ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 4 (2022), 61 – 69

MARCINKIEWICZ'S INTERPOLATION THEOREM FOR LINEAR OPERATORS ON NET SPACES

A.H. Kalidolday, E.D. Nursultanov

Communicated by T.V. Tararykova

Key words: interpolation spaces, net spaces, linear operators.

AMS Mathematics Subject Classification: 46B70.

Abstract. In this paper, we study the interpolation properties of the net spaces $N_{p,q}(M)$. We prove some analogue of Marcinkiewicz's interpolation theorem. This theorem allows to obtain the strong boundedness of linear operators in the net spaces from the weak boundedness of these operators in the net spaces with local nets.

DOI: https://doi.org/10.32523/2077-9879-2022-13-4-61-69

1 Introduction

Let μ be the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and M be an arbitrary system of Lebesgue measurable subsets of $\mathbb{R}^n$. For a function $f(x)$, defined and integrable on each e from M, we define the function

$$\bar{f}(t, M) = \sup_{\substack{e\in M\\ |e|\geqslant t}} \frac{1}{|e|}\left|\int_e f(x)d\mu\right|, \quad t > 0,$$

where the supremum is taken over all $e \in M$, whose measure is $|e| \stackrel{\text{def}}{=} \mu e \geqslant t$. In the case when $\sup\{|e| : e \in M\} = \alpha < \infty$ and $t > \alpha$ we assume that $\bar{f}(t, M) = 0$.

Let parameters p, q satisfy the conditions $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. We define the net spaces $N_{p,q}(M)$, as the set of all functions f, such that for $q < \infty$

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{\frac{1}{p}}\bar{f}(t, M)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,$$

and for $q = \infty$

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}}\bar{f}(t, M) < \infty.$$

These spaces were introduced in the work [18].

Net spaces have found important applications in various problems of harmonic analysis, operator theory and the theory of stochastic processes [2, 3, 4, 22, 23, 19, 24, 21].

Let (A_0, A_1) be a compatible pair of Banach spaces [8],

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}),\ a \in A_0 + A_1,$$

be the Petre functional. For $0 < q < \infty$, $0 < \theta < 1$ the interpolation space is defined by

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

and for $q = \infty$ by

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}.$$

In papers [1, 6, 7, 15, 24, 18, 19, 20, 27, 28] the interpolation properties of the net spaces were investigated and found an application in various problems of analysis. In particular, the following equalities were obtained

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} = N_{p,q}(M), \tag{1.1}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$, $0 < q_0, q_1, q \leqslant \infty$.

In the case when M is the set of all segments in $\mathbb{R}$, or M is the set of all dyadic cubes in $\mathbb{R}^n$. An analogue of equality (1.1) with respect to the anisotropic interpolation method of Fernandez is also proved, when M is the set of all rectangles in $\mathbb{R}^n$. In the case when M is an arbitrary net, only the embedding takes place [18, Theorem 1]

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} \hookrightarrow N_{p,q}(M). \tag{1.2}$$

For some nets M the space $N_{p_1,\infty}(M)$ coincides with the Morrey space. In papers [10, 16, 17, 25] it was shown that the Morrey spaces the equality of form (1.1) does not hold. Therefore for general nets, in the case when M is an arbitrary net equality (1.1) is not true.

In this paper, we study the interpolation properties of these spaces for fairly general nets. We prove a certain analogue of the Marcinkiewicz-type interpolation theorem for linear operators. Note that here we use ideas developed in [11, 12, 13], where an alternative analogue of the Marcinkiewicz-type interpolation theorem for Morrey spaces is obtained. For properties of linear operators in Morrey spaces, see recent papers [5, 14, 26].

Given functions F and G, in this paper $F \lesssim G$ means that $F \leqslant c\, G$ (or $c\, F \geq G$), where c is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

2 Interpolation of the net spaces for local nets

A family of measurable sets $G = \{G_t\}_{t>0}$ is called a local net if it satisfies the following condition: $G_t \subset G_s$ for $t \leqslant s$ and $|G_t| = t$.

