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KOLMOGOROV WIDTHS OF THE INTERSECTION OF A FINITE FAMILY OF SOBOLEV CLASSES

A.A. Vasil'eva

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Abstract. In the present article, order estimates for the Kolmogorov widths of the intersection of Sobolev classes on a John domain are obtained.

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In this paper, the problem of estimating the Kolmogorov widths of the intersection of a finite family of Sobolev classes $W^{r_j}_{p_j}(\Omega)$ $(1 \leqslant j \leqslant s)$ in $L_q(\Omega)$ is studied, where Ω is a John domain. The similar problem on the Kolmogorov widths of intersections of periodic function classes was investigated by Galeev [2, 3, 4]. The idea of the proof of the lower estimate was as follows: by the discretization method the problem was reduced to estimating the Kolmogorov n-widths of intersections of the balls $\big\{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : |x_1|^{p_j} + \cdots + |x_{2n}|^{p_j} \leqslant n^{-p_j\beta_j}\big\}$ in l^{2n}_q (here $\beta_j \in \mathbb{R}$). For some conditions on the parameters, the lower estimate of the widths of the function classes was the same as the upper estimate. However, in [2, 3, 4] no order estimates were obtained if these conditions are not satisfied.

In [10], Galeev's result about the n-widths of intersections of $2n$-dimensional balls was generalized to N-dimensional spaces with $N \geqslant 2n$. In the present paper, by applying the discretization method together with the estimates from [10], we evaluate the orders of the Kolmogorov widths of intersections of Sobolev classes on a John domain for all parameters except some limiting cases (see Theorems 1.2–1.4 below).

Recall some definitions.

- Let X be a normed space, and let $C \subset X$, $n \in \mathbb{Z}_+$. The Kolmogorov n-width of the set C in the space X is defined as follows:

$$d_n(C, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in C} \inf_{y \in L} \|x - y\|;$$

  here $\mathcal{L}_n(X)$ is the family of all linear subspaces in X of dimension at most n. For details, see [5, 8, 9].

- Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \leqslant p \leqslant \infty$, $r \in \mathbb{Z}_+$. Given $f \in L_1^{\rm loc}(\Omega)$, we denote by $\nabla^r f$ the vector of all partial derivatives of order r. If all its components are elements of $L_p(\Omega)$, we write $f \in \mathcal{W}^r_p(\Omega)$. The Sobolev class is defined as follows:

$$W^r_p(\Omega) = \{f \in \mathcal{W}^r_p(\Omega) : \ \|\nabla^r f\|_{L_p(\Omega)} \leqslant 1\};$$

  here $\|\nabla^r f\|_{L_p(\Omega)}$ is the L_p-norm of $|\nabla^r f(\cdot)|$, $|\cdot|$ is the Euclidean norm.

- We denote by $B_a(x)$ the open Euclidean ball of raduis a centered at point x.

  Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We say that Ω is a John domain if there exist a number $a > 0$ and a point $x_* \in \Omega$ such that for each $x \in \Omega$ there are a number $T(x) > 0$ and a curve $\gamma_x : [0,\, T(x)] \to \Omega$ with the following properties:

  (i) γ_x has the natural parametrization in the Euclidean norm on $\mathbb{R}^d$,

  (ii) $\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,

  (iii) $B_{at}(\gamma_x(t)) \subset \Omega$ for all $t \in [0,\, T(x)]$.

  If $\Omega \subset \mathbb{R}^d$ is a John domain, then 1) (see [6, 7]) the conditions ensuring the embedding $\mathcal{W}^r_p(\Omega)$ into $L_q(\Omega)$ are the same as for $\Omega = (0,\, 1)^d$, 2) (see [1]) the order estimates for the Kolmogorov widths of $W^r_p(\Omega)$ in $L_q(\Omega)$ are the same as for $\Omega = (0,\, 1)^d$ (if the orders are known). For some results on the estimates of the Kolmogorov widths of weighted Sobolev spaces on a John domain, see, e.g., [11, 12].

Let $\Omega \subset \mathbb{R}^d$ be a John domain, let $s \geqslant 2$, $r_i \in \mathbb{Z}_+$, $1 \leqslant i \leqslant s$,

$$r_1 < r_2 < \cdots < r_s, \tag{1.1}$$

$1 < p_i \leqslant \infty$, $1 \leqslant i \leqslant s$. We set

$$M = \bigcap_{j=1}^{s} W^{r_j}_{p_j}(\Omega) \tag{1.2}$$

(i.e., M is the intersection of a finite family of Sobolev classes).

