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EXISTENCE OF PERIODIC SOLUTIONS TO A CERTAIN IMPULSIVE DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

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Key words: Carvalho's method, periodic solution, impulsive differential equation, piecewise constant arguments.

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Abstract. This paper concerns the existence of the solutions to a first order nonlinear impulsive differential equation with piecewise constant arguments. Moreover, the periodicity of the solutions is investigated.

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1 Introduction

This paper deals with the impulsive differential equation with piecewise constant arguments of the form

$$x^{'}(t) = \lambda x(t) - g(x[t]) - h(x[t+1]), \quad t \neq k \in \mathbb{Z}^+ = \{1, 2, \dots\},\ t \geq 0, \tag{1.1}$$

$$x(t^-) = dx(t), \quad t = k \in \mathbb{Z}^+, \tag{1.2}$$

where $\lambda > 0$ is a real constant, $d \in \mathbb{R}\backslash\{0, 1\}$, $g, h : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions, $x(k^-) = \lim\limits_{t \to k^-} x(t)$, $x(k) = x(k^+) = \lim\limits_{t \to k^+} x(t)$, i.e., $x(t)$ is right continuous at $t = k$ and $[.]$ denotes the greatest integer function.

On the other hand, the theory of impulsive differential equations has been developed very rapidly. Such equations consist of differential equations with impulse effects and emerge in the modeling of real world problems observed in engineering, physics and biology, etc. The books [8], [24] are good sources for the study of impulsive differential equations and their applications. Moreover, there exist many papers that investigate the behavior of solutions of impulsive differential equations ([9]-[36]). In addition, recently some papers were published dealing with impulsive differential equations and systems ([3]-[22]).

Since the early 1980s, differential equations with piecewise constant arguments have attracted a great deal of attention from researchers in science. Differential equations with piecewise constant arguments appear in diverse areas such as engineering, physics and mathematics. The work [11] covers a systematical study on mathematical models with piecewise constant arguments. Differential equations with piecewise constant arguments are closely related to difference and differential equations. Therefore, they are stated as hybrid dynamical systems [14]. The qualitative works on convergence, oscillation, periodicity and stability of solutions of differential equations with piecewise constant arguments have been done by works [1]-[13]. Also, Wiener's book [34] is a distinguished source with respect to this area.

Carvalho [12] developed a method to obtain conditions for the existence of nonconstant periodic solutions of certain differential equations. After this work, the scalar equation

$$x^{'}(t) = \lambda x(t) - g(x[t])$$

and the system

$$x_1^{'}(t) = -\lambda x_1(t) + g(x_2([t]-1)),$$
$$x_2^{'}(t) = -\lambda x_2(t) + g(x_1([t]-1))$$

were considered by Seifert [32] and proved that this equation and system have a periodic solution with period 2 using the method. Moreover, in 2014, Lafci and Bereketoglu [23] studied the existence of periodic solutions of the following nonlinear impulsive differential system with piecewise constant arguments.

$$x_1^{'}(t) = \lambda x_1(t) - g(x_2[t-1]),$$
$$x_2^{'}(t) = \lambda x_2(t) - g(x_1[t-1]), \quad t \neq k \in \mathbb{Z}^+ = \{1, 2, \dots\},\ t \geq 0,$$
$$x_1(t^-) = dx_1(t), \quad x_2(t^-) = dx_2(t), \quad t = k \in \mathbb{Z}^+.$$

As we know, there are only a few papers ([21]-[29]) on the periodicity of impulsive differential equations with piecewise constant arguments. So, our aim is to search periodic solutions with period 2 of the impulsive differential equation with piecewise constant arguments (1.1)-(1.2) by using Carvalho's method which is given below.

Theorem 1.1. (Carvalho's method, [17]) *If p is a positive integer and $x(k)$ is a periodic sequence of period p, then the following statements hold true:*

(i) If $p > 1$ is odd and $m = \frac{p-1}{2}$, then

$$x(k) = a_0 + \sum_{j=1}^{m} a_j \cos\left(\frac{2jk\pi}{p}\right) + b_j \sin\left(\frac{2jk\pi}{p}\right), \tag{1.3}$$

for all $k \geqslant 1$.

(ii) If p is even and $p = 2m$, then

$$x(k) = a_0 + a_m \cos \pi k + \sum_{j=1}^{m-1} a_j \cos\left(\frac{2jk\pi}{p}\right) + b_j \sin\left(\frac{2jk\pi}{p}\right), \tag{1.4}$$

for all $k \geqslant 1$.

For example, if $p = 2$, then

$$x(k) = a_0 + a_1 \cos \pi k. \tag{1.5}$$

A solution to $(1.1) - (1.2)$ is defined as below.

