ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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13 Kazhymukan St
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117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 4 (2022), 44 – 53

A NOTE ON CAMPANATO'S L^p-REGULARITY WITH CONTINUOUS COEFFICIENTS

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Communicated by P.D. Lamberti

Key words: regularity, elliptic systems, continuous coefficients.

AMS Mathematics Subject Classification: 35B65, 35B45, 35J50.

Abstract. In this note we consider local weak solutions of elliptic equations in variational form with data in L^p. We refine the classical approach due to Campanato and Stampacchia and we prove the L^p-regularity for the solutions assuming the coefficients merely continuous. This result shows that it is possible to prove the same sharp L^p-regularity results that can be proved using classical singular kernel approach also with the variational regularity approach introduced by De Giorgi. This method works for general operators: parabolic, in nonvariational form, of order $2m$.

DOI: https://doi.org/10.32523/2077-9879-2022-13-4-44-53

1 Introduction

The 19^{th} Hilbert problem [13] concerns the analyticity of the solutions of regular problems in the calculus of variations. The attempt to prove such a problem pushed many mathematicians to prove regularity results for elliptic and parabolic *linear* equations. Among them, let us quote the seminal contributions by Schauder [20, 21], who proved Hölder regularity for solutions to equations with Hölder continuous coefficients, and the seminal works of Calderón and Zygmund [2, 3], who proved L^p-regularity for solutions to equations with continuous coefficients by using the singular kernel approach. This technique based on the potentials was adopted by Nash [19] to complete the proof of the 19^{th} Hilbert problem. More specifically, he proved that weak solutions to elliptic and parabolic equations with L^∞ coefficients are Hölder continuous. One year before, De Giorgi in [9] solved the same problem using a completely different and totally new method based on some variational embeddings.

The two approaches are totally different. De Giorgi's approach was extended by Ladyženskaja, Solonnikov and Ural'ceva in [16] to the parabolic case and by Di Benedetto to the *nonlinear* case (see, for instance, [11]). A similar approach holds also in the case of nonvariational operators as proved by Krylov and Safonov [15]. The singular kernel approach is not so flexible as the variational approach. For its application to the *nonlinear* case we cite the fundamental work of Iwaniec [14], which could be considered as the starting point of nonlinear Calderón-Zygmund theory, [18] and references therein. Moreover, as showed by De Giorgi's counterexample [10], general regularity results for systems are not valid (with the exception of the case $N = 2$).

After De Giorgi's breakthrough, the natural question was whether these new techniques could be adapted to retrieve results from Schauder and Calderón-Zygmund results, namely whether the variational technique of De Giorgi, originally conceived for equations with L^∞ coefficients, could be extended to a broader theory which could also include variational and non-variational operators of generic order $2m$ and systems of equations with smooth coefficients. Thanks to an intuition due to De Giorgi, this goal was achieved by Campanato. He first introduced suitable function spaces and thanks to their embedding properties he was able to prove Schauder regularity estimates for local solutions to homogeneous equations with Hölder continuous coefficients.

The use of the variational approach to prove L^p-regularity is less straightforward. This point was (partially) solved by Campanato and Stampacchia [7]. The strategy is the following: first one proves regularity results in the spaces L^2 and BMO (refer to [4]), then the L^p-regularity is proved by making use of the Stampacchia interpolation theorem (as introduced in [22, 23], see also [5]). The theory was then extended by Campanato [6] to the case $m, N \geqslant 1$. The problem was that the BMO regularity requires that the coefficients are not merely continuous (using De Giorgi's techniques requires the Hölder continuity of the coefficients). So, the use of BMO regularity in proving L^p-regularity has as a natural consequence an unnecessary extra regularity assumption on the coefficients, that are assumed to be Hölder continuous instead of merely continuous.

In this short note we prove that the De Giorgi approach works in L^p with the *right* regularity of the coefficients, i.e that Campanato's assumptions can be weakened to just continuous coefficients, recovering in this way the L^p-theory obtained by Calderón and Zygmund by means of singular integrals. Such a result is implicitly proved in other papers (the first one was [8]), but to our knowledge, it was never explicitly stated. Hence, the novelty of this work lies in the fact that we are able to show that Campanato-De Giorgi's approach is equivalent to the one of Calderon-Zygmund. In particular, exploiting the old and original Campanato variational technique developed in the sixties, we prove the same result obtained with the singular integral approach. For more recent techniques concerning regularity in L^p spaces, we refer the reader to [11, 12, 18] and the references therein.