An example of a local net is the set $\{B_t(x)\}_{t>0}$ of all balls centered at the point x.

Lemma 2.1. *(Hardy's inequalities,* [9, p. 25]*) Let $\mu > 0$ and $1 \leqslant \tau \leqslant \infty$. Then the following inequalities hold*

$$\left(\int_0^\infty \left(y^{-\mu} \int_0^y |g(r)| \frac{dr}{r} \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} \leqslant \mu^{-1} \left(\int_0^\infty \left(y^{-\mu} |g(y)| \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}}$$

and

$$\left(\int_0^\infty \left(y^{\mu} \int_y^\infty |g(r)| \frac{dr}{r} \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} \leqslant \mu^{-1} \left(\int_0^\infty \left(y^{\mu} |g(y)| \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}},$$

for all functions g Lebesgue measurable on $(0, \infty)$.

Theorem 2.1. *Let* $0 < p_0 < p_1 < \infty$, $\quad 0 < q_0, q_1, q, \leq \infty$, $\ 0 < \theta < 1$. *If* $G = \{G_t\}_{t>0}$ *is a local net, then*

$$(N_{p_0,q_0}(G), N_{p_1,q_1}(G))_{\theta,q} = N_{p,q}(G),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. From embedding (1.2) we have

$$(N_{p_0,q_0}(G), N_{p_1,q_1}(G))_{\theta,q} \hookrightarrow N_{p,q}(G).$$

Let us show the reverse inclusion. Let $f \in N_{p,q}(M)$ and $t > 0$, χ_{G_t} be the characteristic function of the set G_t. Let $\sigma = \min\{q_0, q_1, q\}$. Using the embedding $N_{p_i,\sigma} \hookrightarrow N_{p_i,q_i}$, $\quad i = 0, 1$ we get

$$\|f\|_{\left(N_{p_0,q_0}(G), N_{p_1,q_1}(G)\right)_{\theta,q}} \lesssim \|f\|_{\left(N_{p_0,\sigma}(G), N_{p_1,\sigma}(G)\right)_{\theta,q}}$$

$$\asymp \left(\int\limits_0^\infty \left(t^{-\theta(\frac{1}{p_0}-\frac{1}{p_1})} K(t^{\frac{1}{p_0}-\frac{1}{p_1}}, f; N_{p_0,\sigma}(G), N_{p_1,\sigma}(G)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_0^\infty \left(t^{-\theta(\frac{1}{p_0}-\frac{1}{p_1})} \left(\|f\chi_{G_t}\|_{N_{p_0,\sigma}} + t^{\frac{1}{p_0}-\frac{1}{p_1}} \|f(1-\chi_{G_t})\|_{N_{p_1,\sigma}} \right) \right)^q \right)^{\frac{1}{q}}. \tag{2.1}$$

Note that

$$\|f\chi_{G_t}\|_{N_{p_0,\sigma}} \leqslant \left(\int_0^t \left(s^{1/p_0} \bar{f}(s,G) \right)^\sigma \frac{ds}{s} \right)^{\frac{1}{\sigma}} + \frac{p_0}{\sigma} t^{1/p_0} \bar{f}(t,G)$$

and

$$\|f(1-\chi_{G_t})\|_{N_{p_0,\sigma}} \asymp \left(\int_t^\infty \left(s^{1/p_1} \overline{f(1-\chi_{G_t})}(s,G) \right)^\sigma \frac{ds}{s} \right)^{\frac{1}{\sigma}} + t^{1/p_1} \bar{f}(t,G).$$

Putting these relations in (2.1), applying the Hardy inequality, we obtain

$$\|f\|_{\left(N_{p_0,q_0}(G), N_{p_1,q_1}(G)\right)_{\theta,q}} \lesssim \|f\|_{N_{p,q}(G)}.$$

□

3 Marcinkiewicz-type interpolation theorem for linear operators

Let $G = \{G_t\}_{t>0}$ be a local net. We define the net $F_{G,\Omega} = \bigcup_{x\in\Omega} G + x$, where $G + x = \{G_t + x\}_{t>0}$. The net $F_{G,\Omega}$ will be called the net generated by a local net G and a set $\Omega \subset \mathbb{R}^n$.

