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The L.N. Gumilyov Eurasian National University
Building no. 3
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The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
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METHODS OF TRIGONOMETRIC APPROXIMATION AND GENERALIZED SMOOTHNESS. II.

S. Artamonov, K. Runovski, H.-J. Schmeisser

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Key words: trigonometric approximation, summability, K-functionals, moduli of smoothness, periodic Besov spaces.

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Abstract. The paper deals with the equivalence of approximation errors in L_p-spaces $(0 < p < \infty)$ with respect to approximation processes, generalized K-functionals and appropriate moduli of smoothness. The results are used to derive various characterizations of periodic Besov spaces by means of constructive approximation and moduli of smoothness. The main focus lies on spaces $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$, where $0 < p < 1,\ 0 < q \le \infty$ and $s > 0$.

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1 Introduction

This survey is a continuation of [32] where a general and unified approach to trigonometric approximation in L_p-spaces with $0 < p \le \infty$ has been presented. Based on the earlier papers [19, 28, 29, 30, 31, 33] we established a general convergence theorem and determined the ranges of convergence for approximation (summation) methods generated by classical kernels. The aim of this second part is twofold. On the one-hand we shall deal with the equivalence of approximation errors for families of linear polynomial operators and Fourier means in terms of generalized K-functionals and adapted moduli of smoothness. Here we follow our earlier papers [2, 3, 20, 23, 24, 34, 35, 36]. On the other-hand we shall use these results in order to give characterizations of periodic Besov spaces $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ in terms of approximation processes, generalized K-functionals and θ-moduli of smoothness, where the focus lies on the case $0 < p < 1$. Some of these results have been already announced in [4].

The paper is organized as follows. In Section 2 we introduce families of linear polynomial operators, Fourier means and interpolation (sampling) means and recall the general convergence theorem. Section 3 is concerned with K-functionals associated with general differential operators generated by homogeneous functions. We establish a Bernstein-type inequality (Theorem 3.2) and deal with the equivalence of approximation errors and appropriate K-functionals in L_p-spaces, $0 < p \le \infty$ (Theorems 3.3, 3.4). Periodic Besov (Nikol'skii, Hölder-Zygmund) spaces $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ are considered in Section 4 for the range of parameters $0 < p \le \infty,\ 0 < q \le \infty,\ 0 \le s < \infty$. We rely both on the classical definition by means of differences and related moduli of smoothness and the Fourier-analytic approach as decomposition spaces. We describe their interrelation (Proposition 4.1) and give characterizations as approximation spaces as well as by means of generalized K-functionals (Propositions 4.2, 4.3). Section 5 is devoted to the characterization of Besov spaces via constructive approximation processes. We present general results based on Sections 3 and 4 (Theorems 5.1, 5.2, 5.3). Applications to summation methods associated with de la Vallée-Poussin and Riesz kernels are presented in Subsections 5.2 and 5.3. The results collected in Sections 4 and 5 are covered as special cases in the more general approach within the framework of spaces with generalized smoothness given in [5]. We shall sketch the proofs for better understanding and reading. The final Section 6 deals with various types of moduli of smoothness, their equivalence with generalized K-functionals and, as a consequence, with related characterizations of periodic Besov spaces by means of various θ-moduli of smoothness (Theorems 6.1, 6.2, Corollary 6.1 as well as Subsections 6.2, 6.3 and 6.4).

2 Preliminaries

2.1 Notations

Unimportant positive constants are denoted by c (sometimes with subscripts). The relation $A(f,\sigma) \lesssim B(f,\sigma)$ means that there exists a positive constant c that does not depend on f and σ such that $A(f,\sigma) \le c\,B(f,\sigma)$. We shall write $A(f,\sigma) \asymp B(f,\sigma)$ if $A(f,\sigma) \lesssim B(f,\sigma) \lesssim A(f,\sigma)$. Symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the sets of natural numbers, integers, real and complex numbers, respectively. Furthermore, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $a_+ = \max(0,a)$ for $a \in \mathbb{R}$. We denote by $L_p(\mathbb{T}^d)$, where $d \in \mathbb{N}$, $0 < p < \infty$, the space of Lebesgue-measurable functions on the d-dimensional torus $\mathbb{T}^d = [0,2\pi)^d$ equipped with the finite norm (quasi-norm if $0 < p < 1$)

$$\| f \|_p = \left(\int\limits_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p} < \infty . \tag{2.1}$$

In the case $p = \infty$ we always consider the space $C(\mathbb{T}^d)$ of continuous functions equipped with the norm

$$\| f \|_\infty = \sup_{x \in \mathbb{T}^d} |f(x)| < \infty . \tag{2.2}$$

As usual, $L_p(\mathbb{R}^d)$ stands for the Lebesgue space on the euclidean d-space $\mathbb{R}^d$. Henceforth we put $\widetilde{p} = \min(1,p)$ and the triangle inequality can be written for all $0 < p \le \infty$ in the form

$$\|f + g\|_p^{\tilde{p}} \leqslant \|f\|_p^{\tilde{p}} + \|g\|_p^{\tilde{p}} .$$

By $\mathcal{S}(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ we denote the Schwartz space of infinitely differentiable rapidly decreasing functions and its dual space of tempered distributions, respectively. Further, $C_0^\infty(\mathbb{R}^d)$ stands for the set of all infinitely differentiable functions with compact support. The Fourier transform F and its inverse F^{-1} are continuous bijective linear mappings of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ onto itself. If $f \in L_1(\mathbb{R}^d)$ then they are given by

$$Ff(\xi) = \int\limits_{\mathbb{R}^d} f(x) e^{-ix\xi} dx\,, \ \xi \in \mathbb{R}^d, \ x\xi = \sum_{j=1}^d x_j \xi_j,$$

and

$$F^{-1} f(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} f(\xi) e^{ix\xi} d\xi = (2\pi)^{-d} F f(-x)\,, \ x \in \mathbb{R}^d,$$

respectively.

By $D'(\mathbb{T}^d)$ we denote the space of periodic distributions and by $f^\wedge(\nu)$, $\nu \in \mathbb{Z}^d$, the Fourier coefficients of $f \in D'(\mathbb{T}^d)$. If $f \in L_1(\mathbb{T}^d)$ then

$$f^\wedge(\nu) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(x)\, e^{-i\nu x}\, dx, \quad \nu \in \mathbb{Z}^d.$$

Let $\sigma \in \mathbb{R},\ \sigma > 0$. We denote by

$$\mathcal{T}_\sigma := \{g:\ g(x) = \sum_{|\nu|<\sigma} c_\nu\, e^{i\nu x},\ c_\nu \in \mathbb{C},\ c_{-\nu} = \overline{c_\nu}\}$$

($|\nu|^2 = \sum_{j=1}^{d} \nu_j^2$) the space of all trigonometric polynomials of (spherical) degree less than σ. Let us put $\mathcal{T} = \bigcup_{\sigma>0} \mathcal{T}_\sigma$ for the space of all trigonometric polynomials. If $f \in L_p(\mathbb{T}^d),\ 0 < p \le \infty$, and $\sigma > 0$ then

$$E_\sigma(f)_p := \inf\{\|f-g\|_p:\ g \in \mathcal{T}_\sigma\} \tag{2.3}$$

stands for its best approximation in $L_p(\mathbb{T}^d)$ by trigonometric polynomials belonging to $\mathcal{T}_\sigma$.

2.2 Approximation methods

We say that a complex-valued continuous function defined on $\mathbb{R}^d$ belongs to the class $\mathcal{K}$ if:

(i) it has compact support, i. e.

$$r(\varphi) := \sup\{|\xi|\ :\ \varphi(\xi) \ne 0\} < \infty\ ,$$

(ii) $\varphi(-\xi) = \overline{\varphi(\xi)},\ \xi \in \mathbb{R}^d$,

(iii) its Fourier transform belongs to $L_1(\mathbb{R}^d)$ and $\varphi(0) = 1$.

The kernels associated with $\varphi \in \mathcal{K}$ are defined as

$$W_n^\varphi(y) = \sum_{\nu\in\mathbb{Z}^d} \varphi\left(\frac{\nu}{n}\right) e^{i\nu y},\quad n \in \mathbb{N},\ y \in \mathbb{T}^d\ . \tag{2.4}$$

The Fourier means (or convolution means) generated by $\varphi \in \mathcal{K}$ are defined as

$$\mathcal{M}_n^\varphi f(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(y)\, W_n^\varphi(x-y)\, dy \tag{2.5}$$

if $f \in L_p(\mathbb{T}^d),\ 1 \le p \le \infty$, and

$$\mathcal{M}_n^\varphi f(x) = \langle f,\, W_n^\varphi(x-\cdot)\rangle \tag{2.6}$$

if $f \in D'(\mathbb{T}^d)$. Note that (2.5) and (2.6) can be reformulated as

$$\mathcal{M}_n^\varphi f(x) = \sum_{\nu\in\mathbb{Z}^d} \varphi\left(\frac{\nu}{n}\right) f^\wedge(\nu)\, e^{i\nu x},\quad n \in \mathbb{N},\ x \in \mathbb{T}^d\ . \tag{2.7}$$

This shows that $\mathcal{M}_n^\varphi f$ is a trigonometric polynomial of order less than $n\, r(\varphi)$. Clearly, the sequence of operators $(\mathcal{M}_n^\varphi)_{n\in\mathbb{N}}$ forms an approximate identity.

A function $\varphi \in \mathcal{K}$ also generates the family of linear polynomial operators $\{\mathcal{L}_n^\varphi\}_{n\in\mathbb{N}}$ given by

$$\mathcal{L}_n^\varphi f(x,\, \lambda) = (2N+1)^{-d} \sum_{k=0}^{2N} f(t_N^k + \lambda)\, W_n^\varphi(x - t_N^k - \lambda)\ . \tag{2.8}$$

Here $(x,\,\lambda)\in\mathbb{T}^d\times\mathbb{T}^d,\ N=[r\,n]\ (r\geq r(\varphi))$, and

$$t_N^k \;=\; \frac{2\pi k}{2N+1}\ (k\in\mathbb{Z}^d), \qquad \sum_{k=0}^{2N} \;=\; \sum_{k_1=0}^{2N}\cdots\sum_{k_d=0}^{2N}\ .$$

These families have been systematically studied in [19, 20, 21, 22, 23, 24, 34] and further papers of the authors. The approximation error of $f-\mathcal{L}_n^\varphi f$ is measured in the space $L_p(\mathbb{T}^d\times\mathbb{T}^d)$, equipped with the quasi-norm

$$\|g\|_{\overline{p}} \;=\; (2\pi)^{-d/p}\left(\int_{\mathbb{T}^d}\|g(\cdot,\,\lambda)\|_p^p\,d\lambda\right)^{1/p}$$

(modification by $\sup\limits_{\lambda\in\mathbb{T}^d}\ldots$ if $p=\infty$).