Suppose that

$$\frac{r_j}{d} - \frac{1}{p_j} < \frac{r_i}{d} - \frac{1}{p_i} \quad \text{for } j > i. \tag{1.3}$$

Let $\{a_n\}_{n\geqslant n_0}$, $\{b_n\}_{n\geqslant n_0}$ be sequences of positive numbers. We write $a_n \asymp b_n$ if there is a number $c \geqslant 1$ such that $c^{-1}a_n \leqslant b_n \leqslant ca_n$ for all $n \geqslant n_0$.

Now we formulate the main result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a John domain, $1 \leqslant q < \infty$, $r_j \in \mathbb{Z}_+$, $1 < p_j \leqslant \infty$, $1 \leqslant j \leqslant s$. Suppose that (1.1) and (1.3) hold, and $\frac{r_1}{d} + \frac{1}{q} - \frac{1}{p_1} > 0$. Let M be defined by (1.2).*

(i) *If $p_i \geqslant q$ for all $i \in \{1,\, \dots,\, s\}$, then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\frac{r_s}{d}}.$$

(ii) *If $q \leqslant 2$, $p_i \leqslant q$ for all $i \in \{1,\, \dots,\, s\}$, then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\frac{r_1}{d}-\frac{1}{q}+\frac{1}{p_1}}.$$

(iii) *If $q > 2$, $p_i \leqslant 2$ for all $i \in \{1,\, \dots,\, s\}$ and $\frac{r_1}{d} \neq \frac{1}{p_1}$, then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\min\left\{\frac{r_1}{d}+\frac{1}{2}-\frac{1}{p_1},\, \frac{q}{2}\left(\frac{r_1}{d}+\frac{1}{q}-\frac{1}{p_1}\right)\right\}}.$$

(iv) *Let* $q \leqslant 2$,

$$I := \{i \in \overline{1,\, s}:\ p_i \geqslant q\} \ne \{1, \ldots, s\},$$
$$J := \{i \in \overline{1,\, s}:\ p_i \leqslant q\} \ne \{1, \ldots, s\}.$$

For $i \in I$, $j \in J$ *we define the numbers* λ_{ij} *by*

$$\frac{1}{q} = \frac{1-\lambda_{ij}}{p_j} + \frac{\lambda_{ij}}{p_i}.$$

Let

$$(i_0,\, j_0) = \underset{i\in I,\, j\in J}{\operatorname{argmax}}((1-\lambda_{ij})r_j/d + \lambda_{ij}r_i/d).$$

Then

$$d_n(M,\, L_q(\Omega)) \asymp n^{-((1-\lambda_{i_0j_0})r_{j_0}/d+\lambda_{i_0j_0}r_{i_0}/d)}.$$

(v) *Let* $q > 2$,

$$I := \{i \in \overline{1,\, s}:\ p_i \geqslant q\} \ne \{1, \ldots, s\}, \tag{1.4}$$
$$J := \{i \in \overline{1,\, s}:\ 2 \leqslant p_i \leqslant q\}, \tag{1.5}$$
$$K := \{i \in \overline{1,\, s}:\ p_i \leqslant 2\} \ne \{1, \ldots, s\}. \tag{1.6}$$

Define the numbers λ_{ij} *and* $\tilde{\lambda}_{ij}$ *by*

$$\frac{1}{q} = \frac{1-\lambda_{ij}}{p_j} + \frac{\lambda_{ij}}{p_i}, \quad i \in I,\ j \in J \cup K,$$
$$\frac{1}{2} = \frac{1-\tilde{\lambda}_{ij}}{p_j} + \frac{\tilde{\lambda}_{ij}}{p_i}, \quad i \in I \cup J,\ j \in K,$$

and set

$$(i_0,\, j_0) = \underset{i\in I,\, j\in J\cup K}{\operatorname{argmax}}((1-\lambda_{ij})r_j/d + \lambda_{ij}r_i/d),$$
$$(i_1,\, j_1) = \underset{i\in I\cup J,\, j\in K}{\operatorname{argmax}}((1-\tilde{\lambda}_{ij})r_j/d + \tilde{\lambda}_{ij}r_i/d).$$