The result we prove here has the same field of application as Campanato's techniques, i.e. it is valid for systems, for elliptic and parabolic operators of order $2m$, for variational, nonvariational and ultraweak solutions (for the definition of ultraweak solution see, for instance, [24]). For the sake of simplicity, here we consider only the case of elliptic equations in variational form of order 2 reduced to the principal part, but, as already stated, this approach works in general.

Let $N \in \mathbb{N}$ and Ω be an open bounded subset of $\mathbb{R}^N$. By $L^p(\Omega)$ we denote the standard Lebesgue space of p-integrable real-valued functions on Ω, and by $W^{k,p}(\Omega)$ we denote the standard Sobolev space of functions in $L^p(\Omega)$ with weak derivatives up to the order k in $L^p(\Omega)$.

We consider linear elliptic equations reduced to the principal part, namely of the form

$$-\sum_{i,j=1}^{N} D_i\left(a_{ij} D_j u\right) = -\sum_{i=1}^{N} D_i f_i + f_0. \tag{1.1}$$

Here $a_{ij}(\cdot)$ are assumed to be bounded and continuous functions in Ω satisfying the ellipticity condition, namely that there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}\, \xi_i \xi_j \geqslant \nu \sum_{i=1}^{N} |\xi|^2 \tag{1.2}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$. We will say that $u \in W^{1,2}(\Omega)$ is a *variational* solution in Ω to (1.1) if

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij} D_j u\, D_i \phi\, dx = \int_\Omega \Big(\sum_{i=1}^{N} f_i\, D_i \phi + f_0 \phi\Big)\, dx, \qquad \forall \phi \in W_0^{1,2}(\Omega),$$

where $F := -\sum_{i=1}^{N} D_i f_i + f_0$ belongs to $W^{-1,2}(\Omega)$. This means that $f_i \in L^2(\Omega)$ for each $i = 1, ..., N$ and $f_0 \in L^q(\Omega)$ where q is the maximum between 1 and $\hat{2} = \frac{2N}{N+2}$.

Given $p > 1$ and $F := -\sum_{i=1}^N D_i f_i + f_0$, where $f_i \in L^p(\Omega)$ for all $i = 1, \ldots, N$ and $f_0 \in L^q(\Omega)$, with q the maximum between 1 and $\hat{p} = \frac{pN}{N+p}$, we define

$$\|F\|_{W^{-1,p}(\Omega)} := \|f_0\|_{L^q(\Omega)} + \sum_{i=1}^N \|f_i\|_{L^p(\Omega)}.$$

The main result of this paper is the following:

Theorem 1.1. *Fix $2 < p < \infty$. Let $u \in W^{1,p}_{loc}(\Omega)$ be a local variational solution to (1.1) with $F \in W^{-1,p}(\Omega)$ Then for any compact set $K \subset \Omega$ the following inequality*

$$\|Du\|_{L^p(K)} \leqslant C \left(\|F\|_{W^{-1,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right) \tag{1.3}$$

holds. Here C is a positive constant that depends on $dist(K, \partial\Omega), p, N, \nu, \|a_{ij}\|_\infty$, the modulus of continuity ω of the functions a_{ij} and the Lebesgue measure of the set Ω.

We recall that Theorem 1.1 is proved in [7] under the assumption of Hölder continuity of the coefficients. Hence, in this note we will assume that (1.3) holds in the case of constant coefficients.

In the sequel, as usual, C will denote a generic positive constant which may change from line to line and also within the same line. We will explicitly write the dependence on the parameters when needed. Moreover, by $B_x(r)$ we denote the ball of radius $r > 0$ centered at the point x.

2 Regularity results for continuous coefficients

Recall that the coefficients a_{ij} are continuous in Ω. In order to localize the solution u we will use a Vitali covering of the compact set $K \subset \Omega$. Hence, we assume that for every $\varepsilon > 0$ there exists a finite family of points $\{x_k\}_{k=1,\ldots,n_\varepsilon} \subset K$ such that

$$K \subset \bigcup_{k=1,\ldots,n_\varepsilon} B_{x_k}(\varepsilon)$$

and

$$B_{x_i}(2\varepsilon) \cap B_{x_j}(2\varepsilon) = \emptyset$$

except for a finite number m_K of indices. Using the Vitali covering we introduce smooth cut-off functions $\theta_k \in C_0^\infty(\mathbb{R}^N), k = 1, \ldots, n_\varepsilon$ such that

$$\theta_k(x) := \begin{cases} 1 & \text{in } B_{x_k}(\varepsilon), \\ 0 & \text{in } B_{x_k}(2\varepsilon)^c. \end{cases}$$

We will use the functions θ_k in order to localize the solution u to the ball $B_{x_k}(2\varepsilon)$.