Moreover, we consider the sampling operators $\mathcal{S}_n^\varphi$ defined on $C(\mathbb{T}^d)$ as

$$\mathcal{S}_n^\varphi f(x) \;=\; (2N+1)^{-d}\sum_{k=0}^{N} f(t_N^k)\,W_n^\varphi(x-t_N^k),\ \ n\in\mathbb{N}. \tag{2.9}$$

Let us recall that

$$\|f-\mathcal{L}_n^\varphi f\|_{\overline{p}}\to 0 \quad\text{if and only if}\quad F\varphi\in L_{\widetilde{p}}(\mathbb{R}^d)\ \ (\widetilde{p}=\min(p,\,1))$$

(see [19, Theorem 4.1]). Under these assumptions (2.8) makes sense for almost all $(x,\,\lambda)$ and $\mathcal{L}_n^\varphi f(\cdot,\,\lambda)$ is a trigonometric polynomial of degree less than $r(\varphi)\,n$ for almost all $\lambda\in\mathbb{T}^d$. Furthermore, if $\varphi\in\mathcal{K}$ then we have the equivalences

$$\|f-\mathcal{M}_n^\varphi f\|_p \;\asymp\; \|f-\mathcal{L}_n^\varphi f\|_{\overline{p}},\ \ n\in\mathbb{N}, \tag{2.10}$$

for all $f\in L_p(\mathbb{T}^d),\ 1\leq p\leq\infty$, as well as

$$\|f-\mathcal{S}_n^\varphi f\|_\infty \;\asymp\; \|f-\mathcal{M}_n^\varphi f\|_\infty,\ \ n\in\mathbb{N}, \tag{2.11}$$

for all $f\in C(\mathbb{T}^d)$. These are proved in [19, Theorem 4.1] and [28, Theorem 6].

3 Generalized K-functionals

Let $\alpha>0$ be a real number. We denote by $\mathcal{H}_\alpha$ the class of all complex-valued continuous functions ψ defined on $\mathbb{R}^d$ satisfying the conditions:

(i) ψ is infinitely differentiable on $\mathbb{R}^d\setminus\{0\}$ and $\psi(\xi)\neq 0$ on $\mathbb{R}^d\setminus\{0\}$,

(ii) $\overline{\psi(\xi)}=\psi(-\xi)$ for $\xi\in\mathbb{R}^d\setminus\{0\}$,

(iii) ψ is homogeneous of degree α, i. e. $\psi(t\xi)\;=\;t^\alpha\,\psi(\xi)$ for all $t>0$ and $\xi\in\mathbb{R}^d\setminus\{0\}$.

If $\psi \in \mathcal{H}_\alpha$ then we define an associated (differential) operator $\psi(\mathcal{D})$ on the space of all trigonometric polynomials $\mathcal{T}$ as

$$(\psi(\mathcal{D})\, g)(x) \;=\; \sum_{\nu\in\mathbb{Z}^d} \psi(\nu) g^\wedge(\nu)\, e^{i\nu x}\;, \quad x \in \mathbb{T}^d. \tag{3.1}$$

The generalized K-functional related to $\psi(\mathcal{D}),\ \psi \in \mathcal{H}_\alpha$, is defined on $L_p(\mathbb{T}^d),\ 0 < p \le \infty$, as

$$K_\psi(f,\, t)_p \;=\; \inf_{g\in\mathcal{T}_{1/t}} (\|f-g\|_p + t^\alpha\, \|\psi(\mathcal{D})g\|_p), \quad t > 0. \tag{3.2}$$

Let us also introduce the functional

$$\boldsymbol{K}_\psi(f,\, t)_p \;=\; \inf\{\, \|f-g\|_p + t^\alpha\, \|\psi(D)g\| :\; g \in \mathcal{T}\,\}\;, \quad 0 < t \le 1\;, \tag{3.3}$$

for $f \in L_p(\mathbb{T}^d)$ if $1 \le p \le \infty$. The following equivalence is proved in [23, Theorem 4.21].

Theorem 3.1. *Let $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$ and let $1 \le p \le \infty$. Then*

$$K_\psi(f,\, t)_p \;\asymp\; \boldsymbol{K}_\psi(f,\, t)_p\;,\; 0 < t \le 1,\; f \in L_p(\mathbb{T}^d). \tag{3.4}$$

Remark 1. The definition (3.2) goes back to [11], where the equivalence of functionals (3.2) related to $\psi(\mathcal{D})g = g^{(k)}$ (k-th derivative) and the moduli of smoothness of order k has been proved on $L_p(\mathbb{T}),\ 0 < p \le \infty$. Note that in contrast to the functional $\boldsymbol{K}_\psi$ defined in (3.3) it makes sense on $L_p(\mathbb{T}^d),\ 0 < p \le \infty$. The functional K_ψ is also called polynomial K_ψ-functional or realization of the functional $\boldsymbol{K}_\psi$ because of the above theorem.

Obviously, $\big(K_\psi(f,\, n^{-1})_p\big)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence and

$$E_n(f)_p \;\le K_\psi(f,\, n^{-1})_p, \quad f \in L_p(\mathbb{T}^d),\; 0 < p \le \infty,\; n \in \mathbb{N}. \tag{3.5}$$

Moreover the following properties are of peculiar interest.

Theorem 3.2. *Let $0 < p \le \infty$ and let $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$.*

(1) *The following inequality*

$$\mathcal{K}_\psi(f,\, \delta t)_p \lesssim \delta^{\alpha+d(\frac{1}{p}-1)_+}\, K_\psi(f,\, t)_p \tag{3.6}$$

holds for all $0 < t, \delta \le 1$ and $f \in L_p(\mathbb{T}^d)$.

(2) *Let $0 < r \le \min(1,\, p)$. Then*

$$K_\psi(f,\, t)_p \;\lesssim\; t^\alpha \left(\sum_{1\le n\le 1/t} n^{\alpha r-1} E_n(f)_p^r \right)^{1/r} \tag{3.7}$$

for all $0 < t \le 1$ and $f \in L_p(\mathbb{T}^d)$.

For a proof of (3.6) see [23, Theorem 4.22]. The Bernstein-type inequality (3.7) is proved in [23, Theorem 4.26].

To formulate the main results we need some further notation. Let v and w be continuous functions on $\mathbb{R}^d$. Let $0 < q \le \infty$ and let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. In the sequel we shall write $v(\cdot) \overset{(q,\,\eta)}{\prec} w(\cdot)$, if $F[\,(\eta v)/w\,]$ belongs to $L_q(\mathbb{R}^d)$. The notation $v(\cdot) \overset{(q,\,\eta)}{\asymp} w(\cdot)$ indicates equivalence which means that $v(\cdot) \overset{(q,\,\eta)}{\prec} w(\cdot)$ and $w(\cdot) \overset{(q,\,\eta)}{\prec} v(\cdot)$ simultaneously.

Theorem 3.3. ([20, Theorem 4.1]) Let $0 < p \le \infty,\ \varphi \in \mathcal{K},\ F\varphi \in L_{\widetilde{p}}(\mathbb{R}^d)$ and let $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$. If $1 - \varphi(\cdot) \overset{(\widetilde{p},\eta)}{\prec} \psi(\cdot)$ for some $\eta \in C_0^\infty(\mathbb{R}^d)$ satisfying $\eta(\xi) = 1$ in a neighborhood of zero then

$$\|f - \mathcal{L}_n^\varphi f\|_{\overline{p}} \ \lesssim\ K_\psi(f,\, n^{-1})_p \tag{3.8}$$

for all $n \in \mathbb{N}$ and $f \in L_p(\mathbb{T}^d)$.

Theorem 3.4. ([20, Theorem 5.1]) Let $0 < p \le \infty,\ \varphi \in \mathcal{K}$ such that $r(\varphi) \le 1$ and $F\varphi \in L_{\widetilde{p}}(\mathbb{R}^d)$. Let $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ and $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\eta(\xi) = 1$ for $|\xi| \le \varrho < 1/2$, $\chi(\xi) = 1$ for $2\varrho \le |\xi| \le 1$, and $\eta(\xi) + \chi(\xi) = 1$ for $|\xi| \le 1$. If $\psi(\cdot) \overset{(\widetilde{p},\eta)}{\prec} 1 - \varphi(\cdot)$ and if there exists $m \in \mathbb{N}$ such that $(\varphi(\cdot))^m \overset{(\widetilde{p},\chi)}{\prec} 1 - \varphi(\cdot)$ then

$$K_\psi(f,\, n^{-1})_p \ \lesssim\ \|f - \mathcal{L}_n^\varphi f\|_{\overline{p}} \tag{3.9}$$

for all $n \in \mathbb{N}$ and $f \in L_p(\mathbb{T}^d)$.

Remark 2. In view of equivalences (2.10) and (2.11) under the assumptions of the above theorems (with $\widetilde{p} = 1$) we can replace $\|f - \mathcal{L}_n^\varphi f\|_{\overline{p}}$ in (3.8) and (3.9) by $\|f - \mathcal{M}_n^\varphi f\|_p$ if $1 \le \infty$ as well as $\|f - \mathcal{S}_n^\varphi f\|_\infty$ if $p = \infty$. Recall that $f \in C(\mathbb{T}^d)$ if $p = \infty$.

4 Besov spaces

The nowadays classical Besov spaces $\mathbb{B}^s_{p,q}$ of functions defined on the Euclidean d-dimensional space $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$ represent scales of spaces which are appropriate to measure the smoothness of functions. The spaces have been introduced and investigated by O.V. Besov in [6] for the range of parameters $s > 0,\ 1 \le p \le \infty$ and $1 \le q \le \infty$. In the particular case $q = \infty$ their definition is due to S.M. Nikol'skii [17]. The case $p = q = \infty$ corresponds to the scale of Hölder-Zygmund spaces. Let us also mention that the spaces $\mathbb{B}^s_{p,p}$ for $1 < p < \infty$ have been introduced in connection with boundary values of functions defined on domains by N. Aronszajn [1], L.N. Slobodeckij [46] and E. Gagliardo [13]. Besov spaces have found various applications, for example, in PDE's, in approximation theory, in computational mathematics as well as in stochastic processes. The aim of this section is to describe the definition via moduli of smoothness as well as the Fourier-analytic approach. For later use we give characterizatioins as approximaten spaces and by means of generalized K-functionals.

4.1 The approach by differences

Let $f \in L_p(\mathbb{T}^d),\ 0 < p \le \infty$ and let $k \in \mathbb{N}$. Then

$$\left(\Delta_h^k f\right)(x) \ =\ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh) \tag{4.1}$$

$(x \in \mathbb{T}^d,\ h \in \mathbb{T}^d)$. We have

$$\left(\Delta_h^1 f\right)(x) \ =\ f(x+h) - f(x) \text{ and } \left(\Delta_h^{k+1} f\right)(x) = \Delta_h^1 \left(\Delta_h^k f\right)(x)\,,\ k \in \mathbb{N}.$$

The modulus of continuity of order k of $f \in L_p(\mathbb{T}^d)$ is defined as

$$\omega_k(f,\, t)_p \ =\ \sup_{|h| \le t} \|\Delta_h^k f\|_p\,,\ \ t > 0\,. \tag{4.2}$$

Definition 1. Let $0 < p \le \infty,\ 0 < q \le \infty,\ s \ge 0$ and let $k \in \mathbb{N}$ such that $k > s$. The space $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ consists of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\|f|\mathbb{B}^s_{p,q}\|_k \;=\; \|f\|_p + \left(\int\limits_0^1 t^{-sq}\omega_k(f,\,t)_p^q\,\frac{dt}{t}\right)^{1/q} \tag{4.3}$$

is finite if $q < \infty$ and

$$\|f|\mathbb{B}^s_{p,\,\infty}\|_k \;=\; \|f\|_p + \sup_{0<t<1}\; t^{-s}\omega_k(f,\,t)_p \tag{4.4}$$

is finite if $q = \infty$

Remark 3. The spaces are independent of the number k, meaning that different values of $k > s$ in (4.3) and (4.4) result in equivalent quasi-norms. Henceforth we shall write $\|f|\mathbb{B}^s_{p,q}\|$ in place of $\|f|\mathbb{B}^s_{p,q}\|_k$. The spaces $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ are quasi-Banach spaces (Banach spaces if $\min(p,\,q) \ge 1$). The integral in (4.3) can be replaced by $\int\limits_0^\delta \ldots \frac{dt}{t}$, the supremum in (4.4) by $\sup\limits_{0<t<\delta} \ldots$, where $0 < \delta \le \infty$ (equivalent definitions and quasi-norms). The definition goes back to S.M. Nikol'skii [17] if $q = \infty,\ 1 \le p \le \infty$ and to O.V. Besov [6] if $1 \le q < \infty,\ 1 \le p \le \infty$. As for the extension to $0 < p < 1$ and $0 < q \le \infty$ and the corresponding proofs we refer to [8, Chapter 2, Section 10, Theorem 10.1]. Note that (4.3) is finite for each $f \in L_p(\mathbb{T}^d)$ if s is negative. Thus the restriction to $s > 0$ is natural (apart from the limiting case $s = 0$ in (4.3)). Moreover, we have $\mathbb{B}^0_{p,\,\infty}(\mathbb{T}^d) = L_p(\mathbb{T}^d)$ ($= C(\mathbb{T}^d)$ if $p = \infty$). Hence the limiting case $s = 0$ is of interest if $q < \infty$, only. The spaces $\mathbb{B}^0_{p,\,q}(\mathbb{T}^d)$ have been considered in [7].