Definition 1. A function x defined on $[0, \infty)$ is said to be a solution to (1.1)-(1.2) if it satisfies the following conditions:

(i) $x : [0, \infty) \to \mathbb{R}$ is continuous for $t \in [0, \infty)$ with the possible exception of the points $t = 1, 2, \dots,$

(ii) x is right continuous and has left-hand limits at the points $t = 1, 2, \dots,$

(iii) $x^{'}(t)$ exists for every $t \in [0, \infty)$ with the possible exception of the points $t = 0, 1, 2, \dots,$ where one-sided derivatives exist,

(iv) $x(t)$ satisfies equation (1.1) on each interval $k < t < k + 1$, $k \in \mathbb{N} = \{0, 1, 2, ...\}$,

(v) $x(t)$ satisfy (1.2) at $t = 1, 2, ...$

2 Main results

In this section, we prove the existence of the solutions and study the periodicity of equation (1.1) with impulse condition (1.2).

Theorem 2.1. *Any solution x to* $(1.1)-(1.2)$ *on the interval* $[0,\infty)$ *has the form*

$$x(t)=x([t])\exp\lambda(t-[t])-[g(x([t])+h(x[t+1])]\frac{\exp\lambda(t-[t])-1}{\lambda},\ t\neq k\in\mathbb{N}, \tag{2.1}$$

and for $t=k$, $k\in\mathbb{N}$, $x(k)$ *satisfies the difference equation*

$$x(k+1)=\frac{\alpha}{d}x(k)-\frac{\beta}{d}[g(x(k))+h(x(k+1))], \tag{2.2}$$

where $\alpha=\exp\lambda$ *and* $\beta=(\exp\lambda-1)/\lambda$.

Proof. In the interval $k<t<k+1$, equation (1.1) can be reduced to the ordinary differential equation

$$x^{'}(t)-\lambda x(t)=-[g(x(k))-h(x(k+1)]. \tag{2.3}$$

Solving equation (2.3), we obtain

$$x(t)=x(k)\exp\lambda(t-k)-[g(x(k))+h(x(k+1)]\frac{\exp\lambda(t-k)-1}{\lambda},\qquad k<t<k+1.$$

Replacing k by $[t]$, we obtain (2.1). At $t=k$, we find the solution of equation (1.1) in the interval $k-1<t<k$ as

$$x(t)=x(k-1)\exp\lambda(t-k+1)-[g(x(k-1))+h(x(k))]\frac{\exp\lambda(t-k+1)-1}{\lambda}.$$

Now, using impulse condition (1.2) with the assumption of right continuity at $t=k$, we find difference equation (2.2). □

It is noted that impulsive differential system $(1.1)-(1.2)$ has a unique solution satisfying the following conditions:

$$x(0)=x_0,\ x(1)=x_1,$$

where x_0 and x_1 are real constants. Also, we note that under the same conditions difference equation (2.2) has a unique solution.

Theorem 2.2. *Let x be a solution to* $(1.1)-(1.2)$. *If $x(k)$ satisfies equation* (2.2) *such that* $x(k+p)=x(k)$ *for all* $k\in\mathbb{N}$, *then we have* $x(t+p)=x(t)$ *for all* $t\in[0,\infty)$, *where p is the least positive integer.*

Proof. In the interval $k<t<k+1$, from (2.1), we get

$$\begin{aligned}x(t+p) &= x(k+p)\exp\lambda(t-k)-[g(x(k+p))+h(x(k+p+1))]\frac{\exp\lambda(t-k)-1}{\lambda}\\ &= x(k)\exp\lambda(t-k)-[g(x(k))+h(x(k+1))]\frac{\exp\lambda(t-k)-1}{\lambda}\\ &= x(t).\end{aligned}$$

□

Theorem 2.3. *Let $x(t)$ be a solution to* $(1.1)-(1.2)$. *If $x(k)$ satisfies equation* (2.2) *such that* $x(k+1)=x(k)$, *then $x(t)$ is a constant.*

Proof. From (2.2) and the definition of β, we get

$$x(k) = \frac{-\beta}{1-\alpha}[g(x(k)) + h(x(k+1))] = \frac{g(x(k)) + h(x(k+1))}{\lambda}$$

and substituting this into (2.1) for $k < t < k+1$, we have

$$\begin{aligned} x(t) &= \frac{g(x(k)) + h(x(k+1))}{\lambda} \exp \lambda(t-k) - [g(x(k)) + h(x(k+1))] \frac{\exp \lambda(t-k) - 1}{\lambda} \\ &= \frac{g(x(k)) + h(x(k+1))}{\lambda}. \end{aligned}$$

□

Theorem 2.4. *Assume that $\lambda > 0$ is a sufficiently small real constant. If g and h are odd functions and there is a number $a > 0$ such that*

$$g(a) - h(a) = a(d+1) \tag{2.4}$$

and

$$g^{'}(a) + h^{'}(a) \neq -d+1,\ g^{'}(a) - h^{'}(a) \neq d+1, \tag{2.5}$$

then there exists a solution x with the least period 2 of (1.1) – (1.2).