Example. Let $\Omega = \mathbb{R}^n$, $G = \{Q_t\}_{t>0}$ be the set of cubes centered at 0 with the length of the edge equal to t, then $F_{G,\Omega} = \{Q_t + x\}_{x\in\mathbb{R}^n}$ is the set of all cubes in $\mathbb{R}^n$.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$, $G = \{G_t\}_{t>0}$ *be a local net,* $F_{G,\Omega} = \bigcup_{x\in\Omega} G + x$. *Let* $0 < p_0 < p_1 < \infty$ *and* $0 < q_0, q_1 \leqslant \infty$, $q_0 \neq q_1$, $0 < \theta < 1$, $1 \leqslant \tau \leqslant \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If the following inequalities hold for a linear operator T *and some* $M_0, M_1 > 0$

$$\|Tf\|_{N_{q_i,\infty}(G+x)} \leqslant M_i \|f\|_{N p_i,1(G+x)}, \quad x \in \Omega, \quad i = 0, 1, \tag{3.1}$$

for all functions $f \in N_{p_i,1}(G + x), (i = 0, 1)$, *then the following inequality holds*

$$\|Tf\|_{N_{q,\tau}(F_{G,\Omega})} \leqslant cM_0^{1-\theta}M_1^{\theta}\|f\|_{N_{p,\tau}(F_{G,\Omega})}, \tag{3.2}$$

for all functions $f \in N_{p,\tau}(F_{G,\Omega})$, *where* $c > 0$ *depends only on the parameters* $p_0, p_1, q_0, q_1, p, q, \tau, \theta$.

Proof. Let $1 \leqslant \tau < \infty$, $f \in N_{p,\tau}(F_{G,\Omega})$, for an arbitrary $x \in \mathbb{R}^n$, $s > 0$ we define the functions

$$f_{0,s} = f\chi_{G_s+x}, \quad f_{1,s} = f - f_{0,s},$$

where χ_{G_s+x} denotes the characteristic function of the set $G_s + x$. It is easy to see that $f_{0,s} \in N_{p_0,1}(G+x)$ and $f_{1,s} \in N_{p_1,1}(G+x)$. Then $f = f_{0,s} + f_{1,s}$ and

$$\sup_{\xi\geqslant t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}Tf(y)dy\right| \leqslant \sup_{\xi\geqslant t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}Tf_{0,s}(y)dy\right|$$
$$+\sup_{\xi\geqslant t}\frac{1}{|G_s|}\left|\int_{G_\xi+x}Tf_{1,s}(y)dy\right| = I_1 + I_2.$$

First, we estimate I_1, according to inequality (3.1) we have

$$I_1 = \sup_{\xi\geqslant t}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}Tf_{0,s}(y)dy\right|$$
$$\leqslant t^{-\frac{1}{q_0}}\sup_{r>0}r^{\frac{1}{q_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}Tf_{0,s}(y)dy\right|$$
$$= t^{-\frac{1}{q_0}}\|Tf_{0,s}\|_{N_{q_0,\infty}(G+x)} \leqslant M_0t^{-\frac{1}{q_0}}\|f_{0,s}\|_{N_{p_0,1}(G+x)}$$

$$= M_0t^{-\frac{1}{q_0}}\left(\int_0^s r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi|}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r}\right).$$

If $\xi \leqslant s$, $y \in G_\xi + x$, we have $f_{0,s}(y) = f(y)\chi_{G_s+x} = f(y)$. If $\xi > s$, then

$$\left|\int_{G_\xi+x}f_{0,s}(y)dy\right| = \left|\int_{G_s+x}f(y)dy\right|.$$

For the first integral, we have the following,

$$\int_0^s r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r}$$
$$\leqslant \int_0^s r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f(y)dy\right|\frac{dr}{r} \leqslant \int_0^s r^{\frac{1}{p_0}}\bar{f}(r, F_{G,\Omega})\frac{dr}{r}.$$