4.2 The Fourier-analytical approach

We follow [41, Chapter 3]. Let $\chi_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp}\chi_0 \;\subset\; \{\xi \in \mathbb{R}^d \,:\, |\xi| \le 1\} \text{ and } \chi_0(\xi) = 1 \text{ if } |\xi| \le \frac{1}{2}\,.$$

For each $j \in \mathbb{N}$ we put

$$\chi_j(\xi) \;=\; \chi_0(2^{-j}\xi) \;-\; \chi_0(2^{-j+1}\xi)\,.$$

Then

$$\sum_{j=0}^{\infty}\chi_j(\xi) \;=\; 1, \quad \xi \in \mathbb{R}^d$$

and $\{\chi_j\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic decomposition of unity.

Given $f \in D'(\mathbb{T}^d)$ we define

$$(\chi_j(\mathcal{D})\,f)(x) \;=\; \sum_{\nu\in\mathbb{Z}^d}\chi_j(\nu)\,f^\wedge(\nu)\,e^{i\nu x}\;, \quad j \in \mathbb{N}_0,\ x \in \mathbb{T}^d\,.$$

Definition 2. Let $s \in \mathbb{R},\ 0 < p \le \infty,\ 0 < q \le \infty$, and let $\{\chi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic decomposition of unity. The space $B^s_{p,q}(\mathbb{T}^d)$ is the collection of all distributions $f \in D'(\mathbb{T}^d)$ such that

$$\|f|B^s_{p,q}\|_\chi \;=\; \left(\sum_{j=0}^{\infty} 2^{sjq}\,\|\chi_j(\mathcal{D})f\|_p^q\right)^{1/q} \;<\; \infty \tag{4.5}$$

if $q < \infty$ and

$$\|f|B^s_{p,\infty}\|_\chi \;=\; \sup_{j\in\mathbb{N}_0} 2^{sj}\,\|\chi_j(\mathcal{D})f\|_p \;<\; \infty \tag{4.6}$$

if $q = \infty$.

Remark 4. The spaces $B^s_{p,q}(\mathbb{T}^d)$ are quasi-Banach spaces (Banach spaces if $\min(p,\,q) \geq 1$). The definition is independent of the choice of χ_0 in the sense of equivalent quasi-norms. Henceforth we shall write $\|f|B^s_{p,q}\|$ in place of $\|f|B^s_{p,q}\|_\chi$.

Proposition 4.1. *Let* $0 < p \leq \infty,\ 0 < q \leq \infty,$ *and* $s \in \mathbb{R},\ s > d(\frac{1}{p} - 1)_+$. *Then*

$$B^s_{p,q}(\mathbb{T}^d) \;=\; \mathbb{B}^s_{p,q}(\mathbb{T}^d)\ . \tag{4.7}$$

A proof can be found in [52, Theorem 2.5.12] in the non-periodic case (spaces on $\mathbb{R}^d$). As far as the periodic case is concerned we refer to [41, Remark 3.5.4/4]. See also [41, Corollary 3.7.1] and Proposition 4.2 below.

Remark 5. If $0 < p < 1$ and $s > d(\frac{1}{p} - 1)$ then

$$B^s_{p,q}(\mathbb{T}^d) \subset L_1(\mathbb{T}^d) \subset L_p(\mathbb{T}^d)$$

for all $q,\ 0 < q \leq \infty$. If $0 < p < 1$ and $s < d(\frac{1}{p} - 1)$ then the periodic Delta-distribution belongs to $B^s_{p,q}(\mathbb{T}^d)$ (cf. [41, Remark 3.5.1/3] and [43, Remark 9]). Hence

$$B^s_{p,q}(\mathbb{T}^d) \;\neq\; \mathbb{B}^s_{p,q}(\mathbb{T}^d) \text{ if } s < d(\frac{1}{p} - 1)$$

In the limiting case $s = d(\frac{1}{p} - 1)_+$ we have that

$$B^s_{p,q}(\mathbb{T}^d) \;\subset\; L_1(\mathbb{T}^d)$$

if and only if $0 < p \leq 1$ and $0 < q < 1$ or $1 < p \leq \infty$ and $0 < q \leq \min(p,\,2)$ (see [45, Theorem 3.3.2] for a proof in the non-periodic case).

The question about the diversity of spaces $\mathbb{B}^s_{p,q}$ and $B^s_{p,q}$ attracted some attention. The necessity of the assumption $s > d(\frac{1}{p} - 1)$ in the case $0 < p < 1$ has been proved in [42, Corollary 3.10] in the non-periodic case. We refer also to the investigations in [15] within a more general framework.

4.3 Besov spaces as approximation spaces

Definition 3. Let $0 < p \leq \infty,\ 0 < q \leq \infty,$ and let $s \geq 0$. The approximation space $\mathbb{A}^s_{p,q}(\mathbb{T}^d)$ consists of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\|f|\mathbb{A}^s_{p,q}\| \;=\; \|f\|_p + \left(\sum_{n=1}^{\infty} n^{sq-1}\,E_n(f)_p^q\right)^{1/q} \;<\; \infty \tag{4.8}$$

if $0 < q < \infty$ and

$$\|f|A^s_{p,\infty}\| \;=\; \|f\|_p + \sup_{n\in\mathbb{N}} n^s\,E_n(f)_p \;<\; \infty \tag{4.9}$$

if $q = \infty$.

Remark 6. By standard arguments using the monotonicty of best approximation we obtain the equivalences

$$\sum_{n=1}^{\infty} n^{sq-1}\, E_n(f)_p^q \;\asymp\; \sum_{j=0}^{\infty} 2^{sjq}\, E_{2^j}(f)_p^q\,. \tag{4.10}$$

and

$$\sup_{n\in\mathbb{N}} n^s\, E_n(f)_p \;\asymp\; \sup_{j\in\mathbb{N}_0} 2^{sj}\, E_{2^j}(f)_p \tag{4.11}$$

for all admitted parameters $0 < p \le \infty,\ 0 < q < \infty$, and $s \ge 0$.

Lemma 4.1. *Let* $0 < p < 1,\ 0 < q \le \infty,\ s > d(\frac{1}{p} - 1)$. *If* $f \in \mathbb{A}^s_{p,q}(\mathbb{T}^d)$ *then* $f \in L_1(\mathbb{T}^d)$.

Proof. Because of

$$\mathbb{A}^{s+\varepsilon}_{p,\infty}(\mathbb{T}^d) \hookrightarrow \mathbb{A}^{s}_{p,q}(\mathbb{T}^d) \hookrightarrow \mathbb{A}^{s}_{p,\infty}(\mathbb{T}^d)$$

we may assume $q < \infty$ without loss of generality. We may choose trigonometric polynomials $g_j \in \mathcal{T}_{2^j}$ such that $\|f - g_j\|_p \le 2\, E_{2^j}(f)_p$. Then $\lim_{j\to\infty} g_j = f$ in $L_p(\mathbb{T}^d)$. By Nikol'skij's inequality for trigonometric polynomials (see, for example, [41, Theorem 3.3.2 and Remark 3.3.2/2]) we get

$$\|\, g_{j+1} - g_j \,\|_1 \;\lesssim\; 2^{jd(\frac{1}{p}-1)}\, \|\, g_{j+1} - g_j \,\|_p\,.$$

Therefore, in view of $s > d(\frac{1}{p} - 1)$ we obtain

$$\sum_{j=0}^{\infty} \|\, g_{j+1} - g_j \,\|_1 \;\lesssim\; \sum_{j=0}^{\infty} 2^{-(s-d(\frac{1}{p}-1))j}\, 2^{sj} \|\, g_{j+1} - g_j \,\|_p$$
$$\lesssim\; \sup_{j\in\mathbb{N}_0} 2^{sj} \|\, g_{j+1} - g_j \,\|_p \;\lesssim\; \Big(\sum_{j=0}^{\infty} 2^{sjq}\, E_{2^j}(f)_p^q\Big)^{1/q} \;<\infty\,.$$

Hence, $(g_j)_j$ is a Cauchy sequence in $L_1(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d)$ and, consequently, $f \in L_1(\mathbb{T}^d)$. □

Proposition 4.2. *Let* $0 < p \le \infty,\ 0 < q \le \infty$, *and let* $s \ge 0$. *Then*

$$\mathbb{A}^s_{p,q}(\mathbb{T}^d) \;=\; \mathbb{B}^s_{p,q}(\mathbb{T}^d)\,. \tag{4.12}$$

A proof in the case $0 < p \le \infty,\ 0 < q \le \infty$ and $s > d(\frac{1}{p} - 1)_+$ based on Proposition 4.1 can be found in [41, Corollary 3.7.1]. The general case follows from the Jackson-type estimate

$$E_{2^j}(f)_p \;\lesssim\; \omega_k(f,\, 2^{-j})_p \tag{4.13}$$

(see [18, formula (5.2.1)] or [8, Chapter 7, Theorem 2.3] ($d = 1$) and [52, Proposition 2.5.12] if $1 \le p \le \infty$ as well as [16] ($d = k = 1$), [48] and [22, Theorem 3.3] if $0 < p < 1$.) as well as the Bernstein-type inequality

$$\omega_k(f,\, 2^{-j})_p \;\lesssim\; 2^{-kj} \left(\sum_{n=1}^{2^j} n^{kr-1} E_n(f)_p^r\right)^{1/r}, \tag{4.14}$$

$r \le \min(p, 1)$ ([16, Theorem 2] if $0 < p < 1$ and, for example, [8, Chapter 7, Theorem 3.1] if $p \ge 1$) by standard arguments.

Remark 7. Proposition 4.2 provides an alternative proof of the independence of Definition 1 of the number $k,\ k > s$, as well as the equivalence of quasi-norms $\| \cdot |\mathbb{B}^s_{p,q}\|_k$ for different values of $k > s$.