Recall that the variational formulation of (1.1) reads as follows:

$$\int_\Omega \sum_{i,j=1}^N a_{ij} D_j u \, D_i \phi \, dx = \int_\Omega \sum_{i=1}^N f_i \, D_i \phi \, dx + \int_\Omega f_0 \, \phi \, dx.$$

For a general number $r \in (1, \infty)$, denote by $r^* := \frac{rN}{N-r}$ the corresponding critical Sobolev exponent. Note that if we take $u \in W^{1,p}(\Omega)$ (thus taking the test functions $\phi \in W^{1,p'}(\Omega)$), the minimal requirement for F so that the integrals above are well defined is that $f_i \in L^p(\Omega)$ and $f_0 \in L^q(\Omega)$ with $q = ((p')^*)' = \hat{p} = \frac{pN}{N+p}$, thanks to the Sobolev embedding (cf. [17, Theorem 12.4]). Note also that $q^* = p$.

Proposition 2.1. *Let $u \in W^{1,2}(\Omega)$ be a variational solution to (1.1) with $F \in W^{-1,2}(\Omega)$. Then, for any compact $K \subset \Omega$ the following estimate*

$$\|Du\|^2_{L^2(K)} \leqslant C\left(\|u\|^2_{L^2(\Omega)} + \|F\|^2_{W^{-1,2}(\Omega)}\right)$$

holds, where $C = C(\mathrm{dist}(K, \partial\Omega), \nu, N, \|a_{ij}\|_{L^\infty(\Omega)})$ is independent of $u \in W^{1,2}(\Omega)$.

Proof. Let $k \in \{1, \dots, n_\varepsilon\}$ be fixed. Using $\theta_k^2 u \in W_0^{1,2}(\Omega)$ as test function, we have

$$\int \sum a_{ij} D_j u\, D_i(\theta_k^2 u) dx = \int \sum a_{ij} D_j u\,(\theta_k u D_i \theta_k + \theta_k D_i(\theta_k u))\, dx.$$

Moreover, using the identity $\theta_k D_j u = D_j(\theta_k u) - u D_j \theta_k$ we get that

$$\begin{aligned}
&\int \sum a_{ij} D_j u\, D_i(\theta_k^2 u) dx \\
&\qquad = \int \sum \left[a_{ij} u D_i \theta_k\,(D_j(\theta_k u) - u D_j\theta_k) + a_{ij} D_i(\theta_k u)\,(D_j(\theta_k u) - u D_j \theta_k)\right] dx \\
&\qquad = \int \sum a_{ij} u (D_i\theta_k) D_j(\theta_k u) dx - \int \sum a_{ij} u^2 (D_i\theta_k)(D_j\theta_k) dx \\
&\qquad\quad + \int \sum a_{ij} D_i(\theta_k u) D_j(\theta_k u) dx - \int \sum a_{ij} u D_i(\theta_k u) D_j\theta_k\, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int \sum a_{ij} D_j(\theta_k u)\, D_i(\theta_k u) dx &= \int \sum a_{ij} D_j u D_i(\theta_k^2 u) - \int \sum a_{ij} u (D_i\theta_k) D_j(\theta_k u) dx \\
&\quad + \int \sum a_{ij} u^2 (D_i\theta_k)(D_j\theta_k) dx + \int \sum a_{ij} u D_i(\theta_k u) D_j\theta_k\, dx =: I_1 + I_2 + I_3 + I_4. \qquad (2.1)
\end{aligned}$$

We note that

$$D(\theta_k^2 u) = (D\theta_k)(\theta_k u) + \theta_k D(\theta_k u),$$

thus

$$\|D(\theta_k^2 u)\|^2_{L^2(\Omega)} \leq C\left(\|\theta_k u\|^2_{L^2(\Omega)} + \|D(\theta_k u)\|^2_{L^2(\Omega)}\right). \qquad (2.2)$$

