defining the constraints are Frechet differentiable, the set of the admissible points is nonempty and the minimizing function is bounded below on the set of admissible points. Under these assumptions we obtain an estimate of the derivative of the Lagrange function. Moreover, we prove the existence of a minimizing sequence $\{x^n\}$ and a sequence of unit Lagrange multipliers $\{\lambda^n\}$ such that the sequence of the values of derivative of the Lagrange function at the point $(x^n, \lambda^n)$ tends zero. This result is a generalization of the known assertion stating that for a bounded below differentiable function $f$ there exists a minimizing sequence $\{x^n\}$ such that the values of the derivative $f'(x^n)$ tend to zero. As an auxiliary tool,there was introduced and studied the property of the directional covering for mappings between normed spaces. There were obtained sufficient conditions of directional covering for Frechet differentiable mappings.

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## 1 Introduction

Let $X$ be a Banach space with the norm $\|\cdot\|$. Denote by $B_X(x,r)$ a closed ball centered at $x \in X$ with radius $r \geq 0$. Let $X^*$ stand for the topological dual to $X$ and stand $\|\cdot\|_*$ for the norm of $X^*$.

Given a positive integer $k$ and Frechet differentiable functions $f_0, f_1, ..., f_k : X \to \mathbb{R}$, consider the optimization problem

$$f_0(x) \to \min, \quad f_1(x) = 0, \quad ..., \quad f_k(x) = 0. \tag{1.1}$$

Define the Lagrange function $L : X \times \mathbb{R}^{k+1} \to \mathbb{R}$ by the formula

$$L(x, \lambda) := \lambda_0 f_0(x) + \lambda_1 f_1(x) + ... + \lambda_k f_k(x), \quad x \in X, \quad \lambda = (\lambda_0, \lambda_1, ..., \lambda_k) \in \mathbb{R}^{k+1}.$$

Denote the set of all admissible points by $\mathcal{M}$, i.e.

$$\mathcal{M} := \{x \in X : f_1(x) = ... = f_k(x) = 0\}.$$

The Lagrange multiplier rule (see, for example, [10, Section 1.2]) states that if a point $\widehat{x} \in X$ is a local solution to problem (1.1), then there exists a nonzero vector $\lambda \in \mathbb{R}^{k+1}$ such that $\dfrac{\partial L}{\partial x}(\widehat{x}, \lambda) = 0$ and $\lambda_0 \geq 0$.

In this paper, we show that if a function $f_0$ is bounded from below on $\mathcal{M} \neq \emptyset$ then there exist sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that $\left\| \frac{\partial L}{\partial x}(x^n, \lambda^n) \right\|_* \to 0$, $f_0(x^n) \to \inf_{x \in \mathcal{M}} f_0(x)$ as $n \to \infty$ and $\|\lambda^n\| = 1$ for every $n$. This result is an analog of the known result for unconstrained optimization problem stating that for a bounded below differentiable functional $f_0$ on $X$ there exists a minimizing sequence $\{x^n\}$ such that $\frac{\partial f_0}{\partial x}(x^n) \to 0$ as $n \to \infty$ (see, for example, [6, Chapter 5, Section 3]).

Moreover, in this paper, we obtain an estimate of the derivative of the Lagrange function. When $X$ is a Hilbert space, similar estimates for the first-order and the second-order derivatives were obtained in [2] and [3]. For the unconstrained optimization problem the estimates of the first-order and the second-order derivatives of the minimizing function were obtained in [7, §2.5.2].

## 2 Main results

Given $x_0 \in \mathcal{M}$ and $R > 0$, denote

$$\gamma(x_0, R) := \inf\{f_0(x) : \ x \in \mathcal{M} \cap B_X(x_0, R)\}.$$

Here $\gamma(x_0, R)$ may take the value $-\infty$. However, in what follows, we will assume that $\gamma(x_0, R) > -\infty$. Note also that $f_0(x_0) - \gamma(x_0, R) \geq 0$ for every $x_0 \in \mathcal{M}$ and $R > 0$.