We can give a further equivalent characterization of $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ using directional moduli of smoothness. Let $e_j,\ j=1,\ldots,d$, be the unit vector in the direction of the j-th coordinate. We introduce the moduli of smoothness

$$\omega_k^{(j)}(f,\,t)_p \;=\; \sup_{|h|\le t} \|\Delta^k_{h,\,j} f\|_p\,, \tag{4.15}$$

where

$$\left(\Delta^k_{h,\,j} f\right)(x) \;=\; \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} f(x+he_j)$$

$(x\in\mathbb{T}^d,\ h\in\mathbb{R},\ j\in\{1,\ldots,d\})$.

Corollary 4.1. *Let $0<p\le\infty,\ 0<q\le\infty,\ s\ge 0$ and let $k\in\mathbb{N},\ k>s$. Then $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ consists of all functions $f\in L_p(\mathbb{T}^d)$ such that*

$$\|f|\mathbb{B}^s_{p,q}\|^*_k \;=\; \|f\|_p + \sum_{j=1}^{d}\left(\int_0^1 t^{-sq}\,\omega_k^{(j)}(f,\,t)_p^q\,\frac{dt}{t}\right)^{1/q} \;<\; \infty \tag{4.16}$$

(standard modification if $q=\infty$). Moreover, $\|f|\mathbb{B}^s_{p,q}\|^*_k$ is an equivalent quasi-norm in $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ for all $k>s$.

Proof. Clearly, $\|f|\mathbb{B}^s_{p,q}\|^*_k \lesssim \|f|\mathbb{B}^s_{p,q}\|_k$. The converse estimate follows from the Jackson-type estimate

$$E_n(f)_p \;\lesssim\; \sum_{j=1}^{d} \omega_k^{(j)}(f,\,n^{-1})_p \quad (n\in\mathbb{N}) \tag{4.17}$$

proved in [12, formula 2.14]. □

Let us mention that Proposition 4.2 and Corollary 4.1 are can be found in [5] as special cases.

4.4 Characterization by generalized K-functionals

Definition 4. Let $0<p\le\infty,\ 0<q\le\infty,\ s\ge 0$, and let $\psi\in\mathcal{H}_\alpha$ $(\alpha>0)$ such that $\alpha>s$. The space $\mathbb{K}^s_{p,q}(\mathbb{T}^d)$ consists of all functions $f\in L_p(\mathbb{T}^d)$ such that

$$\|f|\mathbb{K}^s_{p,q}\|^\psi \;=\; \|f\|_p + \left(\sum_{n=1}^{\infty} n^{sq-1}\,K_\psi(f,\,n^{-1})_p^q\right)^{1/q} \;<\; \infty \tag{4.18}$$

if $0<q<\infty$ and

$$\|f|\mathbb{K}^s_{p,\infty}\|^\psi \;=\; \|f\|_p + \sup_{n\in\mathbb{N}} n^s\,K_\psi(f,\,n^{-1})_p \;<\; \infty \tag{4.19}$$

if $q=\infty$.

Remark 8. By the properties of $\mathcal{K}_\psi$ we have the following analogue of (4.10) and (4.11). Let $\psi\in\mathcal{H}_\alpha$ $(\alpha>0),\ 0<p\le\infty,\ 0<q\le\infty$, and let $s\ge 0$. Then

$$\sum_{n=1}^{\infty} n^{sq-1}\,K_\psi(f,\,n^{-1})_p^q \;\asymp\; \sum_{j=0}^{\infty} 2^{sjq}\,K_\psi(f,\,2^{-j})_p^q \tag{4.20}$$

(modification if $q=\infty$).

Proposition 4.3. *Let $0 < p \le \infty,\ 0 < q \le \infty$, and let $s \ge 0$. Then*

$$\mathbb{K}^s_{p,q}(\mathbb{T}^d) \ = \ \mathbb{B}^s_{p,q}(\mathbb{T}^d)\,. \tag{4.21}$$

Moreover, $\| \cdot |\mathbb{K}^s_{p,q}\|^\psi$, where $\psi \in \mathcal{H}_\alpha$ for some $\alpha > s$ is an equivalent quasi-norm in $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ and $\mathbb{A}^s_{p,q}(\mathbb{T}^d)$.

Proof. Let $\psi \in \mathcal{H}_\alpha$ for some $\alpha > s$. In view of Proposition 4.2, Remark 8, and (3.5) it suffices to prove that

$$\sum_{j=0}^{\infty} 2^{sjq}\, K_\psi(f,\, 2^{-j})_p^q \ \lesssim \ \sum_{j=0}^{\infty} 2^{sjq}\, E_{2^j}(f)_p^q \tag{4.22}$$

for $f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d)$ (modification if $q = \infty$). In order to prove (4.22) we use the Bernstein-type inequality

$$K_\psi(f,\, 2^{-j})_p \ \lesssim \ 2^{-j\alpha} \left(\sum_{n=1}^{2^j} n^{\alpha r - 1}\, E_n(f)_p^r \right)^{1/r} \tag{4.23}$$

($r \le \min(p,\, 1)$) which follows from (3.7). This is the counterpart of (4.14). The right-hand side of (4.23) can be estimated by

$$\begin{aligned} & 2^{-\alpha j} \left(\sum_{\nu=0}^{j-1} \sum_{n=2^\nu}^{2^{\nu+1}} n^{\alpha r - 1}\, E_n(f)_p^r \right)^{1/r} \\ & \lesssim \ 2^{-\alpha j} \left(\sum_{\nu=0}^{j-1} 2^{\alpha(\nu+1) r}\, E_{2^\nu}(f)_p^r\, 2^{-\nu} \sum_{n=2^\nu}^{2^{\nu+1}} 1 \right)^{1/r} \\ & \lesssim \ 2^{-\alpha j} \left(\sum_{\nu=0}^{j-1} 2^{\alpha \nu r}\, E_{2^\nu}(f)_p^r \right)^{1/r} . \end{aligned} \tag{4.24}$$

Choosing r such that $u = \frac{q}{r} > 1$ and using Hölder's inequality with $\frac{1}{u} + \frac{1}{u'} = 1$ we obtain

$$\begin{aligned} \left(\sum_{\nu=0}^{j-1} 2^{\alpha\nu r}\, E_{2^\nu}(f)_p^r \right)^{q/r} \ & \le \left(\sum_{\nu=0}^{j-1} 2^{\varepsilon \nu r u'} \right)^{u/u'} \sum_{\nu=0}^{j-1} 2^{(\alpha-\varepsilon)\nu q}\, E_{2^\nu}(f)_p^q \\ & \lesssim 2^{\varepsilon j q} \sum_{\nu=0}^{j-1} 2^{(\alpha-\varepsilon)\nu q}\, E_{2^\nu}(f)_p^q \end{aligned} \tag{4.25}$$

for $\varepsilon > 0$. Combining (4.14), (4.24) and (4.25) we see that

$$\begin{aligned} & \sum_{j=0}^{\infty} 2^{sjq}\, K_\psi(f,\, 2^{-j})_p^q \ \lesssim \ \sum_{j=0}^{\infty} 2^{(s-\alpha+\varepsilon)jq} \sum_{\nu=0}^{j-1} 2^{(\alpha-\varepsilon)\nu q}\, E_{2^\nu}(f)_p^q \\ & = \sum_{\nu=0}^{\infty} 2^{(\alpha-\varepsilon)\nu q}\, E_{2^\nu}(f)_p^q \sum_{j=\nu}^{\infty} 2^{(s-\alpha+\varepsilon)jq} = \sum_{\nu=0}^{\infty} 2^{(k-\varepsilon)\nu q}\, E_{2^\nu}(f)_p^q \sum_{j=0}^{\infty} 2^{(s-\alpha+\varepsilon)(j+\nu)q} \\ & \lesssim \sum_{\nu=0}^{\infty} 2^{s\nu q}\, E_{2^\nu}(f)_p^q \end{aligned}$$

if $\alpha - s > \varepsilon > 0$. Together with Remark 6 and Remark 8 this proves the lower estimate

$$\|f|\mathbb{K}^s_{p,q}\|^\psi \ \lesssim \ \|f|\mathbb{A}^s_{p,q}\|\,.$$

□

Remark 9. Proposition 4.3 is contained in [5] as a special case. By our knowledge the statement is new, in particular in the case $0 < p < 1$, where classical $\mathcal{K}$-functionals do not make sense. For convenience of the reader we have given details of the proof modifying the arguments of the proof of Proposition 4.2.

Proposition 4.2 and Proposition 4.3 show that the spaces $\mathbb{K}^s_{p,q}(\mathbb{T}^d)$ are independent of the choice of $\psi \in \mathcal{H}_\alpha$, where $\alpha > s$ in the sense of equivalent quasi-norms. According to Proposition 4.1, Proposition 4.2, Proposition 4.3 and Remark 5 we have

$$B^s_{p,q}(\mathbb{T}^d) = \mathbb{B}^s_{p,q}(\mathbb{T}^d) = \mathbb{A}^s_{p,q}(\mathbb{T}^d) = \mathbb{K}^s_{p,q}(\mathbb{T}^d)$$

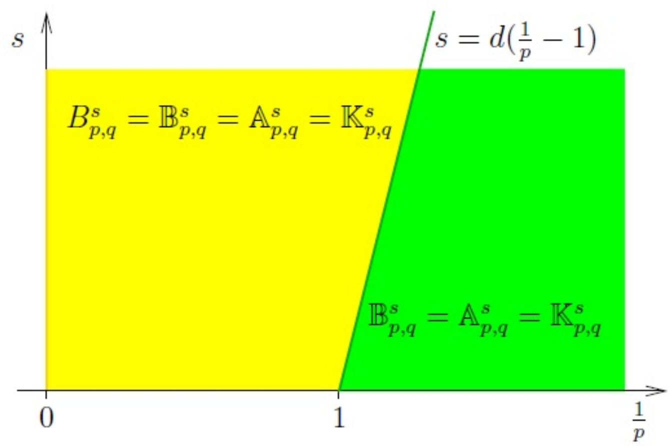
if $s > d(\frac{1}{p} - 1)_+$ and

$$\mathbb{B}^s_{p,q}(\mathbb{T}^d) = \mathbb{A}^s_{p,q}(\mathbb{T}^d) = \mathbb{K}^s_{p,q}(\mathbb{T}^d) \neq B^s_{p,q}(\mathbb{T}^d)$$

if $0 < p < 1,\ 0 < s < d(\frac{1}{p} - 1)$ (see the figure below). Note that

$$\mathbb{B}^0_{p,\,q}(\mathbb{T}^d) \ = \ \mathbb{A}^0_{p,\,q}(\mathbb{T}^d) \ = \ \mathbb{K}^0_{p,\,q}(\mathbb{T}^d)$$

if $0 < p \le \infty$ and $0 < q \le \infty$ (see also the comment in Remark 3).



Remark 10. By similar arguments as in the proof of (4.10) and (4.20), respectively, we can show that

$$\|f|\mathbb{K}^s_{p,q}\|^\psi \ \asymp \ \|f\|_p + \Big(\int_0^1 t^{-sq}\, K_\psi(f,\, t)^q_p\, \frac{dt}{t} \Big)^{1/q} \tag{4.26}$$

for $\psi \in \mathcal{H}_\alpha,\ 0 < p \le \infty,\ 0 < q \le \infty$, and $0 \le s < \alpha$ (modification if $q = \infty$).

