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NONTRIVIAL OUTER DERIVATIONS IN BIMODULES OVER GROUP RINGS

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Abstract. Paper is devoted to the description of outer derivations for bimodules over group rings. Examples of corresponding bimodules with nontrivial outer derivations are constructed and their properties are investigated, in particular, their dependence on generating sets. Various types of groups by type of growth are investigated: polynomial and exponential.

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1 Introduction

Derivations in various associative algebras have been actively studied since the beginning of the last century. In particular, the following question, known as the Johnson Problem (or "Derivation Problem"), is widely known:

Question ([5], question 5.6.B). Is it true that all derivations in $L_1(G)$ are inner?

Hereafter G is a finitely generated generally noncommutative group. A partial answer to this question was given by B. Johnson himself in [8], and the most complete answer was found by V. Losert in [10]. For more details on the history of this problem we refer the reader to [3, 5].

Consider the group ring $\mathbb{C}[G]$ – the space of elements which can be presented as $\sum_{g\in G} x(g)g$, where $x\colon G\to\mathbb{C}$ is a finite function (a function with finite support). The derivation in this case is a linear operator $d:\mathbb{C}[G]\to\mathbb{C}[G]$ satisfying the Leibniz rule

$$d(uv) = d(u)v + ud(v), \quad \forall u, v \in \mathbb{C}[G].$$

Inner derivations are given by the formula

$$d_a : x \to [a, x] = ax - xa, a \in \mathbb{C}[G].$$

The quotient of the space of all derivations by inner derivations is called the space of outer derivations. In this case the algebra of outer derivations is nontrivial (see central derivations from [1] as well as results of [3]).

In this paper, we will consider the case of derivations with respect to analytic structure. Our purpose is to give some examples when there exist nontrivial outer derivations.

Let the ring $\mathbb{C}[G]$ be a normed space with the supremum norm $\|\cdot\|_s$, hence for $\omega = \sum_{g\in G} x(g)g$

$$\|\omega\|_s := \sup_{g\in G} |x(g)|.$$

Consider another norm $\|\cdot\|$ on $\mathbb{C}[G]$. Denote by $\mathcal{A}$ the completion of $\mathbb{C}[G]$ by this norm.

Definition 1. By derivation over $\mathbb{C}[G]$ with values in a bimodule $\mathcal{A}$ we will call a linear bounded operator $d\colon \mathbb{C}[G] \to \mathcal{A}$ such that

$$d(uv) = d(u)v + ud(v), \quad \forall u, v \in \mathbb{C}[G].$$

Denote the space of all such operators by $\mathrm{Der}(\mathcal{A})$.

The boundedness of an operator is understood as the boundedness of its norm

$$\|d\| = \sup_{\omega \neq 0 \in \mathbb{C}[G]} \frac{\|d(\omega)\|}{\|\omega\|_s}.$$

In [2] the structure of derivations in such bimodules was studied ans it was shown that for norms subordinated to supremum norms all derivations are quasi-inner. In this paper, we construct examples of quasi-outer (and hence outer) derivations for norms in which the property for basis elements $\|g\| = 1$ is not satisfied. The following main results are obtained. Definitions can be found in sections 2, 3.

Theorem 1.1. *Let $(G, \mathcal{X})$ be a finitely generated group, $z \in Z(G)$ be a central element of length k_0 and an arbitrary $\alpha > \ln(|\mathcal{X} \cup \mathcal{X}^{-1}|)$ be given. Denote the norm on $\mathbb{C}[G]$ by $\|g\|_{\alpha,\exp} = e^{-\alpha|g|}$. Then any central derivation $d^\tau_z\colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_{\alpha\exp}$ is a continuous operator.*

Central derivations were considered in [1] (Definition 3) and they provide an example of derivations that cannot be inner. Below we give their exact definition (see Definition 6).

Theorem 1.1. *For the groups of polynomial growth of degree n with the norm induced by the function $\xi(k) = \frac{1}{k^{n+\alpha}}$, $\xi(0) = 1$, $\alpha > 1$, all central derivations $d^\tau_z\colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_\xi$ are continuous.*

Thus, we obtain families of norms in which the central derivations are bounded. Hence, the spaces of quasi-outer derivations (containing central derivations) are also nontrivial.

2 Continuity of central derivations and groups growth

First of all, let us introduce some definitions. Hereinafter $G = \langle \mathcal{X} \rangle$ is a group generated by a finite set $\mathcal{X}$.

A *length* $|g|$ of the group element g is a minimal number n such that g can be represented in the form $g = x_1^{\pm 1} \dots x_n^{\pm 1}$, where $x_i \in \mathcal{X}$. Also using the length of the element we can define the metric on the group by setting $\rho(g_1, g_2) = |g_1 g_2^{-1}|$.