**Theorem 2.1.** *Given a point $x_0 \in \mathcal{M}$ and a number $R > 0$, assume that*

$$\gamma(x_0, R) > -\infty.$$

*Then for every $\varepsilon > 0$ there exist vectors $\lambda = (\lambda_0, \lambda_1, ..., \lambda_k) \in \mathbb{R}^{k+1}$ and $\widehat{x} \in \mathcal{M} \cap B_X(x_0, R)$ such that*

$$\|\lambda\| = 1, \qquad \lambda_0 \geq 0, \qquad f_0(\widehat{x}) \leq f_0(x_0),$$
$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \leq (1+\varepsilon)\lambda_0 \frac{f_0(x_0) - \gamma(x_0, R)}{R}. \tag{2.1}$$

Note that if the set $\mathcal{M} \cap B_X(x_0, R)$ contains a point $x$ for which the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly dependent and $f_0(x) \leq f_0(x_0)$, then the proposition of Theorem 2.1 trivially holds. In this case, $\widehat{x} = x$ and the unit vector $\lambda = (\lambda_0, \lambda_1, ..., \lambda_k)$ satisfying the equality $\sum_{i=0}^{k} \lambda_i \frac{\partial f_i}{\partial x}(x) = 0$ and the inequality $\lambda_0 \geq 0$ is the desired one (if $\lambda_0 < 0$ then we take $-\lambda$ instead of $\lambda$). In this case, the left-hand side of (2.1) equals zero and the right-hand side is nonnegative.

If the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly independent on the set $\{x \in \mathcal{M} \cap B_X(x_0, R) : f_0(x) \leq f_0(x_0)\}$ then the proposition of Theorem 2.1 is nontrivial. In this case, inequality (2.1) implies that $\lambda_0 > 0$.

Note also that inequality (2.1) implies the following weaker estimate

$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \leq (1+\varepsilon) \frac{f_0(x_0) - \gamma(x_0, R)}{R}, \tag{2.2}$$

since $\|\lambda\| = 1$.

**Theorem 2.2.** *Assume that the function $f_0$ is bounded from below on $\mathcal{M}$. Then there exist sequences of vectors $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that*

$$\frac{\partial L}{\partial x}(x^n, \lambda^n) \to 0, \quad f_0(x^n) \to \inf_{x \in \mathcal{M}} f_0(x) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|\lambda^n\| = 1 \quad \forall n.$$

The proofs of these theorems are presented in Section 4. Let us now discuss the ideas of proofs of these assertions.

Given a point $x_0 \in X$ and a number $R \geq 0$, we put $v := (-1, 0, 0, ..., 0) \in \mathbb{R}^{k+1}$, $F := (f_0, f_1, ..., f_k)$. Since $\gamma(x_0, R)$ is the infimum of $f_0$ over the admissible set $\mathcal{M}$, the points $F(x_0) + \mu v$, $\mu \geq 0$ do not belong to $F(B_X(x_0, R))$ as $\mu > f_0(x_0) - \gamma(x_0, R)$. If the mapping $F$ is $\bar{\alpha}$-covering in the direction $v$ at every point $x \in \mathcal{M} \cap B_X(x_0, R)$ such that $f_0(x) \leq f_0(x_0)$ (i.e. $\dfrac{\partial F}{\partial x}(x)X = Y$ and $\sup\left\{\alpha \geq 0 : \ \alpha v \in \dfrac{\partial F}{\partial x}(x) B_X(0,1)\right\} \geq \bar{\alpha}$) then $F(x_0) + \mu v \in F(B_X(x_0, R))$ for $\mu \in [0, \alpha R)$. This assertion is Lemma 3.1 below. These reasonings imply that there exists a point $\widehat{x}$ such that $F$ is $\bar{\alpha}$-covering with the constant $\bar{\alpha}$ not exceeding the right-hand side of inequality (2.1). Inequality (2.1) simply follows from this fact (see Lemma 3.2 below). To prove Theorem 2.2, it is enough to take an arbitrary minimizing sequence $\{x_0^n\}$ and apply Theorem 2.1 as $x_0 := x_0^n$, $R := 1$ and $\varepsilon := 1$ for every $n$.

## 3 Auxiliary assertions

In this section, we prove two auxiliary assertions: Lemmas 3.1 and 3.2. In the proof of Lemma 3.1, we will use the following minimum existence conditions from [1, Theorem 3] (see also [8, Lemma 1]).