Remark 11. It follows from (3.4) and (4.26) that

$$\|f|\mathbb{K}^s_{p,q}\|^\psi \ \asymp \ \|f\|_p + \Big(\int_0^1 t^{-sq}\, \boldsymbol{K}_\psi(f,\, t)^q_p\, \frac{dt}{t} \Big)^{1/q} \tag{4.27}$$

for $\psi \in \mathcal{H}_\alpha,\ 1 \le p \le \infty,\ 0 < q \le \infty$, and $0 \le s < \alpha$ (modification if $q = \infty$). In this case $\mathbb{K}^s_{p,q}(\mathbb{T}^d)$ turns out to be a real interpolation space. Namely, let us define the space $X^\psi_p(\mathbb{T}^d)$ as the collection of all functions $f \in L_p(\mathbb{T}^d)$ ($f \in C(\mathbb{T}^d)$ if $p = \infty$) such that

$$\psi(D)\, f(x) \ = \ \sum_{\nu \in \mathbb{Z}^d} \psi(\nu)\, f^\wedge(\nu)\, e^{i\nu x} \ \in L_p(\mathbb{T}^d)$$

(for details and interpretation see [33, Section 3]). The space is equipped with the norm

$$\|f\|_{X_p^\psi} \;=\; \|f\|_p + \|\psi(D)f\|_p \;, \quad 1 \le p \le \infty \;.$$

Note that according to [33, Theorems 4.1 and 4.2] we have

$$X_p^\psi(\mathbb{T}^d) \;=\; H_p^\alpha(\mathbb{T}^d) \quad \text{if} \quad 1 < p < \infty \;,$$

where $H_p^\alpha(\mathbb{T}^d)$ stands for the periodic (fractional) Sobolev space (cf. [41, Subsection 3.5.4]), and

$$B_{p,1}^\alpha(\mathbb{T}^d) \hookrightarrow X_p^\psi(\mathbb{T}^d) \hookrightarrow B_{p,\infty}^\alpha(\mathbb{T}^d)$$

if $1 \le p \le \infty$. We introduce the functional

$$K(t,\, f,\, L_p,\, X_p^\psi) \;=\; \inf\{\|f-g\|_p + t\,\|g\|_{X_p^\psi} \;:\; g \in X_p^\psi(\mathbb{T}^d)\} \;. \tag{4.28}$$

Then it follows from (4.27) that

$$\|f|\mathbb{K}_{p,q}^s\|^\psi \;\asymp\; \|f\|_p + \Big(\int_0^1 t^{-\frac{s}{\alpha}q}\, K(t,\, f,\, L_p,\, X_p^\psi)^q \,\frac{dt}{t}\Big)^{1/q} \tag{4.29}$$

using the density of $\mathcal{T}$ in $L_p(\mathbb{T}^d)$, $1 \le p < \infty$ (in $C(\mathbb{T}^d)$ if $p = \infty$). Equivalence (4.29) implies

$$\mathbb{K}_{p,q}^s(\mathbb{T}^d) \;=\; \big(L_p(\mathbb{T}^d),\, X_p^\psi(\mathbb{T}^d)\big)_{\frac{s}{\alpha},q} \tag{4.30}$$

($\psi \in \mathcal{H}_\alpha$, $0 < s < \alpha$, $1 \le p \le \infty$ and $0 < q \le \infty$), where the right-hand side stands for the real interpolation space in the sense of [52, Definition 2.4.1].

5 Constructive approximation

5.1 General results

Theorem 5.1. (Inverse results) *Let $0 < p,q \le \infty$, $s \ge 0$ and $\varphi \in \mathcal{K}$, where $F\varphi \in L_p(\mathbb{R}^d)$ if $0 < p < 1$. Then the following statements hold.*

(i) *If $f \in L_p(\mathbb{T}^d)$ and*

$$\sum_{n=1}^{\infty} n^{sq-1}\,\|f - \mathcal{L}_n^\varphi f\|_p^q \;<\; \infty \tag{5.1}$$

(standard modification if $q = \infty$), then $f \in \mathbb{B}_{p,q}^s(\mathbb{T}^d)$ and

$$\|f|\mathbb{B}_{p,q}^s\| \;\lesssim\; \|\,f\,\|_p \;+\; \Big(\sum_{n=1}^{\infty} n^{sq-1}\,\|f - \mathcal{L}_n^\varphi f\|_p^q\Big)^{1/q} \;. \tag{5.2}$$

(ii) *If $f \in L_p(\mathbb{T}^d)$ for $1 \leqslant p \leqslant \infty$ or $f \in L_p(\mathbb{T}^d) \cap D'(\mathbb{T}^d)$ for $0 < p < 1$ and*

$$\sum_{n=1}^{\infty} n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q \;<\; \infty \tag{5.3}$$

(standard modification for $q = \infty$), then $f \in \mathbb{B}_{p,q}^s(\mathbb{T}^d)$ and

$$\|f|\mathbb{B}_{p,q}^s\| \;\lesssim\; \|\,f\,\|_p \;+\; \Big(\sum_{n=1}^{\infty} n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q\Big)^{1/q} \;. \tag{5.4}$$

Proof. Taking into account that

$$E_{r(\varphi)n}(f)_p \le \|f - \mathcal{L}_n^\varphi f\|_{\overline{p}}\,,\ f \in L_p(\mathbb{T}^d),\ \ n \in \mathbb{N}\,,$$

we immediately deduce the conclusion of part (i) from Proposition 4.2. Combining Proposition 4.2 and the estimate

$$E_{r(\varphi)n}(f)_p \leqslant \|f - \mathcal{M}_n^\varphi f\|_p\,,\ n \in \mathbb{N}\,,$$

where $f \in L_p(\mathbb{T}^d)$, $1 \le p \le \infty$ or $f \in L_p(\mathbb{T}^d) \cap D'(\mathbb{T}^d)$ if $0 < p < 1$ we obtain the conclusion of part (ii). □

Theorem 5.2. (Direct results) *Let $0 < p, q \le \infty$, $s \ge 0$ and let $\varphi \in \mathcal{K}$ where $F\varphi \in L_p(\mathbb{R}^d)$ if $0 < p < 1$. Suppose there exists $\psi \in \mathcal{H}_\alpha$ $(\alpha > 0)$ such that $1 - \varphi(\cdot) \overset{(\widetilde{p},\eta)}{\prec} \psi(\cdot)$ for some function $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ satisfying $\eta(\xi) = 1$ in a neighborhood of 0. If $0 < s < \alpha$ and $f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d)$ then*

$$\|f\|_p + \left(\sum_{n=1}^\infty n^{sq-1}\,\|f - \mathcal{L}_n^\varphi f\|_{\overline{p}}^q\right)^{1/q} \lesssim \|f|\mathbb{B}^s_{p,q}\| \tag{5.5}$$

(standard modification if $q = \infty$).

Proof. Estimate (5.5) is a corollary of Proposition 4.3 and Theorem 3.3. □

Theorem 5.3. (Direct result) *Let $0 < p < 1$, $0 < q \le \infty$. Let $\varphi \in \mathcal{K}$ and $\psi \in \mathcal{H}_\alpha$ be as in Theorem 5.2. If $0 < s < \alpha$, then*

$$\|f\|_p + \left(\sum_{n=1}^\infty n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q\right)^{1/q} \lesssim \|f|B^s_{p,q}\| \tag{5.6}$$

for all $f \in B^s_{p,q}(\mathbb{T}^d) \cap L_p(\mathbb{T}^d)$ (standard modification if $q = \infty$).

Proof. By standard arguments and $s > 0$ we obtain the estimate

$$\sum_{n=1}^\infty n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q \lesssim \sum_{j=0}^\infty 2^{sjq} \sup_{n=2^j,\dots,2^{j+1}-1} \|\mathcal{M}_{2^j}^\varphi f - \mathcal{M}_n^\varphi f\|^q \tag{5.7}$$

(see also [38, formula (4.6)] for similar arguments). We put

$$\varrho_\tau(\xi) = \varphi(\xi) - \varphi(\tau\xi),\ \ \frac{1}{2} < \tau < 1,\ \xi \in \mathbb{R}^d\ .$$

Then (5.7) implies

$$\sum_{n=1}^\infty n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q \lesssim \sum_{j=0}^\infty 2^{sjq} \sup_{1/2\le\tau\le 1} \Big\| \sum_{\nu\in\mathbb{Z}^d} \varrho_\tau(2^{-j}\nu)\, f^\wedge(\nu)\, e^{i\nu\cdot} \Big\|_p^q\ . \tag{5.8}$$

Note that $\varrho_\tau(0) = 0$ and $\operatorname{supp}\varrho_\tau \subset \{\xi:\ |\xi| \le 2\,r(\varphi)\}$. Let $\{\chi_\ell\}_{\ell=0}^\infty$ be a smooth dyadic decomposition of unity as defined in Subsection 4.2. Recall that

$$\chi_\ell = \chi(2^{-\ell}\xi) \quad \text{with} \quad \chi(\xi) = \chi_0(\xi) - \chi_0(2\xi),\ \ell \in \mathbb{N}\ .$$

By definition

$$\operatorname{supp}\chi(2^{-\ell}(\xi)\,\cdot) \subset \{\xi:\ 2^{\ell-2} \le |\xi \le 2^\ell\}\,.$$

If $|\nu| \ge 1$ we have that $\sum\limits_{\ell=1}^{\infty} \chi(2^{-\ell}\xi) = 1$ and

$$\varrho_\tau(2^{-j}\nu) = \sum_{\ell=1}^{\infty} \varrho_\tau(2^{-j}\nu)\,\chi(2^{-\ell})\nu) = \sum_{\ell=1}^{j+L} \varrho_\tau(2^{-j}\nu)\,\chi(2^{-\ell}\nu) = \sum_{\ell=-j+1}^{L} \varrho_\tau(2^{-j}\nu)\,\chi(2^{-(\ell+j)}\nu)$$

for some $L \in \mathbb{N}$. We put

$$f_\ell(x) = \sum_{\nu\in\mathbb{Z}^d} \psi(2^{-\ell}\nu)\,\chi(2^{-\ell}\nu)\,f^\wedge(\nu)\,e^{i\nu x}\ , \quad \ell \in \mathbb{N}.$$

By the homogeneity of ψ we obtain

$$\begin{aligned}\left|\sum_{\nu\in\mathbb{Z}^d} \varrho_\tau(2^{-j}\nu)\,f^\wedge(\nu)\,e^{i\nu x}\right| &= \left|\sum_{\nu\in\mathbb{Z}^d}\sum_{\ell=-j+1}^{L} \varrho_\tau(2^{-j}\nu)\,\chi(2^{-(\ell+j)}\nu)\,f^\wedge(\nu)\,e^{i\nu x}\right| \\ &\le \sum_{\ell=-j+1}^{L}\left|\sum_{\nu\in\mathbb{Z}^d} \frac{\varrho_\tau(2^{-j}\nu)}{\psi(2^{-(\ell+j)}\nu)}\,\psi(2^{-(\ell+j)}\nu)\,\chi(2^{-(\ell+j)}\nu)\,f^\wedge(\nu)\,e^{i\nu x}\right| \\ &\le \left(\sum_{\ell=-j+1}^{L} 2^{\alpha\ell r}\left|\sum_{\nu\in\mathbb{Z}^d} \frac{\varrho_\tau(2^{-j}\nu)}{\psi(2^{-j}\nu)}\,f^\wedge_{\ell+j}(\nu)\,e^{i\nu x}\right|^r\right)^{1/r}\end{aligned}$$

for $r \le \min(1, p, q)$. This leads to the estimate

$$\Big\|\sum_{\nu\in\mathbb{Z}^d} \varrho_\tau(2^{-j}\nu) f^\wedge(\nu) e^{i\nu x}\Big\|_p^r \le \sum_{\ell=-j+1}^{L} 2^{\alpha\ell r}\Big\|\sum_{\nu\in\mathbb{Z}^d} \frac{\varrho_\tau(2^{-j}\nu)}{\psi(2^{-j}\nu)} f^\wedge_{\ell+j}(\nu) e^{i\nu x}\Big\|_p^r\ . \tag{5.9}$$