For the second integral, we have

$$\int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi}\left|\int_{G_\xi+x}f_{0,s}(y)dy\right|\frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi}\left|\int_{G_s+x}f(y)dy\right|\frac{dr}{r}$$
$$= \left|\int_{G_s+x}f(y)dy\right|\int_s^\infty r^{\frac{1}{p_0}}\sup_{\xi\geqslant r}\frac{1}{|G_\xi|}\frac{dr}{r} = \left|\int_{G_s+x}f(y)dy\right|\int_s^\infty r^{\frac{1}{p_0}-1}\frac{dr}{r}$$
$$= p_0's^{\frac{1}{p_0}}\frac{1}{|G_s|}\left|\int_{G_s+x}f(y)dy\right| \leqslant p_0's^{\frac{1}{p_0}}\bar{f}(s, F_{G,\Omega}).$$

Thus, we get

$$I_1 \lesssim M_0 t^{-\frac{1}{q_0}} \bigg(\int_0^s r^{\frac{1}{p_0}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + s^{\frac{1}{p_0}} \bar{f}(s, F_{G,\Omega}) \bigg).$$

We estimate I_2, in a similar way, applying inequality (3.1), and we obtain

$$\begin{aligned} I_2 &= \sup_{s \geqslant t} \frac{1}{|G_s|} \bigg| \int_{G_s+x} T f_{1,s}(y) dy \bigg| \\ &\leqslant t^{-\frac{1}{q_1}} \sup_{r>0} r^{\frac{1}{q_1}} \sup_{s \geqslant r} \frac{1}{|G_s|} \bigg| \int_{G_s+x} T f_{1,s}(y) dy \bigg| = t^{-\frac{1}{q_1}} \| T f_{1,s}(y) \|_{N_{q_1,\infty}(G+x)} \\ &\leqslant M_1 t^{-\frac{1}{q_1}} \| f_{1,s} \|_{N_{p_1,1}(G+x)} = M_1 t^{-\frac{1}{q_1}} \bigg(\int_0^\infty r^{\frac{1}{p_1}} \sup_{s \geqslant r} \frac{1}{|G_s|} \bigg| \int_{G_s+x} f_{1,s}(y) dy \bigg| \frac{dr}{r} \bigg) \\ &= M_1 t^{-\frac{1}{q_1}} \bigg(\int_0^s r^{\frac{1}{p_1}} \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg| \int_{G_\xi+x} f_{1,s}(y) dy \bigg| \frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg| \int_{G_\xi+x} f_{1,s}(y) dy \bigg| \frac{dr}{r} \bigg) \\ &= M_1 t^{-\frac{1}{q_1}} \Big(J_1 + J_2 \Big). \end{aligned}$$

To estimate J_1, J_2 note that

$$\bigg| \int_{G_\xi+x} f_{1,s}(y) dy \bigg| = \begin{cases} 0, \ \xi \leqslant s, \\ \Big| \int_{G_\xi+x \setminus G_s+x} f(y) dy \Big|, \ \xi > s \end{cases}$$

$$= \begin{cases} 0, \ \xi \leqslant s, \\ \Big| \int_{G_\xi+x} f(y) dy - \int_{G_s+x} f(y) dy \Big| \leqslant \Big| \int_{G_\xi+x} f(y) dy \Big| + \Big| \int_{G_s+x} f(y) dy \Big|, \ \xi > s. \end{cases}$$