We will denote by $B_k(g_0)$ the set of elements $g \in G$ such that $\rho(g_0, g) \leq k$.

Definition 2. The *growth function* $f = f_G^{\mathcal{X}}$ of the group G with respect to the set $\mathcal{X}$ is the function $f\colon \mathbb{N} \to \mathbb{N}$ that maps the number k to the number of elements in $B_k(1_G)$, that is, $k \mapsto |B_k(1_G)|$.

Let us see how the growth function of the group depends on the choice of the generating set. The following assertion is well known.

Proposition 2.1 ([12], Lemma 37.1)**.** *Let $f_G^{\mathcal{X}}$ and $f_G^{\mathcal{Y}}$ be growth functions of the finitely generated group G with respect to the generating sets $\mathcal{X} = \{x_1, \dots, x_l\}$ and $\mathcal{Y} = \{y_1, \dots, y_m\}$. Then there exists a constant $C > 0$ such that $f_G^{\mathcal{Y}}(k) \leq f_G^{\mathcal{X}}(Ck)$ for all k.*

Define a binary relation on the set of functions with natural numbers as a domain: $f \sim g$ if for some constants $C > 0$ and $D > 0$: $f(k) \leq g(Ck)$ and $g(k) \leq f(Dk)$ for all k. Evidently, this relation is an equivalence relation.

Definition 3. The equivalence class of the growth function $f_G^{\mathcal{X}}$ of G is called the *growth type* of G. According to the proposition 2.1, the growth type of G is well defined.

It is known that free groups can be regarded as word groups [11]. That is, each element g in a free group can be identified with the equivalence class of its expression through generators.

Definition 4. The word is called *reduced* if it does not contain any subwords of the form aa^{-1} for $a \in \mathcal{X}^{\pm 1}$.

Free groups provide examples of groups of exponential growth.

Example 1. Free group F_m of rank $m > 1$ has an exponential growth function in case when $\mathcal{X}$ freely generates F_m.

See [12, §37], for details.

We will work with bimodules obtained by providing group rings with a norm. Recall that group ring is an abelian group $\bigoplus_G \mathbb{C}$, where standard $R-$ basis vectors identified with elements of G. The multiplicative structure is defined by the extension of the group multiplication

$$\left(\sum_{g\in G} a_g g\right) \cdot \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}\left(\sum_{h\in G} a_h b_{h^{-1}g}\right) g. \tag{2.1}$$

To find out examples of nontrivial outer derivations we will generate norms by functions.

Definition 5. We say that a norm is *induced* by a positive function f if it has the form

$$\|\sum_{g\in G} a_g g\| = \sum_{g\in G} |a_g| f(|g|). \tag{2.2}$$

We are mainly interested in norms induced by exponential and polynomial functions.

Let us obtain reverse triangle inequality for the length of the product of two elements. Let a, $b \in G$. Then

$$|ab| \geq \left||a| - |b|\right|. \tag{2.3}$$

Let $ab = c$, $|a| = k$, $|b| = l$, $|c| = m$. Then we must prove that $m \leq |k - l|$. Writing all the elements through the generators in the shortest possible way, we have

$$x_{i_1} \dots x_{i_k} \cdot x_{j_1} \dots x_{j_l} = x_{s_1} \dots x_{s_m}. \tag{2.4}$$

Since words are written in the shortest form, we see that

$$x_{i_1} \dots x_{i_k} = x_{s_1} \dots x_{s_m} \cdot x_{j_l}^{-1} \dots x_{j_1}^{-1} \Rightarrow k \leq m + l \Rightarrow m \geq k - l. \tag{2.5}$$

Similarly, getting the inequality with the opposite sign, we obtain $m \geq |k - l|$.

Now let us consider the central derivations.

Definition 6. [[1], Definition 3] Recall that the *central derivation* d_z^τ is a linear operator defined via $g \mapsto \tau(g) g z$. Here $\tau\colon G \to (\mathbb{C}, +)$ is a homomorphism, $z \in Z(G)$ is some central element of G.

It is known that for the algebraic case the subspace generated by all central derivations is a subalgebra, this is shown in [1].

We mostly interesting in the case of mapping from $\mathbb{C}[G]_{\text{sup}}$ to $\mathbb{C}[G]_{\text{exp}}$. We can regard G as $G = \bigcup_k S_k$, where S_k are words of length k.

Let us find out if the central derivation is continuous, as the mapping $d_z^\tau\colon \mathbb{C}[G]_{\text{sup}} \to \mathbb{C}[G]_{\text{exp}}$.

Lemma 2.1. *Consider a group G with generated set $\mathcal{X}$. Let a positive nonincreasing function f satisfy the following condition:*

$$\forall k_0\ \exists C>0 : \forall k>k_0 \hookrightarrow f(k-k_0)<Cf(k). \tag{2.6}$$

Then if the operator $L\colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_f$, defined by $g \mapsto |g|g$ is bounded then an arbitrary central derivation is bounded.