Note that $f_{\ell+j}$ is a trigonometric polynomial of order less than $2^{\ell+j} \le 2^{L+j}$ and that

$$\sum_{\nu\in\mathbb{Z}^d} \frac{\varrho_\tau(2^{-j}\nu)}{\psi(2^{-j}\nu)}\,e^{i\nu x}$$

is a trigonometric polynomial of order less than $2^{j+1}\,r(\varphi)$. According to the Fourier multiplier theorem [41, pp. 150/151] or [26, Theorem 3.2] we can estimate

$$\Big\|\sum_{\nu\in\mathbb{Z}^d} \frac{\varrho_\tau(2^{-j}\nu)}{\psi(2^{-j}\nu)}\,f^\wedge_{\ell+j}(\nu)\,e^{i\nu x}\Big\|_p \lesssim \|f_{\ell+j}\|_p\ .$$

Here we used that

$$\sup_{1/2\le\tau\le1} \Big\|F\Big[\frac{\varrho_\tau}{\psi}\Big]\Big\|_p = C < \infty \tag{5.10}$$

(see [5, formula (4.33)]). Together with (5.8) and (5.9) we arrive at

$$\sum_{n=1}^{\infty} n^{sq-1}\,\|f - \mathcal{M}_n^\varphi f\|_p^q \lesssim \sum_{j=0}^{\infty} 2^{sjq}\Big(\sum_{\ell=-j+1}^{L} 2^{\alpha\ell r}\,\|f_{\ell+j}\|_p^r\Big)^{q/r}\ . \tag{5.11}$$

Setting $f_\ell = 0$ if $\ell = 0, -1, -2, \ldots$ and using $s < \alpha$ estimate (5.11) implies

$$\sum_{n=1}^{\infty} n^{sq-1} \|f - \mathcal{M}_n^{\varphi} f\|_p^q \lesssim \Big(\sum_{\ell=-\infty}^{L} 2^{(\alpha-s)\ell r} \Big(\sum_{j=0}^{\infty} 2^{s(\ell+j)q} \|f_{\ell+j}\|_p^q \Big)^{r/q} \Big)^{q/r}$$

$$\lesssim \sum_{j=1}^{\infty} 2^{sjq} \| \sum_{\nu \in \mathbb{Z}^d} \psi(2^{-j}\nu)\, \chi(2^{-j}\nu)\, f^\wedge(\nu)\, e^{i\nu x} \|_p^q . \tag{5.12}$$

We put $\widetilde{\chi}_1 = \chi_0 + \chi(\frac{\cdot}{2})$ and

$$\widetilde{\chi}_j = \chi(2^{-j+1}\cdot) + \chi(2^{-j}\cdot) + \chi(2^{-j-1}\cdot)$$

if $j = 2, 3, \ldots$. Taking into account that $F[\psi\,\chi] \in L_p(\mathbb{R}^d)$ and applying again the multiplier theorem [41, pp. 150-151] or [26, Theorem 3.2] we see that

$$\Big\| \sum_{\nu \in \mathbb{Z}^d} \psi(2^{-j}\nu)\, \chi(2^{-j}\nu)\, f^\wedge(\nu)\, e^{i\nu x} \Big\|_p = \Big\| \sum_{\nu \in \mathbb{Z}^d} [\psi \cdot \chi](2^{-j}\nu)\, \widetilde{\chi}_j(\nu)\, f^\wedge(\nu)\, e^{i\nu x} \Big\|_p$$

$$\lesssim \Big\| \sum_{\nu \in \mathbb{Z}^d} \widetilde{\chi}_j(\nu)\, f^\wedge(\nu)\, e^{i\nu x} \Big\|_p .$$

Inserting this estimate into (5.12) yields (5.6) and proves the theorem. □

For variants of Theorem 5.3 we refer also to [38], [39] and to [40] (non-periodic one-dimensional case). The following corollary is a consequence of Theorems 5.1, 5.2, 5.3, Proposition 4.1 as well as the equivalences (2.10) and (2.11).

Corollary 5.1. *Let $0 < p \le \infty,\ 0 < q \le \infty$ and let $\varphi \in \mathcal{K}$ be as in Theorem 5.2. Then the following statements hold.*

(1) *If $0 \le s < \alpha$ then*

$$f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d) \iff f \in L_p(\mathbb{T}^d) \ \ \textit{and} \ \ \sum_{n=1}^{\infty} n^{sq-1} \|f - \mathcal{L}_n^{\varphi} f\|_p^q < \infty. \tag{5.13}$$

(2) *If $1 \le p \le \infty$ and $0 \le s < \alpha$ then*

$$f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d) \iff f \in L_p(\mathbb{T}^d) \ \ \textit{and} \ \ \sum_{n=1}^{\infty} n^{sq-1} \|f - \mathcal{M}_n^{\varphi} f\|_p^q < \infty. \tag{5.14}$$

(3) *If $0 \le s < \alpha$ then*

$$f \in \mathbb{B}^s_{\infty,q}(\mathbb{T}^d) \iff f \in C(\mathbb{T}^d) \ \ \textit{and} \ \ \sum_{n=1}^{\infty} n^{sq-1} \|f - \mathcal{S}_n^{\varphi} f\|_\infty^q < \infty. \tag{5.15}$$

(4) *If $0 < p < 1$ and $d(\frac{1}{p} - 1) < s < \alpha$ then*

$$f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d) \iff f \in L_p(\mathbb{T}^d) \cap D'(\mathbb{T}^d) \ \ \textit{and} \ \ \sum_{n=1}^{\infty} n^{sq-1} \|f - \mathcal{M}_n^{\varphi} f\|_p^q < \infty. \tag{5.16}$$

(Standard modification if $q = \infty$, $C(\mathbb{T}^d)$ in place of $L_p(\mathbb{T}^d)$ if $p = \infty$).

Note that generalizations of Theorems 5.1, 5.2, 5.3 and Corollary 5.1 to Besov spaces with generalized smoothness are given in [5].

5.2 Means and families of de la Vallée-Poussin type

Let $\varphi \in \mathcal{K}$ be a function for which $\varphi(\xi) = 1$ for ξ, satisfying the inequality $|\xi| \le \delta < r(\varphi)$ for some $\delta > 0$.

Theorem 5.4. *Let $0 < p \le \infty$, $0 < q \le \infty$ and let $\varphi \in \mathcal{K}$ with $F\varphi \in L_{\widetilde{p}}(\mathbb{R}^d)$. If φ has the above property then the statements of Corollary 5.1 hold true with $\alpha = \infty$.*

This follows from the fact that the condition $(1-\varphi) \overset{(\widetilde{p},\eta)}{\prec} \psi$ in Theorem 5.2 and Theorem 5.3 is satisfied for all $\psi \in \mathcal{H}_\alpha$, where α is an arbitrary positive number. Alternatively, (5.13), (5.14) and (5.15) follow from the equivalence

$$E_{n\,r(\varphi)}(f)_p \lesssim \|f - \mathcal{L}_n^\varphi f\|_{\overline{p}} \lesssim E_n(f)_p\,, \quad n \in \mathbb{N}$$

(see [25, Theorem 1] for the upper estimate), Proposition 4.2 as well as the equivalences (2.10) and (2.11). As far as (5.16) is concerned we refer also to [38, Theorem 7]. Let us mention that $F\varphi \in L_p(\mathbb{R}^d)$, $0 < p < 1$, if $\varphi \in B^{\varkappa}_{2,\infty}(\mathbb{R}^d)$ where $\varkappa > d(\frac{1}{p} - \frac{1}{2})$ (cf. [52, Remark 1.5.2/1]).

Classical de la Vallée-Poussin means and families are generated by the function

$$\varphi(\xi) = \begin{cases} 1 & , \quad |\,\xi\,| \leqslant 1 \\ 2 - |\,\xi\,| & , \quad 1 < |\,\xi\,| \leqslant 2 \\ 0 & , \quad |\,\xi\,| > 2 \end{cases} . \tag{5.17}$$

In this case we have $F\varphi \in L_p(\mathbb{R})$ if and only if $p > \frac{1}{2}$.

Corollary 5.2. *Let $\frac{1}{2} < p \le \infty$ and $0 < q \le \infty$. If φ is defined by (5.17) then the statements of Corollary 5.1 hold with $d = 1$ and $\alpha = \infty$.*

5.3 Riesz means and families

Riesz means and families with indices $\alpha, \beta > 0$ in the d-dimensional case are generated by $\varphi_{\alpha,\beta}(\xi) = (1 - |\,\xi\,|^\alpha)_+^\beta$, $\xi \in \mathbb{R}^d$. We define

$$p_{\alpha,\beta} = \begin{cases} 2d/(d+2\beta+1) & , \quad \beta \geqslant 0,\, \alpha \in \mathbb{E} \\ 2d/(d+2\beta+1) & , \quad 0 \leqslant \beta < \alpha + (d-1)/2,\, \alpha \notin \mathbb{E} \\ d/(d+\alpha) & , \quad \beta \geqslant \alpha + (d-1)/2,\, \alpha \notin \mathbb{E} \end{cases} \tag{5.18}$$

Here $\mathbb{E}$ is the set of all even natural numbers. Let $\beta > (d-1)/2$. It is shown in [31, Theorem 2.1] that

(i) $F\varphi_{\alpha,\beta} \in L_p(\mathbb{R}^d)$ if and only if $p_{\alpha,\beta} < p \le \infty$,

(ii) $1 - \varphi_{\alpha,\beta} \overset{(\widetilde{p},\eta)}{\prec} \psi$ for $\psi(\xi) = |\xi|^\alpha$ if $p_{\alpha,\beta} < p \le \infty$.

Obviously, $p_{\alpha,\beta} < 1$ if $\alpha > 0$ and $\beta > \frac{d-1}{2}$.

Theorem 5.5. *Let $0 < q \le \infty$, $\alpha > 0$ and let $\beta > \frac{d-1}{2}$, where $p_{\alpha,\beta}$ is defined by (5.18).*

(1) *If $p_{\alpha,\beta} < p \le \infty$ and $0 \le s < \alpha$ then (5.13) holds with $\varphi_{\alpha,\beta}$ in place of φ.*

(2) *If* $1 \le p \le \infty$ *and* $0 \le s < \alpha$ *then* (5.14) *holds with* $\varphi_{\alpha,\beta}$ *in place of* φ.

(3) *If* $0 \le s < \alpha$ *then* (5.15) *holds with* $\varphi_{\alpha,\beta}$ *in place of* φ.

(4) *If* $p_{\alpha,\beta} < p < 1$ *and* $d(\frac{1}{p} - 1)_+ < s < \alpha$ *then it holds* (5.16) *with* $\varphi_{\alpha,\beta}$ *in place of* φ.