By definition of variational solution, and using the Peter-Paul inequality, we have that

$$\begin{aligned}
|I_1| &= \left| \int \left(\sum f_i D_i(\theta_k^2 u) + f_0(\theta_k^2 u)\right) dx \right| \\
&\leq C\left(c(\varepsilon)\|F\|^2_{W^{-1,2}(B_{x_k}(2\varepsilon))} + \varepsilon\|\theta_k^2 u\|^2_{W^{1,2}(\Omega)}\right) \\
&\leqslant C\left(\|F\|^2_{W^{-1,2}(B_{x_k}(2\varepsilon))} + \|\theta_k^2 u\|^2_{L^2(\Omega)} + \varepsilon\|D(\theta_k^2 u)\|^2_{L^2(\Omega)}\right) \\
&\leq C\left(\|F\|^2_{W^{-1,2}(B_{x_k}(2\varepsilon))} + \|u\|^2_{L^2(B_{x_k}(2\varepsilon))} + \varepsilon\|D(\theta_k u)\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

where in the last inequality we have used (2.2). Again using the Peter-Paul inequality we also have that

$$|I_2| \leqslant C\Big(\|u\|^2_{L^2(B_{x_k}(2\varepsilon))} + \varepsilon\|D(\theta_k u)\|^2_{L^2(\Omega)}\Big),$$

and a similar estimate holds for the term I_4. Finally, it is readily seen that

$$|I_3| \leqslant C\|u\|^2_{L^2(B_{x_k}(2\varepsilon))}.$$

Therefore, from (2.1) and ellipticity assumption (1.2) we get

$$\nu\|D(\theta_k u)\|^2_{L^2(\Omega)} \leqslant C_1\|F\|^2_{W^{-1,2}(B_{x_k}(2\varepsilon))} + C_2\|u\|^2_{L^2(B_{x_k}(2\varepsilon))} + C_3\varepsilon\|D(\theta_k u)\|^2_{L^2(\Omega)}.$$

Hence, taking $\varepsilon > 0$ small enough

$$\|Du\|^2_{L^2(B_{x_k}(\varepsilon))} \leq \|D(\theta_k u)\|^2_{L^2(\Omega)} \leqslant C\left(\|F\|^2_{W^{-1,2}(B_{x_k}(2\varepsilon))} + \|u\|^2_{L^2(B_{x_k}(2\varepsilon))}\right).$$

Summing up over the elements of the Vitali covering we can conclude that

$$\|Du\|^2_{L^2(K)} \leqslant m_K\, C\left(\|u\|^2_{L^2(\Omega)} + \|F\|^2_{W^{-1,2}(\Omega)}\right)$$

which is the desired estimate. □

Remark 1. A careful inspection of the passages in the previous proof reveals that the final constant $C > 0$ depends only on the ellipticity constant ν, the dimension N, the L^∞-norm of the coefficients a_{ij} and the distance $\operatorname{dist}(K, \partial\Omega)$ of the compact set K from the boundary of Ω.

Recall that $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent.

Proposition 2.2. *Fix $2 < p \leq 2^*$ if $N > 2$, $2 < p < \infty$ if $N \leq 2$. Let $u \in W^{1,p}(\Omega)$ be a local variational solution to* (1.1) *with $F \in W^{-1,p}(\Omega)$. Then for any compact set $K \subset \Omega$ the following inequality*

$$\|Du\|_{L^p(K)} \leqslant C\left(\|F\|_{W^{-1,p}(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{2.3}$$

holds. Here C is a positive constant that depends on dist$(K, \partial\Omega), p, N, \nu, \|a_{ij}\|_\infty$ and the modulus of continuity ω of the functions a_{ij}.