Further,

$$\begin{aligned} J_1 &\leqslant \int_0^s r^{\frac{1}{p_1}} \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg(\bigg| \int_{G_\xi+x} f(y) dy \bigg| + \bigg| \int_{G_s+x} f(y) dy \bigg| \bigg) \frac{dr}{r} \\ &\leqslant \int_0^s r^{\frac{1}{p_1}} \bigg(\bar{f}(s, F_{G,\Omega}) + \bigg| \int_{G_s+x} f(y) dy \bigg| \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg) \frac{dr}{r} \leqslant 2 \bar{f}(s, F_{G,\Omega}) \int_0^s r^{\frac{1}{p_1}} \frac{dr}{r} = 2 p_1 s^{\frac{1}{p_1}} \bar{f}(s, F_{G,\Omega}) \end{aligned}$$

and

$$\begin{aligned} J_2 &\leqslant \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg(\bigg| \int_{G_\xi+x} f(y) dy \bigg| + \bigg| \int_{G_s+x} f(y) dy \bigg| \bigg) \frac{dr}{r} \\ &\leqslant \int_s^\infty r^{\frac{1}{p_1}} \bigg(\bar{f}(s, F_{G,\Omega}) + \bigg| \int_{G_s+x} f(y) dy \bigg| \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \bigg) \frac{dr}{r} \leqslant \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} \\ &+ \bigg| \int_{G_s+x} f(y) dy \bigg| \int_s^\infty r^{\frac{1}{p_1}} \sup_{\xi \geqslant r} \frac{1}{|G_\xi|} \frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + \bigg| \int_{G_s+x} f(y) dy \bigg| \frac{s^{\frac{1}{p_1}-1} p_1}{p_1 - 1} \\ &\lesssim \int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + s^{\frac{1}{p_1}} \bar{f}(s, F_{G,\Omega}). \end{aligned}$$

Combining the obtained estimates, we get the following estimate

$$I_2 \lesssim M_1 t^{\frac{1}{q_1}} \left(\int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + s^{\frac{1}{p_1}} \bar{f}(s, F_{G,\Omega}) \right).$$

So, we have

$$\sup_{s \geqslant t} \frac{1}{|G_s|} \left| \int_{G_s+x} Tf(y)dy \right| \lesssim M_0 t^{-\frac{1}{q_0}} \left(\int_0^s r^{\frac{1}{p_0}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + s^{\frac{1}{p_0}} \bar{f}(s, F_{G,\Omega}) \right)$$
$$+ M_1 t^{-\frac{1}{q_1}} \left(\int_s^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} + s^{\frac{1}{p_1}} \bar{f}(s, F_{G,\Omega}) \right).$$

Assume that $s = ct^\gamma$, where $\gamma = \left(\frac{1}{q_1} - \frac{1}{q_0} \right) \Big/ \left(\frac{1}{p_1} - \frac{1}{p_0} \right)$. Then, taking into account the estimates obtained above, we get

$$\|Tf\|_{N_{q,\tau}(F_{G,\Omega})} = \left(\int_0^\infty \left(t^{\frac{1}{q}} \sup_{\substack{s \geqslant t \\ x \in \mathbb{R}^n}} \frac{1}{|G_s|} \left| \int_{G_s+x} f(x)dx \right| \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}$$
$$\lesssim M_0 A_1 + M_0 A_2 + M_1 A_3 + M_1 A_4,$$

where

$$A_1 = \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_0}} \int_0^{ct^\gamma} r^{\frac{1}{p_0}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}},$$

$$A_2 = \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_0}} (ct^\gamma)^{\frac{1}{p_0}} \bar{f}(ct^\gamma, F_{G,\Omega}) \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}},$$

$$A_3 = \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_1}} \int_{ct^\gamma}^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}$$

and

$$A_4 = \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_1}} (ct^\gamma)^{\frac{1}{p_1}} \bar{f}(ct^\gamma, F_{G,\Omega}) \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}.$$

Using the following change of the variable $ct^\gamma = y$, we get

$$A_1 = \gamma^{-\frac{1}{\tau}} c^{-\theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} B_1, \quad A_2 = \gamma^{-\frac{1}{\tau}} c^{-\theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} B_2$$
$$A_3 = \gamma^{-\frac{1}{\tau}} c^{(1-\theta)\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} B_2, \quad A_4 = \gamma^{-\frac{1}{\tau}} c^{(1-\theta)\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} B_3,$$

where

$$B_1 = \left(\int_0^\infty \left(y^{\theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} \int_0^y r^{\frac{1}{p_0}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}},$$

