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Short communications

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MULTIPLIERS OF FOURIER-HAAR SERIES IN LORENTZ SPACES

N.T. Tleukhanova, A.N. Bashirova

Communicated by V.S. Guliyev

Key words: Fourier series, Haar system, Fourier series multipliers, Lorentz spaces.

AMS Mathematics Subject Classification: 42B05, 46E30.

Abstract. This article provides a complete description of the multipliers of the Fourier series along the Haar system in the Lorentz spaces $L_{p,r}$. Necessary and sufficient conditions are obtained ensuring that $\{\lambda_k^j\}_{k=0,j=1}^{\infty,2^k} \in m(L_{p,r} \to L_{q,s})$. This work generalizes and supplements the result of work [8].

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1 Introduction

Let X, Y be normed spaces of functions, defined on the segment $[0,1]$, such that $X, Y \hookrightarrow L_1$. Let $\{\varphi_k\}$ be an orthonormal system of X. For a function $f \in X$ define the Fourier series

$$f \sim \sum_{k=1}^{\infty} a_k \varphi_k,$$

where a_k are the Fourier coefficients of the function f along the system $\{\varphi_k\}$. We say that a sequence of complex numbers $\lambda = \{\lambda_k\}$ is a Fourier multiplier from the space X to the space Y, if for any function $f \in X$ with the Fourier series

$$\sum_{k=1}^{\infty} a_k \varphi_k$$

there is a function $f_\lambda \in Y$, whose Fourier series is

$$\sum_{k=1}^{\infty} \lambda_k a_k \varphi_k$$

and the operator $\Lambda f = f_\lambda$ is a bounded operator from X to Y.

The set $m(X \to Y)$ of all multipliers defined in this way is a normed linear space with the norm

$$\|\lambda\|_{m(X \to Y)} = \|\Lambda\|_{X \to Y}.$$

The theory of Fourier series multipliers has its source in the theorem of M. Riss [25], where it is shown that the characteristic function χ_A, when A is a segment from $\mathbb{Z}$, is a multiplier of the trigonometric Fourier series in the $L_p[0, 2\pi)$, that is

$$\|S_A(f)\|_{L_p} \le C\|f\|_{L_p}, \tag{1.1}$$

where C does not depend on the choice of the segment A from $\mathbb{Z}^n$ and the function f from $L_p(\mathbb{T}^n)$. In the general case, when A is an arbitrary finite subset of $\mathbb{Z}^n$, the constant C in (1.1) will depend essentially on the geometric properties of the set A [22].

For trigonometric series, the fundamental theorem of Marcinkiewicz is known [9]. Further development of the theory of multipliers of Fourier series can be found in the works [12]–[15], [17], [18], [20].

We will be interested in the multipliers of Fourier series along the Haar system.

The Haar system is a system of functions $\chi = \{\chi_k^j(x)\}_{k=0,j=1}^{\infty,2^k}$, $x \in [0,1]$, in which $\chi_1(x) \equiv 1$, and the function $\chi_k^j(x)$, where $k = 0, 1, \ldots$, $j = 1, 2, \ldots, 2^k$ is defined as:

$$\chi_k^j(x) = \begin{cases} 2^{\frac{k}{2}}, & \dfrac{2j-2}{2^{k+1}} < x < \dfrac{2j-1}{2^{k+1}}, \\ -2^{\frac{k}{2}}, & \dfrac{2j-1}{2^{k+1}} < x < \dfrac{2j}{2^{k+1}}, \\ 0, & x \notin \left(\dfrac{j-1}{2^k}; \dfrac{j}{2^k}\right). \end{cases}$$

Set of indeces (k,j) corresponding to the Haar functions will be denoted by Ω.

The Fourier-Haar series of a function $f \in L_1[0,1]$ is the series of the form

$$\sum_{k=0}^{\infty}\sum_{j=1}^{2^k} a_k^j(f)\chi_k^j(x),$$

where $a_k^j(f) = (f, \chi_k^j)$ are the Fourier-Haar coefficients of the function $f(x)$, which are calculated using the following formulas

$$a_k^j(f) = 2^k\left(\int_{\frac{j-1}{2^k}}^{\frac{j-\frac{1}{2}}{2^k}} f(x)dx - \int_{\frac{j-\frac{1}{2}}{2^k}}^{\frac{j}{2^k}} f(x)dx\right).$$