**Theorem 3.1.** *Given a complete metric space $(M, \rho)$, a lower semicontinuous function $U : M \to \mathbb{R}_+$ and a number $\alpha > 0$, assume that the function $U$ satisfies the Caristi-like condition*

$$\forall\, x \in M : \quad U(x) > 0 \quad \exists\, x' \in M \setminus \{x\} : \quad U(x') + \alpha\rho(x, x') \leq U(x). \tag{3.1}$$

*Then for every $x_0 \in M$ there exists a point $\bar{x} \in M$ such that $U(\bar{x}) = 0$ and $\rho(x_0, \bar{x}) \leq \alpha^{-1} U(x_0)$.*

Let $Y$ be a finite-dimensional linear space with a norm $\|\cdot\|$. Denote by $Y^*$ a dual space to $Y$. We denote the value of the functional $\lambda \in Y^*$ on the vector $y \in Y$ by $\langle \lambda, y \rangle$. An analogous notation we will use for the functionals from $X^*$. Denote the unit sphere in $Y$ by $S$, i.e.

$$S := \{v \in Y : \ \|v\| = 1\}.$$

For an arbitrary linear bounded operator $A : X \to Y$ we denote by $A^* : Y^* \to X^*$ the adjoint operator to $A$. For an arbitrary vector $v \in S$ we put

$$\operatorname{cov}(A|v) := \sup\{\alpha \geq 0 : \quad \alpha v \in A B_X(0, 1)\}.$$

It is a straightforward task to ensure that $\operatorname{cov}(A|v) > 0$ if and only if $v \in AX$.

**Lemma 3.1.** *Given a Frechet differentiable mapping $F : X \to Y$, vectors $x_0 \in X$, $v \in S$ and a number $R > 0$, assume that*

**(i)** $\bar{\alpha} := \inf\left\{\operatorname{cov}\left(\dfrac{\partial F}{\partial x}(x) \,\middle|\, v\right) : \ x \in B_X(x_0, R), \ \ F(x) \in F(x_0) + \operatorname{cone}\{v\}\right\} > 0;$

**(ii)** $\dfrac{\partial F}{\partial x}(x) X = Y \ \forall\, x \in B_X(x_0, R) : F(x) \in F(x_0) + \operatorname{cone}\{v\}.$

*Then*

$$F(x_0) + \alpha r v \in F(B_X(x_0, r)) \quad \forall\, r \in [0, R], \quad \forall\, \alpha \in (0, \bar{\alpha}).$$

*Proof.* Fix an arbitrary $r \in [0, R]$ and $\alpha \in (0, \bar{\alpha})$. Put

$$M := \{x \in X : F(x) - F(x_0) - s\alpha r v = 0, \ \ \|x - x_0\| \le sr, \ \ s \in [0,1]\}.$$

Obviously, the set $M$ is nonempty, since it contains the point $x_0$. Moreover, $M$ is closed, since $F$ is continuous. Define a functional $U : M \to \mathbb{R}$ by the formula

$$U(x) = \|F(x) - F(x_0) - \alpha r v\|, \quad x \in M. \tag{3.2}$$

To prove the lemma it is enough to show that there exists a point $\bar{x} \in M$ such that $U(\bar{x}) = 0$. To prove this assertion we will apply Theorem 3.1.

Obviously, the functional $U$ is continuous and nonnegative. So, it is enough to prove that $U$ satisfies the Caristi-like condition (3.1).

Fix an arbitrary $x \in M$ such that $U(x) > 0$ and show that there exists a point $x' \in M \setminus \{x\}$ such that

$$U(x') + \alpha\|x - x'\| \le U(x). \tag{3.3}$$

The definition of $M$ implies that there exists $t \in [0,1]$ such that

$$F(x) = F(x_0) + t\alpha r v, \quad \|x - x_0\| \le tr. \tag{3.4}$$

Since $U(x) = \|F(x) - F(x_0) - \alpha r v\| > 0$, we have $t < 1$.

Put $A := \dfrac{\partial F}{\partial x}(x)$. It follows from the assumption (**i**) that $\operatorname{cov}(A|v) \ge \bar{\alpha} > 0$. Hence, $\bar{\alpha} > (\alpha + \bar{\alpha})/2$ by virtue of the choice of $\alpha$. The definition of $\operatorname{cov}(A|v)$ implies that there exists a vector $e \in B_X(0,1)$ such that

$$Ae = \frac{\alpha + \bar{\alpha}}{2} v.$$

Since $AX = Y$ by virtue of **(ii)**, we have that there exists a linear operator $R : Y \to X$ such that

$$e = R\Big(\frac{\alpha + \bar{\alpha}}{2} v\Big) \quad \text{and} \quad ARy \equiv y. \tag{3.5}$$

Since the mapping $F$ is differentiable, we have

$$F(x + \xi) = F(x) + A\xi + o(\xi), \quad \xi \in X, \tag{3.6}$$

where $o : X \to Y$ is a continuous mapping such that there exists $\delta > 0$, for which the following relation takes place

$$\|o(\xi)\| \le \frac{\bar{\alpha} - \alpha}{\|R\|(\bar{\alpha} + \alpha)}\|\xi\| \quad \forall \xi \in B_X(0, \delta). \tag{3.7}$$

Reducing $\delta$ we obtain that

$$0 < \delta < r - tr. \tag{3.8}$$

Note that when we reduce $\delta$, relation (3.7) remains true.