Again the theorem is a consequence of Corollary 5.1 and the above properties of $\varphi_{\alpha,\beta}$. If $\alpha = 2$ the theorem refers to Bochner-Riesz means and families. The case $d = \alpha = \beta = 1$ corresponds to classical Fejér means and families.

A statement with respect to part (4) of the theorem can be found in [39, Theorem 7, part (i)]. Note that the result in [39] is not correct in the case $\beta > \alpha + \frac{d-1}{2}$ if α is not an even number (see also the comments in [31, page 135]).

6 Generalized moduli of smoothness

Analyzing different moduli of smoothness we observe their equivalence with functionals K_ψ defined and investigated in Subsection 4.4 for appropriate homogeneous functions $\psi \in \mathcal{H}_\alpha$ and operators $\psi(D)$, respectively. According to Proposition 4.3 this immediately leads to characterizations of Besov spaces $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$ with $0 \le s < \alpha$ and $0 < q \le \infty$ under certain restrictions with respect to the parameter p. We present a general result based on papers [34] (case $d = 1$) and [24] (case $d > 1$). Finally we give a list of examples covering also known results.

6.1 θ-moduli - general results

We say that a function $\theta : \mathbb{R}^d \to \mathbb{C}$ which is continuous and 2π-periodic with respect to each variable belongs to the class $\mathcal{G}$ if

$$\begin{aligned} \theta(-\xi) &= \overline{\theta(\xi)}\ , \quad \xi \in \mathbb{R}^d \\ \theta(0) &= 0, \quad \theta(\xi) \ne 0 \quad \text{if} \quad 0 < |\xi| < 2\pi\ , \\ \theta^\wedge(0) &= -1 \quad \text{and} \quad \sum_{\nu \in \mathbb{Z}^d} |\theta^\wedge(\nu)| < \infty\ . \end{aligned}$$

Following [34] and [24] we define the θ-modulus of smoothness of $f \in L_p(\mathbb{T}^d)$, $0 < p \le \infty$ by

$$\omega_\theta(f, t)_p = \sup_{0 \le h \le t} \| \Delta_h^{(\theta)} f \|_p\ , \quad t \ge 0, \tag{6.1}$$

where

$$(\Delta_h^{(\theta)} f)(x) = \sum_{\nu \in \mathbb{Z}^d} \theta^\wedge(\nu)\, f(x + h\,\nu) \tag{6.2}$$

is the θ-difference. The modulus (6.1) is well-defined in $L_p(\mathbb{T}^d)$ if

$$\|\theta^\wedge\|_{\ell_{\widetilde{p}}} = \Big(\sum_{\nu \in \mathbb{Z}^d} |\theta^\wedge(\nu)|^{\widetilde{p}} \Big)^{1/\widetilde{p}} < \infty \tag{6.3}$$

and then we have

$$\omega_\theta(f, t)_p \le \|\theta^\wedge\|_{\ell_{\widetilde{p}}}\, \|f\|_p\ . \tag{6.4}$$

Obviously, $\omega_\theta(f, \cdot)_p$ is an increasing function on $[0, \infty)$ and it holds $\omega_\theta(f, 0)_p = 0$. In the multivariate case ($d > 1$) the classical modulus of smoothness of order k as defined in (4.2) is not covered within the framework (6.1). However, let us emphasize that θ-moduli possess analogous properties. For details we refer to [24, Lemma 4 and 5, Theorem 1].

Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a smooth function satisfying $\eta(\xi) = 1$ for $|\xi| \le \delta$ and $\eta(\xi) = 0$ for $|\xi| \ge 2\delta$, where $0 < \delta < \frac{\pi}{2}$. We shall write

$$\eta_*(\xi) \;=\; \sum_{k\in\mathbb{Z}^d} \eta(\xi + 2\pi k)$$

for its periodic extension. Let $\alpha > 0$ and $0 < p \le \infty$. The class $\mathcal{G}(\alpha, p)$ consists of all functions $\theta \in \mathcal{G}$ satisfying (6.3) and the following conditions.

(i) There exist $\psi \in \mathcal{H}_\alpha$ and η as above such that $\psi(\cdot) \overset{(\widetilde{p},\eta)}{\asymp} \theta(\cdot)$.

(ii) It holds

$$\big((1 - \eta_*(\cdot))/\theta(\cdot)\big)^\wedge \in \ell_{\widetilde{p}}(\mathbb{Z}^d)\,. \tag{6.5}$$

Moreover , we set ($\mathbb{E}$ is the set of even natural numbers)

$$\Omega_d = \begin{cases} \mathbb{N} \times (0,\infty] \bigcup \big\{(\alpha,p) : \alpha \notin \mathbb{N},\, p > 1/(1+\alpha)\big\} & , \quad d = 1 \\ \mathbb{E} \times (0,\infty] \bigcup \big\{(\alpha,p) : \alpha \notin \mathbb{E},\, p > d/(d+\alpha)\big\} & , \quad d > 1 \end{cases}. \tag{6.6}$$

Lemma 6.1. *The class $\mathcal{G}(\alpha,p)$ is not empty if and only if $(\alpha,p) \in \Omega_d$.*

Proof. Let first $(\alpha,p) \notin \Omega_d$. Then $0 < p \leqslant d/(d+\alpha)$ and $\alpha \notin \mathbb{N}$ if $d = 1$, $\alpha \notin \mathbb{E}$ if $d > 1$. We assume (to the contrary) that there exists a certain function $\theta \in \mathcal{G}(\alpha,p)$. Then by Lemma 1 in [24] we get

$$\begin{aligned} \|\, F(\psi\tau)\,\|_p \;&=\; \|\, F(\psi\tau\eta)\,\|_p \;=\; \|\, F(((\eta\psi)/\theta)\,(\theta\tau))\,\|_p \\ &\leqslant\; c\,\|(((\eta\psi)/\theta)\,(\theta\tau))_*^\wedge\|_{\ell_p} = c\,\|((\eta\psi)/\theta)_*^\wedge * \theta_*^\wedge * \tau_*^\wedge\|_{\ell_p} \\ &\leqslant\; c\,\|\,((\eta\psi)/\theta)_*^\wedge\,\|_{\ell_p}\,\|\,\theta_*^\wedge\,\|_{l_p}\,\|\,\tau_*^\wedge\,\|_{\ell_p} \\ &\leqslant\; c_1\,\|\, F((\eta\psi)/\theta)\,\|_p \;<\; \infty\,, \end{aligned}$$

where ψ belongs to $\mathcal{H}_\alpha$, η is the test function introduced above and τ is a test function, whose support is concentrated in the set $\{\,\xi : \eta(\xi) = 1\,\}$, and satisfying $\tau(0) \neq 0$. Hence, $F(\psi\tau)$ belongs to $L_p(\mathbb{R}^d)$ and this implies that ψ should be a polynomial in view of the statement on the Fourier transform of homogeneous functions multiplied with test functions given in [26]. In order to get a contradiction it remains to notice that the class $\mathcal{H}_\alpha$, where α satisfies the above conditions, does not contain polynomials.

Let now $(\alpha,p) \in \Omega_d$. First we consider the case $d = 1$. For $\alpha \in \mathbb{N}$, $0 < p \leqslant \infty$ the class $\mathcal{G}(\alpha,p)$ contains the generator $\theta(\xi) = -(1 - e^{i\xi})^\alpha$ of the classical modulus of smoothness of order α. In the case $\alpha \notin \mathbb{N}$, $p > 1/(\alpha+1)$ the function $\theta(\xi) = (|\,\xi\,|^\alpha \eta(\xi))_* + (1 - \eta_*(\xi))$, where η is as above, belongs to $\mathcal{G}(\alpha,p)$ due to the result in [26] mentioned above. If $d > 1$, then for $\alpha \in \mathbb{E}$, $0 < p \leqslant \infty$, as well as for $\alpha \notin \mathbb{E}$ and $p > d/(d+\alpha)$ the class $\mathcal{G}(\alpha,p)$ contains the function $\theta(\xi) = (|\,\xi\,|^\alpha \eta(\xi))_* + (1 - \eta_*(\xi))$. □

Theorem 6.1. *Let $0 < p \le \infty$, $0 < q \le \infty$ and let $0 \le s < \infty$. Let $\theta \in \mathcal{G}$ such that conditions* (6.3) *and* (6.5) *are satisfied. If there exists $\psi \in \mathcal{H}_\alpha$ such that $\psi \overset{(\widetilde{p},\eta)}{\prec} \theta$ for some $\alpha > 0$ then*

$$\|f|\mathbb{B}^s_{p,q}\| \;\lesssim\; \|f\|_p + \left(\int_0^1 t^{-sq}\,\omega_\theta(f,\,t)_p^q\,\frac{dt}{t}\right)^{1/q} \tag{6.7}$$

for all $f \in L_p(\mathbb{T}^d)$ (standard modification if $q = \infty$).

Proof. Under the assumptions of the theorem we have the Jackson-type estimate

$$E_n(f)_p \lesssim \omega_\theta(f, n^{-1})_p, \quad n \in \mathbb{N}, \quad f \in L_p(\mathbb{T}^d) \tag{6.8}$$

(see [24, Theorem 1]). Since $\omega_\theta(f, \cdot)_p$ is non-decreasing, we get

$$E_{2^j}(f)_p \lesssim \omega_\theta(f, 2^{-j})_p \lesssim \omega_\theta(f, t)_p$$

for all $t,\ 2^{-j} \le t \le 2^{-j+1} \quad (j \in \mathbb{N}_0)$. This implies

$$\int_{2^{-j}}^{2^{-j+1}} t^{-sq} E_{2^j}(f)_p^q \frac{dt}{t} \lesssim \int_{2^{-j}}^{2^{-j+1}} t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t}$$

and, consequently,

$$2^{sjq} E_{2^j}(f)_p^q \lesssim \int_{2^{-j}}^{2^{-j+1}} t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t} .$$

By summation and (6.4) we obtain

$$\sum_{j=0}^{\infty} 2^{sjq} E_{2^j}(f)_p^q \lesssim \int_0^2 t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t} \lesssim \|f\|_p + \int_0^1 t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t} .$$

This proves (6.7) taking into account Proposition 4.2 and Remark 6. □

Theorem 6.2. *Let $0 < p \le \infty,\ 0 < q \le \infty$, and let $0 < \alpha < \infty$. Let $\theta \in \mathcal{G}$ such that condition (6.3) is satisfied. Suppose there exists $\psi \in \mathcal{H}_\alpha$ such that $\theta \overset{(\widetilde{p},\eta)}{\prec} \psi$. If $0 \le s < \alpha$ then it holds*

$$\int_0^1 t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t} \lesssim \|f|\mathbb{B}_{p,q}^s\| \tag{6.9}$$

for all $f \in \mathbb{B}_{p,q}^s(\mathbb{T}^d)$ (standard modification if $q = \infty$).