Proof. We make use of the Vitali covering of the compact set K again. Note that here we assume that

$$B_{x_i}(3\varepsilon) \cap B_{x_j}(3\varepsilon) = \emptyset$$

except for a finite number m_K of indices. Thus, fix $k \in \{1, \ldots, n_\varepsilon\}$ and $\phi \in W^{1,p'}_0(\Omega)$ (note that $\theta_k u \in W^{1,p}(\Omega)$). Since by assumption u is a variational solution, taking $\theta_k\phi \in W^{1,p'}_0(\Omega)$ as a test function we have that

$$\int \left(\sum f_i D_i(\theta_k\phi) + f_0(\theta_k\phi)\right) dx = \int \sum a_{ij} D_j u\, D_i(\theta_k\phi)\, dx.$$

Moreover

$$\begin{aligned}\int \sum a_{ij} D_j u\, D_i(\theta_k\phi)\, dx &= \int \sum a_{ij} D_j u (D_i\theta_k)\phi\, dx + \int \sum a_{ij}(D_j u)\theta_k(D_i\phi)\, dx \\ &= \int \sum a_{ij} D_j u(D_i\theta_k)\phi\, dx + \int \sum a_{ij} D_j(\theta_k u)(D_i\phi)\, dx - \int \sum a_{ij}(D_j\theta_k)u(D_i\phi)\, dx.\end{aligned}$$

Therefore we have that

$$\begin{aligned}
&\int \sum a_{ij} D_j(\theta_k u) D_i\phi \, dx \\
&= \int \sum a_{ij}(D_j u) D_i(\theta_k\phi) dx - \int \sum a_{ij} D_j u (D_i\theta_k)\phi \, dx + \int \sum a_{ij}(D_i\phi)(D_j\theta_k) u \, dx \\
&= \int \sum f_i D_i(\theta_k\phi) dx + \int f_0(\theta_k\phi) dx - \int \sum a_{ij} D_j u (D_i\theta_k)\phi \, dx \\
&\quad + \int \sum a_{ij}(D_i\phi)(D_j\theta_k) u \, dx \\
&= \int \sum \Big(f_i \, D_i\theta_k - a_{ij}(D_i\theta_k)(D_j u) \Big) \phi \, dx + \int f_0\theta_k\phi \, dx \\
&\quad + \int \sum \Big(f_i\theta_k + a_{ij}(D_i\theta_k) u \Big) D_i\phi \, dx.
\end{aligned}$$

Since this holds for every $\phi \in W_0^{1,p'}(\Omega)$, the function $\theta_k u$ is a variational solution in Ω of the following problem

$$\sum_{i=1}^N D_i \left(a_{ij}(x) D_j(\theta_k u) \right) = \tilde{F} := \sum_{i=1}^N D_i \tilde{f}_i + \tilde{f}_0,$$

where

$$\tilde{f}_0 := f_0\theta_k + \sum_{i,j=1}^N \Big(f_i \, D_i\theta_k - a_{ij}(D_i\theta_k)(D_j u) \Big)$$

and

$$\tilde{f}_i := f_i\theta_k + \sum_{j=1}^N a_{ij}(D_i\theta_k) u, \quad \text{for every } i = 1, \ldots, N.$$

Since $p < 2^*$ it is not difficult to see that $q < 2$. Moreover, note that if $S \subset \mathbb{R}^N$ has finite Lebesgue measure $|S| < \infty$, and $1 \le r \le t$ then $L^t(S) \subset L^r(S)$ and

$$\|g\|_{L^r(S)} \le |S|^{\frac{t-r}{tr}} \|g\|_{L^t(S)} \quad \text{for all } g \in L^t(S).$$

From this observation we have that $\tilde{f}_0 \in L^q(\Omega)$ since $f_0 \in L^q(\Omega)$, $f_i \in L^p(\Omega)$, $D_j u \in L^p_{loc}(\Omega)$, and we have the following inequality

$$\begin{aligned}
\|\tilde{f}_0\|_{L^q(\Omega)} &\le C \left(\|f_0\|_{L^q(B_{x_k}(2\varepsilon))} + \sum_{i=1}^N \|f_i\|_{L^q(B_{x_k}(2\varepsilon))} + \|Du\|_{L^q(B_{x_k}(2\varepsilon))} \right) \\
&\le C \left(\|f_0\|_{L^q(B_{x_k}(2\varepsilon))} + \sum_{i=1}^N \|f_i\|_{L^p(B_{x_k}(2\varepsilon))} + \|Du\|_{L^2(B_{x_k}(2\varepsilon))} \right) \\
&\le C \left(\|F\|_{W^{-1,p}(B_{x_k}(2\varepsilon))} + \|u\|_{L^2(B_{x_k}(3\varepsilon))} + \|F\|_{W^{-1,2}(B_{x_k}(3\varepsilon))} \right) \\
&\le C \left(\|F\|_{W^{-1,p}(B_{x_k}(3\varepsilon))} + \|u\|_{L^2(B_{x_k}(3\varepsilon))} \right).
\end{aligned}$$