Define the functions $h_0,\, h_1,\, h_2 : [1,\, q/2] \to \mathbb{R} \cup \{-\infty\}$ *by*

$$h_0(t) = \begin{cases} t\left((1-\lambda_{i_0j_0})\frac{r_{j_0}}{d} + \lambda_{i_0j_0}\frac{r_{i_0}}{d}\right) & \text{if } I \ne \varnothing, \\ -\infty & \text{if } I = \varnothing, \end{cases}$$

$$h_1(t) = \begin{cases} t\left((1-\tilde{\lambda}_{i_1j_1})\frac{r_{j_1}}{d} + \tilde{\lambda}_{i_1j_1}\frac{r_{i_1}}{d} - \frac{1}{2}\right) + \frac{1}{2} & \text{if } K \ne \varnothing, \\ -\infty & \text{if } K = \varnothing, \end{cases}$$

$$h_2(t) = \begin{cases} \max_{j\in J}\varphi_j(t) & \text{if } J \ne \varnothing, \\ -\infty & \text{if } J = \varnothing, \end{cases}$$

where

$$\varphi_j(t) = t\left(\frac{r_j}{d} - \frac{1}{2}\cdot\frac{1/p_j - 1/q}{1/2 - 1/q}\right) + \frac{1}{2}\cdot\frac{1/p_j - 1/q}{1/2 - 1/q}.$$

We set

$$h = \max\{h_0,\, h_1,\, h_2\}.$$

Suppose that the function h *has a unique minimum point* $t_* \in [1,\, q/2]$. *Then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-h(t_*)}.$$

Remark 1. If condition (1.3) fails, we can reduce the problem to estimating the widths of the intersection of a subfamily of Sobolev classes, for which (1.3) holds.

Consider two particular cases when there is an explicit formula for $h(t_*)$.

Theorem 1.3. *Suppose that the conditions of Theorem* 1.2 *hold. Let* $q > 2$, $\{1, \ldots, s\} = I \cup K$, *the sets* I *and* K *are defined by* (1.4), (1.6), $I \neq \{1, \ldots, s\}$, $K \neq \{1, \ldots, s\}$. *Let*

$$(i_0, j_0) = \mathrm{argmax}_{i\in I,\, j\in K}((1-\lambda_{ij})r_j/d + \lambda_{ij}r_i/d),$$

$$(i_1, j_1) = \mathrm{argmax}_{i\in I,\, j\in K}((1-\tilde{\lambda}_{ij})r_j/d + \tilde{\lambda}_{ij}r_i/d).$$

(i) *If* $(1-\tilde{\lambda}_{i_1j_1})\frac{r_{j_1}}{d} + \tilde{\lambda}_{i_1j_1}\frac{r_{i_1}}{d} - \frac{1}{2} > 0$, *then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-(1-\tilde{\lambda}_{i_1j_1})r_{j_1}/d - \tilde{\lambda}_{i_1j_1}r_{i_1}/d}.$$

(ii) *If* $(1-\tilde{\lambda}_{i_1j_1})\frac{r_{j_1}}{d} + \tilde{\lambda}_{i_1j_1}\frac{r_{i_1}}{d} - \frac{1}{2} < 0$, *then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\hat{\theta}},$$

where

$$\hat{\theta} = \frac{1}{2}\cdot\frac{(1-\lambda_{i_0j_0})\frac{r_{j_0}}{d} + \lambda_{i_0j_0}\frac{r_{i_0}}{d}}{\frac{1}{2} - (1-\tilde{\lambda}_{i_1j_1})\frac{r_{j_1}}{d} - \tilde{\lambda}_{i_1j_1}\frac{r_{i_1}}{d} + (1-\lambda_{i_0j_0})\frac{r_{j_0}}{d} + \lambda_{i_0j_0}\frac{r_{i_0}}{d}}.$$

Theorem 1.4. *Let the conditions of Theorem* 1.2 *hold,* $q > 2$, $\{1, \ldots, s\} = J$ *(see* (1.5)*), and let* $\frac{r_1}{d} + \frac{1}{q} - \frac{1}{p_1} > 0$.