Proof. Since L is continuous, we get

$$\|L\| = \sum_{k=0}^{\infty} k|S_k|f(k) < +\infty. \tag{2.7}$$

Now consider an arbitrary central derivation $d_z^\tau\colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_f$. Let us show that d_z^τ is bounded.

Indeed, let $\left\|\sum_{g\in G} a_g g\right\|_{\sup} = 1$. Then

$$\left\|d_z^\tau\left(\sum_{g\in G} a_g g\right)\right\|_f = \left\|\sum_{g\in G} a_g \tau\left(g\right) gz)\right\|_f \leq \sum_{g\in G} |a_g|\, \|\tau(g)gz\|_f\,. \tag{2.8}$$

The set of generators is finite, so there exists $M = \max_{x\in\mathcal{X}} |\tau(x)|$. Then for any $g \in S_k$, we get $|\tau(g)| \leq Mk$. Now we can continue the chain of inequalities.

$$\begin{aligned}\sum_{g\in G} |a_g|\, \|\tau\left(g\right) gz\|_f &= \sum_{k=0}^{\infty}\sum_{g\in S_k} |a_g|\, \|\tau\left(g\right) gz\|_f \leq C\sum_{k=0}^{\infty}\sum_{g\in S_k} k|a_g|\, \|gz\|_f \\ &\leq M\sum_{k=0}^{\infty} k\left|S_k\right| f(|k-k_0|) \leq M\sum_{k=0}^{k_0} k|S_k|f(|k-k_0|) + CM\sum_{k=k_0+1}^{\infty} k\left|S_k\right| f(k).\end{aligned} \tag{2.9}$$

Here, we use the inequality $|a_g| \leq 1$, and apply inequality (2.3) to estimate the norm of $\|gz\|_f$, the last inequality is obtained by applying condition (2.6).

Now we can see that $\sum_{k=0}^{k_0} k\left|S_k\right| f(|k-k_0|)$ does not depend on the original element $\sum_{g\in G} a_g g$, and

$$\sum_{k=k_0+1}^{\infty} k\left|S_k\right| f(k) \leq CM\left\|L\right\|. \tag{2.10}$$

Thus, we have shown that the central derivation is continuous.

□

Proposition 2.2. *Suppose f satisfies the conditions of Lemma 2.1. Then the right-multiplication by $u \in \mathbb{C}[G]$ is a continuous operator $r_u\colon \mathbb{C}[G]_f \to \mathbb{C}[G]_f$.*

Proof. Since the linear combination of continuous operators is continuous, to prove the proposition it suffices to prove that r_{g_0} is continuous for arbitrary $g_0 \in G$.

Let $u = \sum_{g\in G} a_g g \in \mathbb{C}[G]_f$ such that $\|u\|_f \leq 1$. The assumption of Lemma 2.1 can be rewritten as

$$\forall k_0\ \exists C>0 : \forall k \hookrightarrow\ f(k) < Cf(k+k_0). \tag{2.11}$$

So, we obtain the following estimates:

$$\begin{aligned}\|r_{g_0}u\|_f &= \sum_{g\in G} |a_g|\, f\left(|gg_0|\right) \leq \sum_{g\in G} |a_g|\, f\left(||g|-|g_0||\right) \\ &\leq C\sum_{g\in G} |a_g|\, f\left(||g|-|g_0||+|g_0|\right) \leq C\sum_{g\in G} |a_g|\, f\left(|g|\right) \leq C.\end{aligned} \tag{2.12}$$

Here, the penultimate inequality follows from (2.11). Since f in non-increasing and $||g|-|g_0||+|g_0| \geq |g|$, we obtain the last inequality. Thus, $\|r_{g_0}\| < C$. Note that here the constant C depends on g_0. □

Analogous reasoning shows the continuity of the left-multiplication.

Example 2. Assumption (2.6) holds for the following functions:

$$f_{exp}(k) = e^{-\alpha k}, \qquad f_{pol}(k) = \begin{cases} f(k) = \frac{1}{k^n}, & k > 0 \\ 1, & k = 0 \end{cases}. \tag{2.13}$$

Let us pass to the main result of this section.