Observe that in the second-to-last passage in the above inequality we have made use of Proposition 2.1, which yields

$$\|Du\|_{L^2(B_{x_k}(2\varepsilon))} \leqslant C \left(\|u\|_{L^2(B_{x_k}(3\varepsilon))} + \|F\|_{W^{-1,2}(B_{x_k}(3\varepsilon))} \right). \tag{2.4}$$

Moreover, by the Sobolev embedding (cf. [17, Theorem 12.4]) and (2.4) we also get that

$$\begin{aligned}\|u\|_{L^p(B_{x_k}(2\varepsilon))} &\leq C\|u\|_{W^{1,2}(B_{x_k}(2\varepsilon))} \leq C\left(\|u\|_{L^2(B_{x_k}(3\varepsilon))} + \|F\|_{W^{-1,2}(B_{x_k}(3\varepsilon))}\right) \\ &\leq C\left(\|u\|_{L^2(B_{x_k}(3\varepsilon))} + \|F\|_{W^{-1,p}(B_{x_k}(3\varepsilon))}\right). \end{aligned} \tag{2.5}$$

Therefore for every $i = 1, \dots, N$ we have that $\tilde{f}_i \in L^p(\Omega)$ since $f_i \in L^p(\Omega)$ and $u \in L^p_{loc}(\Omega)$, and the following inequality

$$\|\tilde{f}_i\|_{L^p(\Omega)} \leq C\big(\|f_i\|_{L^p(B_{x_k}(2\varepsilon))} + \|u\|_{L^p((B_{x_k}(2\varepsilon))}\big) \leq C\Big(\|F\|_{W^{-1,p}(B_{x_k}(2\varepsilon))} + \|u\|_{L^2(B_{x_k}(2\varepsilon))}\Big)$$

holds. By the above computations we get that

$$\|\tilde{F}\|_{W^{-1,p}(\Omega)} \leq C\left(\|F\|_{W^{-1,p}(B_{x_k}(3\varepsilon))} + \|u\|_{L^2(B_{x_k}(3\varepsilon))}\right). \tag{2.6}$$

Freezing the coefficients with respect to the point x_k, we obtain

$$\sum_{i=1}^{N} D_i\Big(a_{ij}(x_k)D_j(\theta_k u)\Big) = \tilde{F} - \sum_{i,j=1}^{N} D_i\Big((a_{ij}(x_k) - a_{ij}(x))D_j(\theta_k u)\Big). \tag{2.7}$$

Define $G \in W^{1,p}(\Omega)$ as the right-hand side of (2.7). Using the Campanato-Stampacchia estimate for Hölder continuous coefficients (refer to (4.1) in [7]), which in particular holds for constant coefficients, and inequality (2.6), we obtain

$$\begin{aligned}\|D(\theta_k u)\|_{L^p(\Omega)} &\leq C\|G\|_{W^{-1,p}(\Omega)} \leq C\left(\|\tilde{F}\|_{W^{-1,p}(\Omega)} + \omega(2\varepsilon)\|D(\theta_k u)\|_{L^p(\Omega)}\right) \\ &\leq C\left(\|F\|_{W^{-1,p}(B_{x_k}(3\varepsilon))} + \|u\|_{L^2(B_{x_k}(3\varepsilon))} + \omega(2\varepsilon)\|D(\theta_k u)\|_{L^p(\Omega)}\right).\end{aligned}$$

Here $\omega(\cdot)$ denotes the modulus of continuity of the coefficients a_{ij}. Taking $\varepsilon > 0$ small enough we get

$$\|Du\|_{L^p(B_{x_k}(\varepsilon))} \leq \|D(\theta_k u)\|_{L^p(\Omega)} \leq C\left(\|F\|_{L^p(B_{x_k}(3\varepsilon))} + \|u\|_{L^2(B_{x_k}(3\varepsilon))}\right).$$

Summing up over the elements of the Vitali covering we finally get

$$\|Du\|_{L^p(K)} \leqslant m_K\, C\left(\|F\|_{W^{-1,p}(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

which is the desired estimate. □

Remark 2. Upon careful inspection of the passages in the previous proof, one realizes that the constant C in (2.3) depends on the same parameters $\mathrm{dist}(K, \partial\Omega), \nu, N, \|a_{ij}\|_{L^\infty(\Omega)}$ as in Proposition 2.1, and on the additional parameters p and $\omega(\cdot)$, the modulus of continuity in Ω of the coefficients a_{ij}.