(i) *Let* $\frac{r_s}{d} > \frac{1}{2}\cdot\frac{1/p_s - 1/q}{1/2-1/q}$. *Then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\frac{r_s}{d}}.$$

(ii) *Let* $\frac{r_1}{d} < \frac{1}{2}\cdot\frac{1/p_1 - 1/q}{1/2-1/q}$. *Then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\frac{q}{2}\left(\frac{r_1}{d}+\frac{1}{q}-\frac{1}{p_1}\right)}.$$

(iii) *Let* $\frac{r_s}{d} < \frac{1}{2}\cdot\frac{1/p_s - 1/q}{1/2-1/q}$, $\frac{r_1}{d} > \frac{1}{2}\cdot\frac{1/p_1 - 1/q}{1/2-1/q}$. *We set*

$$\varphi_j(t) = t\left(\frac{r_j}{d} - \frac{1}{2}\cdot\frac{1/p_j - 1/q}{1/2-1/q}\right) + \frac{1}{2}\cdot\frac{1/p_j - 1/q}{1/2-1/q},\ 1 \leqslant j \leqslant s,$$

$h(t) = \max_{1\leqslant j\leqslant s}\varphi_j(t)$. *Suppose that* h *has a unique minimum point* t_* *on* $[1, q/2]$. *Then*

$$d_n(M,\, L_q(\Omega)) \asymp n^{-\theta_{i_*j_*}},$$

where

$$\theta_{ij} = \frac{\frac{r_j}{d}\left(\frac{1}{p_i} - \frac{1}{q}\right) - \frac{r_i}{d}\left(\frac{1}{p_j} - \frac{1}{q}\right)}{\left(\frac{r_j}{d} - \frac{r_i}{d}\right)\left(1 - \frac{2}{q}\right) + \frac{1}{p_i} - \frac{1}{p_j}},$$

and the indices $i_* \neq j_*$ *are such that* $\varphi_{i_*}(t_*) = \varphi_{j_*}(t_*) = h(t_*)$.

The algorithm for construction of the function h in Theorem 1.4.

There exist a partition $1 = t_0 < t_1 < \dots < t_m = \frac{q}{2}$ of $[1, q/2]$ and a sequence $\{j_l\}_{j=1}^m \subset \{1, \dots, s\}$ such that $h|_{[t_{l-1}, t_l]} = \varphi_{j_l}|_{[t_{l-1}, t_l]}$. The points t_l and the indices j_l are defined by induction.

First we notice that $\varphi_s(1) > \varphi_j(1)$, $1 \leqslant j \leqslant s-1$; hence $h(t) = \varphi_s(t)$ in a neighbourhood of 1. We set $t_0 = 1$, $j_1 = s$.

Suppose that the points $1 = t_0 < t_1 < \dots < t_{l-1} < \frac{q}{2}$ and the indices $\{j_k\}_{k=1}^l$ are found, and, in addition, $h(t) = \max_{1 \leqslant j \leqslant j_l} \varphi_j(t)$ for $t \in [t_{l-1}, q/2]$,

$$\varphi_{j_l}(t_{l-1}) > \varphi_j(t_{l-1}), \quad 1 \leqslant j < j_l.$$

Hence $h(t) = \varphi_{j_l}(t)$ in a right neighbourhood of t_{l-1}.

If $j_l = 1$, then $h(t) = \varphi_1(t)$ for $t \in [t_{l-1}, q/2]$, and the construction is finished.

Let $j_l > 1$. We set

$$t_l = \min\{t \in (t_{l-1}, q/2) : \exists j \in \overline{1, j_l - 1} : \varphi_{j_l}(t) = \varphi_j(t)\},$$

$$j_{l+1} = \min\{j \in \overline{1, j_l - 1} : \varphi_{j_l}(t_l) = \varphi_j(t_l)\}.$$

Then $h(t) = \varphi_{j_l}(t)$ for $t_{l-1} \leqslant t \leqslant t_l$. We claim that

$$h(t) = \max_{1 \leqslant j \leqslant j_{l+1}} \varphi_j(t) \text{ for } t_l \leqslant t \leqslant q/2,$$

$$\varphi_{j_{l+1}}(t_l) > \varphi_j(t_l), \quad 1 \leqslant j < j_{l+1}.$$

Remark 2. Applying this algorithm we can similarly construct the function h_2 in Theorem 1.2; after that it is easy to construct the function h and to find its minimum point.

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Anastasia Andreevna Vasil'eva
Moscow Center for Fundamental and Applied Mathematics
M.V. Lomonosov Moscow State University
GSP-1, 1 Leninskiye gory, MSU, Main building,
faculty of mechanics and mathematics,
119991 Moscow, Russian Federation
E-mail: vasilyeva_nastya@inbox.ru

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