Consider the equation

$$\xi = R(\alpha\delta v - o(\xi))$$

with the unknown $\xi \in B_X(0, \delta)$. Define a mapping $\Phi : B_X(0,\delta) \to B_X(0,\delta)$ by the formula

$$\Phi(\xi) := R(\alpha\delta v - o(\xi)), \quad \xi \in B_X(0,\delta).$$

This mapping is well-defined, i.e. $\|R(\alpha\delta v - o(\xi))\| \le \delta$ for every $\xi \in B_X(0,\delta)$, since

$$\|R(\alpha\delta v - o(\xi))\| \le \|\alpha\delta R v\| + \|Ro(\xi)\| \overset{(3.5)}{\le} \frac{2\alpha\delta}{\alpha + \bar{\alpha}} + \|Ro(\xi)\| \overset{(3.7)}{\le}$$

$$\overset{(3.7)}{\leq} \frac{2\alpha\delta}{\alpha+\bar{\alpha}} + \frac{\bar{\alpha}-\alpha}{\bar{\alpha}+\alpha}\|\xi\| \leq \frac{2\alpha\delta}{\alpha+\bar{\alpha}} + \frac{\bar{\alpha}-\alpha}{\bar{\alpha}+\alpha}\delta = \delta \quad \forall\, \xi \in B_X(0,\delta).$$

Moreover, the mapping $\Phi$ is compact and continuous since $o(\cdot)$ is continuous and the linear operator $R: Y \to X$ has a finite-dimensional image (recall that the space $Y$ is finite-dimensional). Thus, the Schauder fixed-point theorem (see, for example, [11, Section 2.1]) implies that there exists a point $\xi' \in B_X(0,\delta)$ such that $\xi' = \Phi(\xi')$. Therefore,

$$\xi' = R(\alpha\delta v - o(\xi')), \quad \|\xi'\| \leq \delta. \tag{3.9}$$

Put

$$x' := x + \xi'. \tag{3.10}$$

Let us show that $x' \in M \setminus \{x\}$. We have

$$F(x') - F(x_0) = \alpha r\left(t + \frac{\delta}{r}\right)v, \quad \|x_0 - x'\| \leq r\left(t + \frac{\delta}{r}\right), \tag{3.11}$$

since

$$F(x') \overset{(3.10)}{=} F(x+\xi') \overset{(3.6)}{=} F(x) + A\xi' + o(\xi') \overset{(3.9)}{=} F(x) + AR(\alpha\delta v - o(\xi')) + o(\xi') \overset{(3.5)}{=}$$

$$\overset{(3.5)}{=} F(x) + \alpha\delta v \overset{(3.4)}{=} t\alpha r v + \alpha\delta v + F(x_0) = \alpha r v\left(t + \frac{\delta}{r}\right) + F(x_0);$$

$$\|x_0 - x'\| \leq \|x_0 - x\| + \|x - x'\| \overset{(3.4)}{\leq} tr + \|x - x'\| \overset{(3.10)}{=} tr + \|\xi'\| \overset{(3.9)}{\leq} r\left(t + \frac{\delta}{r}\right).$$

It follows from (3.8) that the inequality $t + \dfrac{\delta}{r} < 1$ takes place. Therefore, relation (3.11) and the definition of $M$ implies $x' \in M$. Moreover, $Rv \neq 0$ by virtue of (3.5). Therefore, $\xi' \neq 0$ by virtue of (3.9). So, relation (3.10) implies that $x' \neq x$. Hence, we have $x' \in M \setminus \{x\}$.