Proof. Under the assumptions of the theorem we have the inequality

$$\omega_\theta(f, 2^{-j})_p \lesssim K_\psi(f, 2^{-j})_p, \quad j \in \mathbb{N}_0 . \tag{6.10}$$

This follows from the proof of [24, Theorem 2]. It yields the estimates

$$\int_0^1 t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t} \le \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} t^{-sq} \omega_\theta(f, t)_p^q \frac{dt}{t}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{sjq} \omega_\theta(f, 2^{-j})_p^q \lesssim \sum_{j=0}^{\infty} 2^{sjq} K_\psi(f, 2^{-j})_p^q .$$

This implies (6.9) according to Proposition 4.3 and Remark 8. □

Corollary 6.1. *Let $0 < q \le \infty$, $(\alpha, p) \in \Omega_d$ and let $\theta \in \mathcal{G}(\alpha, p)$. If $0 \le s < \alpha$ then it holds $f \in \mathbb{B}^s_{p,q}(\mathbb{T}^d)$ if and only if $f \in L_p(\mathbb{T}^d)$ such that*

$$\|f|\mathbb{B}^s_{p,q}\|_\theta \;=\; \|f\|_p + \left(\int\limits_0^1 t^{-sq}\, \omega_\theta(f,\, t)_p^q \, \frac{dt}{t} \right)^{1/q} \;<\; \infty \tag{6.11}$$

(standard modification if $q = \infty$). Moreover, $\| \cdot |\mathbb{B}^s_{p,q}\|_\theta$ is an equivalent quasi-norm in $\mathbb{B}^s_{p,q}(\mathbb{T}^d)$.

Remark 12. Let us emphasize that under the assumptions of Corollary 6.1 the equivalence

$$\omega_\theta(f,\, n^{-1})_p \;\asymp\; K_\psi(f,\, n^{-1})_p, \quad n \in \mathbb{N},\; f \in L_p(\mathbb{T}^d)\;. \tag{6.12}$$

even holds. This is proved in [24, Theorem 2] and in [34, Theorem 3].

6.2 Moduli of fractional order

Let $\alpha > 0$ be be a non-integer real number and let $f \in L_p(\mathbb{T}),\; 0 < p \le \infty$. If $t > 0$ we put

$$\omega_\alpha(f,\, t)_p \;=\; \sup_{0 \le h \le t} \left\| \sum_{\nu=1}^{\infty} \binom{\alpha}{\nu} (-1)^{\nu-1} f(\cdot + \nu h) \;-\; f(\cdot) \right\|_p , \tag{6.13}$$

where

$$\binom{\alpha}{\nu} \;=\; \frac{\alpha(\alpha-1)\,\cdots\,(\alpha-\nu+1)}{\nu\,!}\;.$$

This modulus is called modulus of smoothness of fractional order α and has been systematically investigated, for example, in [44], [49] and [50] in the case $1 \le p \le \infty$. It coincides with $\omega_{\theta_\alpha}(f,\,\cdot)_p$ for $\theta_\alpha(\xi) = -(1 - e^{i\xi})^\alpha,\; \xi \in \mathbb{R}$, where $z^\alpha = |z|^\alpha\, e^{i\alpha \arg z},\; z \in \mathbb{C},\; -\pi < \arg z \le \pi$. It is related to the functional K_{ψ_α}, where $\psi_\alpha(\xi) = (i\xi)^\alpha,\; \xi \in \mathbb{R}$. Note that $\psi_\alpha(D)$ is the Weyl derivative of order α. In this case it holds the equivalence (6.12) if $\frac{1}{1+\alpha} < p \le \infty$. For details we refer to [34, Lemma 5].

Corollary 6.2. *Let $0 < q \le \infty$ and let $\alpha > 0$ be a non-integer real number. If $0 \le s < \alpha$ and $\frac{1}{1+\alpha} < p \le \infty$ then $f \in L_p(\mathbb{T})$ belongs to $\mathbb{B}^s_{p,q}(\mathbb{T})$ if and only if*

$$\int\limits_0^1 t^{-sq}\, \omega_\alpha(f,\, t)_p^q \, \frac{dt}{t} \;<\; \infty \tag{6.14}$$

(standard modification if $q = \infty$).

6.3 Moduli related to Riesz derivatives

Let us consider $\psi_\alpha(\xi) = |\xi|^\alpha,\; \alpha > 0,\; \xi \in \mathbb{R}$. Then the associated operator $\psi_\alpha(D)$ corresponds to the Riesz derivative of order α (fractional derivative if α is non-integer, usual derivative if α) is an even natural number). The related K-functional (3.2) will be denoted by $K_{\langle\alpha\rangle}$. Here we focus our interest on the case when α is non-integer. There are different moduli of smoothness which are equivalent to $K_{\langle\alpha\rangle}$ on $L_p(\mathbb{T})$ for the range of parameters $\frac{1}{1+\alpha} < p \le \infty$. We give list of examples.

The modulus

$$\widetilde{\omega}(f,\, t)_p \;=\; \sup_{0 \le h \le t} \left\| \frac{4}{\pi^2} \sum_{\nu \in \mathbb{Z}} \frac{f(x + (2\nu+1)h)}{(2\nu+1)^2} \;-\; f(x) \right\|_p \tag{6.15}$$

is generated by $\widetilde{\theta}(\xi) = -\frac{2}{\pi}\,|\xi|, \quad -\pi \le \xi \le \pi$. Relation (6.12) holds with $K_{\langle 1\rangle}$ for $\widetilde{\omega}$ if $\frac{1}{2} < p \le \infty$ (see [34, Theorem 6]). Let $\frac{1}{2} < p \le \infty$ and let $0 \le s < 1$. Then (6.11) with $\widetilde{\omega}$ holds for $f \in L_p(\mathbb{T})$ if and only if $f \in \mathbb{B}^s_{p,q}(\mathbb{T})$.

The modulus

$$\overline{\omega}(f,\, t)_p \;=\; \sup_{0\le h\le t} \Big\| \frac{3}{\pi^2} \sum_{\nu\in\mathbb{Z},\ \nu\ne 0} \frac{f(x+\nu h)}{\nu^2} \;-\; f(x) \Big\|_p \tag{6.16}$$

generated by $\overline{\theta}(\xi) = \frac{3}{\pi^2}\,\xi^2 - \frac{3}{\pi}, \quad 0 \le \xi \le 2\pi$, has been introduced and treated in [2] and [3]. Let $\frac{1}{2} < p \le \infty$ and let $0 \le s < 1$. Then we have equivalence (6.12) with $K_{\langle 1\rangle}$ for $\overline{\omega}$ and (6.11) with $\overline{\omega}$ holds for $f \in L_p(\mathbb{T})$ if and only if $f \in \mathbb{B}^s_{p,q}(\mathbb{T})$.

The modulus

$$\widetilde{\omega}_{\langle\alpha\rangle}(f,\, t)_p \;=\; \sup_{0\le h\le t} \Big\| \sum_{\nu\in\mathbb{Z},\ \nu\ne 0} \frac{f(\cdot+\nu h)}{|\nu|^{\alpha+1}} \Big\|_p, \tag{6.17}$$

where $0 < \alpha < 1$, is generated by

$$\widetilde{\theta}_{\langle\alpha\rangle}(\xi) \;=\; \sum_{\nu\in\mathbb{Z},\ \nu\ne 0} |\nu|^{-\alpha-1}\left(e^{i\nu\xi} - 1\right), \quad \xi \in \mathbb{R}.$$

Equivalence (6.12) is proved for $K_{\langle\alpha\rangle}$ and $\widetilde{\omega}_{\langle\alpha\rangle}$ in [35] for $\frac{1}{1+\alpha} < p \le \infty$. Let $\frac{1}{\alpha+1} < p \le \infty$ and let $0 \le s < \alpha$. Then (6.11) with $\widetilde{\omega}_{\langle\alpha\rangle}$ holds for $f \in L_p(\mathbb{T})$ if and only if $f \in \mathbb{B}^s_{p,q}(\mathbb{T})$.

The modulus

$$\omega_{\langle\alpha\rangle}(f,\, t)_p \;=\; \sup_{0\le h\le t} \Big\| \sum_{\nu\in\mathbb{Z},\ \nu\ne 0} \Big(- \frac{a_{|\nu|}(\alpha)}{a_0(\alpha)} \Big) f(\cdot+\nu h) \;-\; f(\cdot) \Big\|_p, \tag{6.18}$$

where

$$a_m(\alpha) \;=\; \sum_{j=m}^{\infty} \frac{(-1)^{j+1}}{2^{2j}} \binom{\alpha/2}{j} \binom{2j}{j-m}, \quad m \in \mathbb{N}_0,$$

is generated by

$$\theta_{\langle\alpha\rangle}(\xi) \;=\; \frac{1}{a_0(\alpha)}\,\Big|\sin\frac{\xi}{2}\Big|^{\alpha}, \quad \xi \in \mathbb{R}.$$

If $\alpha \notin \mathbb{E}$ and $\frac{1}{\alpha+1} < p \le \infty$ then equivalence (6.12) holds for $K_{\langle\alpha\rangle}$ and $\widetilde{\omega}_{\langle\alpha\rangle}$. This can be found in [34, Theorem 7]. Under these assumptions and $0 \le s < \alpha$ (6.11) with $\omega_{\langle\alpha\rangle}$ holds for $f \in L_p(\mathbb{T})$ if and only if $f \in \mathbb{B}^s_{p,q}(\mathbb{T})$.

6.4 Moduli related to the Laplacian

Given $m \in \mathbb{N}$ we define the modulus

$$\omega_{m,\,d}(f,\, t)_p \;=\; \sup_{0\le h\le t} \Big\| \frac{\sigma_m}{d} \sum_{j=1}^{d} \sum_{\nu=-m,\,\nu\ne 0}^{m} \frac{(-1)^j}{\nu^2} \binom{2m}{m-|\nu|} f(x+\nu h e_j) \;-\; f(x) \Big\|_p \tag{6.19}$$

$(t > 0,\ f \in L_p(\mathbb{T}^d),\ 0 < p \le \infty)$, where

$$\sigma_m \;=\; \Big(2 \sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu^2} \Big(\binom{2m}{m-\nu} \Big) \Big)^{-1}$$

and where $\{e_j\}_{j=1}^d$ denotes the standard basis in $\mathbb{R}^d$. It is generated by the function

$$\theta_{m,d}(\xi) \;=\; \frac{1}{d} \sum_{j=1}^{d} \theta_m(\xi_j), \quad \xi \in \mathbb{R}^d,$$

where

$$\theta_m(\xi_j) \;=\; \gamma_m \int_0^{\xi_j} \int_0^{x} \left(\sin^{2m}(\tau/2) - a_m \right) d\tau \, dx$$

with

$$a_m \;=\; 2^{-2m} \binom{2m}{m}, \quad \gamma_m \;=\; -2^{2m} \, \sigma_m \, .$$

This modulus has been considered in [10] (the case $m = 1$) and in [36] (general case). It is equivalent to the K-functional (3.2) associated with the Laplacian ($\psi(\xi) = |\xi|^2$) in the sense of (6.12) if and only if

$$p \;>\; p_{m,d} \;=\; \begin{cases} \frac{d}{d+2(m+1)}, & d = 2, \; m \in \mathbb{E} \\ \frac{d}{2m}, & \text{otherwise} \end{cases}$$

(cf. [36, Theorem 5.3]). Consequently, the statement of Corollary 6.1 with $\omega_{m,\,d}(f,\,\cdot)_p$ in place of $\omega_\theta(f,\,\cdot)_p$ holds if $p_{m,\,d} < p \le \infty$ and $0 \le s < 2$.

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Sergei Artamonov
National Research University Higher School of Economics
Moscow, Russian Federation
E-mails: sergei.artamonov@gmail.com

Konstantin Runovski
Lomonosov Moscow State University
Moscow, Russian Federation
E-mail: k_runov@mail.ru

Hans-Jürgen Schmeisser
Friedrich-Schiller University
Jena, Germany
E-mail: mhj@uni-jena.de

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