Proof. A solution x of (1.1) – (1.2) is given by (2.1). From Theorem 2.2, it is known that

$$x(t+2) = x(t) \text{ for all } t \in [0, \infty) \tag{2.6}$$

provided that

$$x(k+2) = x(k) \text{ for all } k \in \mathbb{N}, \tag{2.7}$$

where $x(k)$ is a solution of (2.2). So, we should only prove that (2.7) is true. Because of Theorem 1.1, we can choose a solution of (2.2) as

$$x(k) = a_0(\alpha) + a_1(\alpha) \cos \pi k, \tag{2.8}$$

where $\alpha = e^{\lambda}$, a_i, $i = 0, 1$ are real-valued functions. Substituting (2.8) into (2.2), we obtain

$$a_0(1 - \frac{\alpha}{d}) - a_1(1 + \frac{\alpha}{d}) \cos \pi k + \frac{\beta}{d}[g(a_0 + a_1 \cos \pi k) + h(a_0 - a_1 \cos \pi k)] = 0. \tag{2.9}$$

If equation (2.9) is satisfied for $k = 0$ and $k = 1$, then it holds for all $k \in \mathbb{N}$. So, putting $k = 0$ and $k = 1$ into (2.9), we get the following system

$$\begin{aligned} a_0(1 - \frac{\alpha}{d}) - a_1(1 + \frac{\alpha}{d}) + \frac{\beta}{d}[g(a_0 + a_1) + h(a_0 - a_1)] &= 0, \\ a_0(1 - \frac{\alpha}{d}) + a_1(1 + \frac{\alpha}{d}) + \frac{\beta}{d}[g(a_0 - a_1) + h(a_0 + a_1)] &= 0. \end{aligned} \tag{2.10}$$

For $\lambda = 0$, it is $\alpha = 1$ and also $\beta(1) = 1$. Hence, (2.10) reduces to the system

$$\begin{aligned} a_0(1 - \frac{1}{d}) - a_1(1 + \frac{1}{d}) + \frac{1}{d}[g(a_0 + a_1) + h(a_0 - a_1)] &= 0, \\ a_0(1 - \frac{1}{d}) + a_1(1 + \frac{1}{d}) + \frac{1}{d}[g(a_0 - a_1) + h(a_0 + a_1)] &= 0. \end{aligned} \tag{2.11}$$

Since g and h are odd and satisfy (2.4), system (2.11) has a solution as $a_0 = 0$, $a_1 = a$. Therefore, equation (2.2) has a periodic solution with period 2 as

$$x(k) = a\cos \pi k. \tag{2.12}$$

This means that (1.1) − (1.2) has a solution of least period 2 for $\lambda = 0$.

Now, let λ be sufficiently small. Again, we should find a solution $x(k)$ of equation (2.2) of form (2.8). To fulfill this, we use the Implicit Function Theorem to show that there exists a $\delta > 0$ such that there are functions $a_0(\alpha)$ and $a_1(\alpha)$ which are continuous for $0 \le \alpha - 1 < \delta$ and $a_0(1) = 0$, $a_1(1) = a$. Putting (2.8) into (2.2), we find

$$\begin{aligned} a_0(\alpha)(1-\frac{\alpha}{d}) - a_1(\alpha)(1+\frac{\alpha}{d})\cos \pi k + \frac{\beta(\alpha)}{d}[g(a_0(\alpha) + a_1(\alpha)\cos \pi k) \\ +h(a_0(\alpha) - a_1(\alpha)\cos \pi k)] = 0. \end{aligned} \tag{2.13}$$

Again, if this equation holds for $k = 0$ and $k = 1$, then it will be satisfied for all $k \in \mathbb{N}$. Substituting $k = 0$ and $k = 1$ into (2.14), respectively, we obtain the system

$$\begin{aligned} a_0(\alpha)(1-\frac{\alpha}{d}) - a_1(\alpha)(1+\frac{\alpha}{d}) + \frac{\beta(\alpha)}{d}[g(a_0(\alpha)+a_1(\alpha)) + h(a_0(\alpha)-a_1(\alpha))] &= 0, \\ a_0(\alpha)(1-\frac{\alpha}{d}) + a_1(\alpha)(1+\frac{\alpha}{d}) + \frac{\beta(\alpha)}{d}[g(a_0(\alpha)-a_1(\alpha)) + h(a_0(\alpha)+a_1(\alpha))] &= 0. \end{aligned} \tag{2.14}$$

Since g and h are odd, the Jacobian determinant of system (2.14) at $\alpha = 1$ is

$$\begin{aligned} J &= \frac{1}{d^2}\begin{vmatrix} d-1+g'(a)+h'(a) & -d-1+g'(a)-h'(a) \\ d-1+g'(a)+h'(a) & d+1-g'(a)+h'(a) \end{vmatrix} \\ &= \frac{2}{d^2}(d^2-1+2dh'(a)+2g'(a)-g'^2(a)+h'^2(a)) \\ &= \frac{2}{d^2}(g'(a)-h'(a)-1-d)(g'(a)+h'(a)-1+d). \end{aligned}$$

From (2.5), we obtain that $J \neq 0$ at $\alpha = 1$. So, for sufficiently small $\lambda > 0$, there is a $\delta > 0$ such that there exist functions $a_0(\alpha)$, $a_1(\alpha)$ that are continuous on $[1, 1+\delta)$ and form a solution of system (2.14) such that $a_0(1) = 0$, $a_1(1) = a$. □

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