$$B_2 = \left(\int_0^\infty \left(y^{\frac{1}{p}} \bar{f}(y, F_{G,\Omega}) \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} = \|f\|_{N_{p,\tau}(F_{G,\Omega})},$$

$$B_3 = \left(\int_0^\infty \left(y^{-(1-\theta)\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} \int_y^\infty r^{\frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \frac{dr}{r} \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}}$$

and

$$B_4 = \left(\int_0^\infty \left(y^{\frac{1}{p}} \bar{f}(y, F_{G,\Omega}) \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} = \|f\|_{N_{p,\tau}(F_{G,\Omega})}.$$

To estimate B_1, B_3 we apply Hardy's inequalities from Lemma 2.1, we obtain

$$B_1 \lesssim \left(\int_0^\infty \left(y^{\theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right) + \frac{1}{p_0}} \bar{f}(r, F_{G,\Omega}) \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} \lesssim \|f\|_{N_{p,\tau}(F_{G,\Omega})},$$

$$B_3 \lesssim \left(\int_0^\infty \left(y^{(\theta-1)\left(\frac{1}{p_1} - \frac{1}{p_0}\right) + \frac{1}{p_1}} \bar{f}(r, F_{G,\Omega}) \right)^\tau \frac{dy}{y} \right)^{\frac{1}{\tau}} \lesssim \|f\|_{N_{p,\tau}(F_{G,\Omega})}.$$

Thus, from the obtained estimates we come to the following estimate,

$$\|Tf\|_{N_{q,\tau}(F_{G,\Omega})} \lesssim \left(M_0 c^{-\theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} + M_1 c^{(1-\theta)\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} \right) \|f\|_{N_{p,\tau}(F_{G,\Omega})},$$

where the corresponding constants depend only on $p_0, p_1, q_0, q_1, p, q, \tau$ and θ.

Let $c = \left(\frac{M_1}{M_0} \right)^{\frac{p_0 p_1}{p_1 - p_0}}$, then

$$\|Tf\|_{N_{q,\tau}(F_{G,\Omega})} \lesssim M_0^{1-\theta} M_1^\theta \|f\|_{N_{p,\tau}(F_{G,\Omega})}.$$

Therefore, we got the required estimate. □

Acknowledgments

This research was supported by the Ministry of Education and Science of the Republic of Kazakhstan (project no. AP14870758).

References

[1] T.U. Aubakirov, E.D. Nursultanov, *The Hardy-Littlewood theorem for Fourier-Haar series.* Math. Notes, 73 (2003), no. 3, 314–320.

[2] R. Akylzhanov, M. Ruzhansky, $L_p - L_q$ *multipliers on locally compact groups.* J. Fun. Anal., 278 (2020), no. 3, 1-49.

[3] R. Akylzhanov, M. Ruzhansky, *Net spaces on lattices, Hardy-Littlewood type inequalities, and their converses.* Eurasian Math. J., 8 (2017), no. 3, 10–27.

[4] R. Akylzhanov, M. Ruzhansky, E.D. Nursultanov, *Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and* $L_p - L_q$ *Fourier multipliers on compact homogeneous manifolds.* J. Math. Anal. Appl., 479 (2019), no. 2, 1519–1548.

[5] O.G. Avsyankin, *On integral operators with homogeneous kernels in Morrey spaces*, Eurasian Math. J., 12 (2021), no. 1, 92–96.

[6] A.N. Bashirova, A.H. Kalidolday, E.D. Nursultanov, *Interpolation theorem for anisotropic net spaces.* Russian Mathematics (Izvestiya VUZ. Matematika), 8 (2021), 1–13.

[7] A.N. Bashirova, E.D. Nursultanov, *On the inequality of different metrics for multiple Fourier-Haar series.* Eurasian Math. J., 12 (2021), no. 3, 90–93.

[8] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Springer, Berlin, 1976.

[9] O.V. Besov, V.P. Il'in, S.M. Nikośkii, *Integral representations of functions and imbedding theorems.* Izdat. "Nauka", Moscow, 1975 (in Russian).