Let us prove that (3.3) holds. We have

$$U(x') \overset{(3.2)}{=} \|F(x') - F(x_0) - \alpha r v\| \overset{(3.11)}{=} \|t\alpha r v + \alpha\delta v - \alpha r v\| = \Big\|\big((r - tr) - \delta\big)\alpha v\Big\| \overset{(3.8)}{=}$$

$$\overset{(3.8)}{=} \Big\|(r - tr)\alpha v\Big\| - \|\delta \cdot \alpha v\| = \|t\alpha r v - \alpha r v\| - \|\alpha\delta v\| \overset{(3.4)}{=} \|F(x) - F(x_0) - \alpha r v\| - \|\alpha\delta v\| \overset{(3.9)}{\leq}$$

$$\overset{(3.9)}{\leq} \|F(x) - F(x_0) - \alpha r v\| - \alpha\|\xi'\| \overset{(3.2)}{=} U(x) - \alpha\|\xi'\| \overset{(3.10)}{=} U(x) - \alpha\|x - x'\|.$$

So, it is shown that there exists a point $x' \in M \setminus \{x\}$ such that relation (3.3) holds. Therefore, the Caristi-like condition (3.1) holds for the function $U$.

It is shown that all the assumptions of Theorem 3.1 hold. This theorem implies that there exists a point $\bar{x} \in M$ such that $U(\bar{x}) = 0$. The definitions of the set $M$ and the functional $U$ imply that $\bar{x} \in B_X(x_0, r)$ and $F(x_0) + \alpha r v = F(\bar{x})$. Therefore, $F(x_0) + \alpha r v \in F(B_X(x_0, r))$. □

**Lemma 3.2.** *Given a linear bounded operator $A: X \to Y$ and a vector $v \in S$, there exists a nonzero functional $\lambda \in Y^*$ such that*

$$\|A^*\lambda\|_* \leq -\langle \lambda, v\rangle \operatorname{cov}(A|v).$$

*Here, obviously, $\langle \lambda, v\rangle \leq 0$.*

*Proof.* Put $c := \operatorname{cov}(A|v)$. The point $cv$ does not belong to the interior of the set $AB_X(0,1)$. Otherwise, the inclusion $(\delta + c)v \in AB_X(0,1)$ takes place fo a sufficiently small $\delta > 0$, so $\operatorname{cov}(A|v) > c$ in contradiction to the definition of $c$. Moreover, the set $AB_X(0,1) \subset Y$ is convex.

By the finite-dimensional separability theorem (see, for example [4, Theorem 4.6]) there exists a nonzero $\lambda \in Y^*$ such that $\langle \lambda,\, Ax\rangle \geq \langle \lambda, v\rangle c$ for any $x \in B_X(0,1)$. Therefore, $\langle A^*\lambda,\, x\rangle \geq \langle \lambda, v\rangle c$ for every $x \in B_X(0,1)$. So, $-\|A^*\lambda\|_* \geq \langle \lambda, v\rangle c$. Therefore, $\|A^*\lambda\|_* \leq -\langle \lambda, v\rangle c$. □

## 4 Proofs of the main results

*Proof of Theorem* 2.1. Take an arbitrary $\varepsilon > 0$. Consider the set

$$M := \{x \in B_X(x_0, R) \cap \mathcal{M} : \quad f_0(x) \le f_0(x_0)\}.$$

Two cases may occur: either there exists a point $x \in M$ such that the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly dependent or these vectors are linearly independent for every $x \in M$. In the first case, the point $\widehat{x} = x$ is the desired one (see the comments after the formulation of Theorem 2.1).

Consider the second case: the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly independent for every $x \in M$. Then the Lagrange multiplier rule imply that the point $x_0$ is not a point of local minimum of $f_0$ under the constraints $f_1(x) = ... = f_k(x) = 0$ (see, for example, [9] or [5]). Thus,

$$f_0(x_0) > \gamma(x_0, R). \tag{4.1}$$

Put $Y := \mathbb{R}^{k+1}$, $v := (-1, 0, ..., 0) \in Y$. Define a mapping $F : X \to Y$ by the formula

$$F(x) := (f_0(x), f_1(x), ..., f_k(x)), \quad x \in X.$$

Obviously, the mapping $F$ is differentiable and satisfies the assumption **(ii)** of Lemma 3.1. Indeed, if $F(x) \in F(x_0) + \text{cone}\{v\}$ for some $x \in B_X(x_0, R)$, then by virtue of the choice of $v$ we have $f_0(x) \le f_0(x_0)$ and $x_0, x \in \mathcal{M}$, where $\mathcal{M} = \{\xi : f_1(\xi) = ... = f_k(\xi) = 0\}$. Thus, the vectors $\frac{\partial f_i}{\partial x}(x)$, $i = \overline{0, k}$ are linearly independent.