Proof of Theorem 1.1. By Proposition 2.2 we obtain that

$$\|Du\|_{L^p(K)} \leqslant C\left(\|F\|_{W^{-1,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right)$$

holds for $2 < p \leq 2^*$. Reproducing the same argument for $2^* < p \leq 2^{**}$ and so on, one can prove the validity of (1.3) for any $p \in (2, +\infty)$. □

Remark 3. By duality arguments, one also can prove similar estimates for any $p \in (1,2)$. Indeed, if $p \geqslant 2$ and $p' = p/(p-1) \in (1,2)$ is the conjugate exponent of p, inequality (1.3) holds also replacing p by p', assuming additionally that $F \in W^{-1,p'}(\Omega)$. Recall that

$$\|Du\|_{L^{p'}(K)} = \sup_{\substack{g_i \in L^p(\Omega) \\ \operatorname{supp} g_i \subset K \\ \|g_i\|_{L^p(\Omega)} \leq 1}} \int \sum_{i=1}^N (D_i u) g_i \, dx,$$

where $K \subset\subset \Omega$. We fix $(g_1, \dots, g_N) \in L^p(\Omega) \times \cdots \times L^p(\Omega)$ such that $\|g_i\|_{L^p(\Omega)} \leqslant 1$ and $\operatorname{supp} g_i \subset K$ for each $i = 1, \dots, N$. Consider the local variational solution $w \in W_0^{1,p}(K)$ to the Dirichlet problem in K associated with

$$-\sum_{i,j=1}^N D_i(a_{ji} D_j w) = -\sum_{i=1}^N D_i g_i =: G$$

(such a solution exists since by assumption the coefficients $a_{ji}(\cdot)$ satisfy the ellipticity condition, thus the operator associated with them is coercive, and one can apply Lax-Milgram theory). Since w satisfies the Dirichlet boundary conditions, and (1.3) holds for $p > 2$, we have that

$$\|Dw\|_{L^p(K)} \leq C \|G\|_{W^{-1,p}(\Omega)}.$$

Using the fact that u is a variational solution to (1.1) we get that

$$\begin{aligned}
\int \sum_{i=1}^N (D_i u) g_i \, dx &= \int \sum_{i,j=1}^N a_{ji} D_j w \, D_i u \, dx = \int \sum_{i=1}^N f_i (D_i w) \, dx + \int f_0 w \, dx \\
&\leq \sum_{i=1}^N \|f_i\|_{L^{p'}(K)} \|D_i w\|_{L^p(K)} + \|f_0\|_{L^q(K)} \|w\|_{L^{p^*}(K)} \\
&\leq C \left(\sum_{i=1}^N \|f_i\|_{L^{p'}(\Omega)} + \|f_0\|_{L^q(\Omega)} \right) \|Dw\|_{L^p(K)} + \|f_0\|_{L^q(\Omega)} \|w\|_{L^p(K)} \\
&\leq C \|F\|_{W^{-1,p'}(\Omega)} \|Dw\|_{L^p(K)} \\
&\leq C \|F\|_{W^{-1,p'}(\Omega)} \|G\|_{W^{-1,p}(K)} \\
&\leq C \|F\|_{W^{-1,p'}(\Omega)}
\end{aligned}$$

where q is such that $q^* = p'$. Here we have first used the Sobolev embedding $W^{1,p} \subset L^{p^*}$, then the Poincare inequality for functions in $W_0^{1,p}$ (cf. [1, Corollary 9.19]). Finally, taking the supremum with respect to all $g_i \in L^p(\Omega)$ such that $\|g_i\| \leq 1$ and $\operatorname{supp} g_i \subset K$, we obtain

$$\|Du\|_{L^{p'}(K)} \leq C \|F\|_{W^{-1,p'}(\Omega)}.$$

Acknowledgments

The authors are members of the 'Gruppo Nazionale per l'Analisi matematica, la Probabilità e le loro Applicazioni' (GNAMPA) of the 'Istituto Nazionale di Alta Matematica' (INdAM). The third author was partially supported by the Fondazione Ing. Aldo Gini during the preparation of this paper.

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Received: 30.03.2022