[10] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Campanato and block spaces.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 4 (1999), 31–40.

[11] V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov, *Marcinkiewicz-type interpolation theorem and estimates for convolutions for Morrey-type spaces.* Eurasian Math. J., 9 (2018), no. 2, 82–88.

[12] V.I. Burenkov, E.D. Nursultanov, *Interpolation theorems for nonlinear operators in general Morrey-type spaces and their applications.* Proc. Steklov Inst. Math., 312 (2021), 124–149.

[13] V.I. Burenkov, E.D. Nursultanov, *Interpolation theorems for nonlinear Urysohn integral operators in general Morrey-type spaces.* Eurasian Math. J., 11 (2020), no. 4, 87–94.

[14] V.I. Burenkov, M.A. Senouci, *Boundedness of the generalized Riesz potential in local Morrey type spaces*, Eurasian Math. J., 12 (2021), no. 4, 92–98.

[15] A.H. Kalidolday, E.D. Nursultanov, *Interpolation properties of certain classes of net spaces.* arXiv:2107.10633 [math.FA]

[16] P.G. Lemarié-Rieusset, *Multipliers and Morrey spaces.* Potential Analysis, Springer 38 (2013), no. 3, 741–752.

[17] P.G. Lemarié-Rieusset, *Erratum to: Multipliers and Morrey spaces.* Potential Analysis, Springer 41 (2014), no. 4, 1359–1362.

[18] E.D. Nursultanov, *Net spaces and inequalities of Hardy-Littlewood type.* Sb. Math., 189 (1998), no. 3, 399–419.

[19] E.D. Nursultanov, *On the coefficients of multiple Fourier series in* L_p *- spaces.* Izv. Math., 64 (2000), no. 1, 93–120.

[20] E.D. Nursultanov, *Application of interpolational methods to the study of properties of functions of several variables.* Mathematical Notes, 75 (2004), 341–351.

[21] E.D. Nursultanov, T.U. Aubakirov, *Interpolation methods for stochastic processes spaces.* Abstr. Appl. Anal., 2013 (2013), 1–12.

[22] E.D. Nursultanov, A.G. Kostyuchenko, *Theory of control of "catastrophes"* . Russ. Math. Surv., 53 (1998), no. 3, 628–629.

[23] E.D. Nursultanov, N.T. Tleukhanova, *Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces.* Func. Anal. Appl., 34 (2000), no. 2, 151–153.

[24] E.D. Nursultanov, S. Tikhonov, *Net spaces and boundedness of integral operators.* J. Geom. Anal., 21 (2011), 950–981.

[25] A. Ruiz, L. Vega, *Corrigenda to "Unique continuation for Schrodinger operators" and a remark on interpolation of Morrey spaces.* Publicacions Matem'atiques, 39 (1995), 405–411.

[26] M.A. Senouci,*Boundedness of Riemann-Liouville fractional integral operator in Morrey spaces*. Eurasian Math. J., 12 (2021), no. 1, 82–91.

[27] N.T. Tleukhanova, A.Bakhyt, *On trigonometric Fourier series multipliers in* $\lambda_{p,q}$ *spaces*. Eurasian Math. J., 12 (2021), no. 1, 103–106.

[28] N.T. Tleukhanova, K.K. Sadykova,*O'Neil-type inequalities for convolutions in anisotropic Lorentz spaces*. Eurasian Math. J., 10 (2019), no. 3, 68–83.

Aitolkyn Hunvaikyzy Kalidolday
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Kazakhstan
E-mails: aitolkynnur@gmail.com

Erlan Dautbakovich Nursultanov
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Kazakhstan

Department of Mathematics and Informatics
M.V.Lomonosov Moscow State University (Kazakhstan branch)
11 Kazhimukan St,
010010 Astana, Kazakhstan
and
Institute of Mathematics and Mathematical Modeling
125 Pushkin St,
050010 Almaty, Kazakhstan
E-mail: er-nurs@yandex.kz

Received: 25.07.2021