Put

$$\alpha_0 := (1 + \varepsilon)(f_0(x_0) - \gamma(x_0, R))R^{-1}.$$

It follows from (4.1) that $\alpha_0 > 0$. Let us show that there exists a point $\widehat{x} \in B_X(x_0, R)$ such that

$$F(\widehat{x}) \in F(x_0) + \text{cone}\{v\} \quad \text{and} \quad \text{cov}\left(\frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v\right) < \alpha_0. \tag{4.2}$$

Consider to the contrary that

$$\bar{\alpha} := \inf\left\{\text{cov}\left(\frac{\partial F}{\partial x}(x) \,\middle|\, v\right) : x \in B_X(x_0, R), \;\; F(x) \in F(x_0) + \text{cone}\{v\}\right\} \ge \alpha_0. \tag{4.3}$$

Then the assumption **(i)** of Lemma 3.1 holds, since $\alpha_0 > 0$.

Put

$$\alpha := (1 + 2^{-1}\varepsilon)(f_0(x_0) - \gamma(x_0, R))R^{-1}.$$

It follows from relations (4.1) and $\alpha < \alpha_0 \le \bar{\alpha}$ that $\alpha \in (0, \bar{\alpha})$. Therefore, Lemma 3.1 implies that

$$F(x_0) + \alpha R v \in F(B_X(x_0, R)).$$

Therefore, there exists a point $x \in B_X(x_0, R)$ such that $F(x_0) + \alpha R v = F(x)$. Then the definition of the vector $v$ implies that

$$f_0(x_0) - \left(1 + \frac{\varepsilon}{2}\right)(f_0(x_0) - \gamma(x_0, R)) = f_0(x), \quad f_i(x) = 0, \quad i = \overline{1, k}.$$

So, relation (4.1) and the definition of the mapping $F$ imply that $f_0(x) < \gamma(x_0, R)$ and $x \in \mathcal{M} \bigcap B_X(x_0, R)$ which contradicts the definition of $\gamma$. Relation (4.2) is proved.

Applying Lemma 3.2 to the linear operator $A := \dfrac{\partial F}{\partial x}(\widehat{x})$ and the vector $v = (-1, 0, ..., 0)$ we obtain that there exists a vector $\lambda \in Y^*$ such that $\|\lambda\| = 1$ and

$$\left\| \left( \frac{\partial F}{\partial x}(\widehat{x}) \right)^* \lambda \right\|_* \le -\langle \lambda, v \rangle \operatorname{cov} \left( \frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right) = \lambda_0 \operatorname{cov} \left( \frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right). \tag{4.4}$$

This inequality and the equality $\dfrac{\partial L}{\partial x}(\widehat{x}, \lambda) = \left( \dfrac{\partial F}{\partial x}(\widehat{x}) \right)^* \lambda$, imply that

$$\left\| \frac{\partial L}{\partial x}(\widehat{x}, \lambda) \right\|_* \le \lambda_0 \operatorname{cov} \left( \frac{\partial F}{\partial x}(\widehat{x}) \,\middle|\, v \right).$$

The definition of $\alpha_0$ and the strict inequality in (4.2) imply (2.1). The inclusion (4.2) implies that $f_0(\widehat{x}) \le f_0(x_0)$. Inequality (4.4) and the relation $A^*\lambda \ne 0$ imply that $\lambda_0 \ge 0$. So, the vectors $\widehat{x}$ and $\lambda$ are the desired ones. □

*Proof of Theorem* 2.2. Put

$$\gamma_0 := \inf_{x \in \mathcal{M}} f_0(x), \quad \varepsilon := 1.$$

Take an arbitrary sequence $x_0^n \subset \mathcal{M}$ such that $f_0(x_0^n) \to \gamma_0$. Applying Theorem 2.1 at the point $x_0 = x_0^n$ as $R = 1$ we obtain that there exist sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ such that $\|\lambda^n\| = 1$ for every $n$, $f_0(x^n) \le f_0(x_0^n)$ for every $n$ and

$$\left\| \frac{\partial L}{\partial x}(x^n, \lambda^n) \right\|_* \le 2 \frac{f_0(x_0^n) - \gamma(x_0^n, R)}{R} \le 2 \frac{f_0(x_0^n) - \gamma_0}{R} \to 0$$

as $n \to \infty$. The constructed sequences $\{x^n\} \subset \mathcal{M}$ and $\{\lambda^n\} \subset \mathbb{R}^{k+1}$ are the desired ones. □

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