Consider a sequence $\lambda = \{\lambda_k^j\}_{k=0,j=1}^{\infty,2^k}$. Any sequence λ generates the operator Λ, called the multiplier, which is defined on polynomials along the Haar system as follows:

$$\Lambda\left(\sum_{k=0}^{N}\sum_{j=1}^{2^k} a_k^j(f)\chi_k^j(x)\right) = \sum_{k=0}^{N}\sum_{j=1}^{2^k} \lambda_k^j a_k^j(f)\chi_k^j(x).$$

According to the classical Paley-Marcinkiewicz theorem [6], if $1 < p < \infty$ and $\sup_{(k,j)\in\Omega}|\lambda_k^j| < \infty$, then

$$\|f_\lambda\|_{L_p} \le c_p\,\|f\|_{L_p} \tag{1.2}$$

for all $f \in L_p$.

The exact value $c_p = \max(p, p') - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ was found by D. Burkholder [4]. Multipliers along the Haar system were studied in [7], [10], [24] and other works.

According to Yano's theorem [24], if $1 < p < q < \infty$, then

$$\|\lambda\|_{m(L_p \to L_q)} \asymp \sup_{(k,j)\in\Omega} |\lambda_k^j| 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{1.3}$$

where the equivalence constants depend only on p, q.

The questions about the boundedness of multipliers along the Haar system in more general spaces are addressed in [2], [5], [8], [21], [23].

In [2] I.B. Bryskin, O.V. Lelond, E.M. Semenov showed that if the multiplier Λ acts from L_p to L_q, $1 < p < q < \infty$, then Λ acts also from $L_{p,r}$ to $L_{q,r}$ for all $1 \le r \le \infty$. In particular Λ acts from L_p to $L_{q,p}$.

Moreover, in order that

$$\|\lambda\|_{m(L_p \to L_{q,r})} \asymp \sup_{(k,j)\in\Omega} |\lambda_k^j| 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{1.4}$$

it is necessary and sufficient that $r \ge p$.

In [8] O.V. Lelond, E.M. Semenov, S.N. Uksusov proved the following statement: let $1 < p < q < \infty$, $1 \le r, s \le \infty$, in order that

$$\|\lambda\|_{m(L_{p,r} \to L_{q,s})} \asymp \sup_{(k,j)\in\Omega} |\lambda_k^j| 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{1.5}$$

it is necessary and sufficient that $r \le s$.

A description of the class of Fourier-Haar series multipliers $m\left(L_{p,r} \to L_{q,s}\right)$ at $r > s$ remains an open question.

In this paper we consider this case $r > s$.

2 Main result

Let f be a measurable function taking almost everywhere finite values,

$$m(\sigma, f) = \mu\left(\{x : x \in [0,1], |f| > \sigma\}\right)$$

be its distribution function. The function

$$f^*(t) = \inf\left\{\sigma : m(\sigma, f) \le t\right\}, \quad t > 0$$

is called a non-increasing rearrangement of the function f.

Let $0 < p < \infty$, $0 < r \le \infty$. The Lorentz spaces $L_{p,r}[0,1]$ are defined as the spaces of all measurable f functions defined on $[0,1]$ for which

if $r < \infty$

$$\|f\|_{L_{p,r}} = \left(\int_0^1 \left(t^{\frac{1}{p}} f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} < \infty,$$

if $r = \infty$

$$\|f\|_{L_{p,\infty}} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty.$$

Theorem 2.1. *Let $1 < p < q < \infty$, $0 < r, s \le \infty$, where $\frac{1}{\tau} = \left(\frac{1}{s} - \frac{1}{r}\right)_+ = \max\left\{\frac{1}{s} - \frac{1}{r}, 0\right\}$. If $0 < \tau < \infty$, then*

$$\|\lambda\|_{m(L_{p,r} \to L_{q,s})} \asymp \left(\sum_{k=0}^{\infty} \left(2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \sup_{1\le j\le 2^k} \left|\lambda_k^j\right|\right)^{\tau}\right)^{\frac{1}{\tau}},$$

if $\tau = +\infty$, the expression on the right is replaced by the $\sup_{\substack{0\le k\le\infty \\ 1\le j\le 2^k}} 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} |\lambda_k^j|$.

The proof of this theorem is based on theorem 2.1 from [1] and theorem 3 from [11]. We also used interpolation methods for Lorentz spaces and interpolation properties of the Lebesgue and net spaces [3], [16], [19].

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