Theorem 2.1. *Let $(G, \mathcal{X})$ be a finitely generated group, $z \in Z(G)$ be a central element of length k_0 and an arbitrary $\alpha > \ln(|\mathcal{X} \cup \mathcal{X}^{-1}|)$ be given. Denote the norm on $\mathbb{C}[G]$ by $\|g\|_{\alpha,\exp} = e^{-\alpha|g|}$. Then any central derivation $d_z^\tau \colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_{\alpha\exp}$ is a continuous operator.*

Proof. Denote $m_0 = |\mathcal{X} \cup \mathcal{X}^{-1}|$. Let us verify that the assumptions of Lemma 2.1 are satisfied. The number of elements of length k is less or equal than m_0^k. Thus, we obtain

$$\|L\| = \sum_{k=0}^{\infty} k\,|S_k|\,e^{-k} = \sum_{k=1}^{\infty} k m_0^k e^{-k} = \sum_{k=0}^{\infty} k \exp\left(-\left(\alpha - \ln m_0\right)\right)^k < +\infty. \tag{2.14}$$

□

In other words, for a finitely generated group there exists a norm of exponential form such that any central derivation is bounded. Note that for groups of polynomial growth the theorem can be improved. Recall that a group G has a *polynomial growth* if its growth function is bounded above by a polynomial function.

Proposition 2.3. *If G is a group of a polynomial growth, then any central derivation is bounded for any $\alpha > 0$.*

Proof. Let us show that the assumptions of Lemma 2.1 are satisfied, that is, the continuity of the operator L. From the fact that the growth function of the group is bounded by the polynomial function $f_G^{\mathcal{X}}(k) \leq P(k)$, we get that the number of elements of length k is bounded by the same number. So,

$$\|L\| = \sum_{k=0}^{\infty} k\,|S_k|\,e^{-k} \leq \sum_{k=0}^{\infty} k \left|f_G^{\mathcal{X}}(k)\right| e^{-k} \leq \sum_{k=0}^{\infty} k\,|P(k)|\,e^{-k} < +\infty. \tag{2.15}$$

□

The assumption $\alpha > \ln(|\mathcal{X} \cup \mathcal{X}^{-1}|)$ in Theorem 2.1 is essential, as the following example shows.

Example 3. Consider the free group $F_4 = \langle x_1, \ldots, x_4 \rangle$. Define the homomorphism τ on the generators by $\tau(x_i) = 1$, set $z = e$. Let $u_k \in \mathbb{C}[G]$ be the sum with unit coefficients of group elements such that their length is equal to k, containing no inverse letters in the reduced form. We see that $\forall k\ \|u_k\|_{\sup} = 1$ and

$$\|d_z^\tau(u_k)\|_{\alpha\exp} = k4^k e^{-\alpha k}. \tag{2.16}$$

When $\alpha < \ln 4$ the norm of the image can be arbitrarily large as $k \to \infty$. Therefore, the central derivation is not necessary continuous when $\alpha < \ln 4$.

Let us proceed to a more detailed study of central derivations when a group has a polynomial growth.

3 Groups of polynomial growth

A significant achievement is the characterization of groups of polynomial growth [7]. Mikhail Gromov has proved that these groups are exactly groups that contain a nilpotent subgroup of finite index.

Definition 7. It is said that a group G has exponential groups growth if its growth function $f(k)$ is bounded below by a function of the form c^k, where $c > 1$ (then any equivalent function is bounded below by some exponential function $c_1^k, c_1 > 1$).

Note that in [6] were found examples of intermediate growth groups, that is, groups whose growth rate is faster than any polynomial but slower than any exponential function c^k, $c > 1$.

Consider a group of polynomial growth $f_G^{\mathcal{X}}(k) \leq C(k^n + 1)$. Our goal is to find a function ξ that with respect to the norm defined by this function $\|g\|_\xi = \xi(|g|)$ all central derivations are continuous.

Theorem 3.1. *For the groups of polynomial growth of degree n with the norm induced by the function $\xi(k) = \frac{1}{k^{n+\alpha}}$, $\xi(0) = 1$, $\alpha > 1$, all central derivations $d_z^\tau \colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_\xi$ are continuous.*

Proof. It suffices to show that the operator L is bounded, then the theorem follows from Lemma 2.1. We need to verify that

$$\sum_{k=0}^{\infty} k\,|S_k|\,\xi(k) < \infty. \tag{3.1}$$

Let us show that the sum $\sum_{k=1}^{N} k\,|S_k|\,\xi(k)$ is uniformly bounded for $N \in \mathbb{N}$. Noting that $|S_k| = f_G^{\mathcal{X}}(k) - f_G^{\mathcal{X}}(k-1)$, we have

$$\sum_{k=1}^{N} k\,|S_k|\,\xi(k) = \sum_{k=1}^{N-1} f_G^{\mathcal{X}}(k)\left(k\xi\left(k\right) - \left(k+1\right)\xi\left(k+1\right)\right) + N f_G^{\mathcal{X}}(N)\xi(N) - f_G^{\mathcal{X}}(0)\xi(1). \tag{3.2}$$

The term $N f_G^{\mathcal{X}}(N)\xi(N)$ is uniformly bounded for $N \in \mathbb{N}$ since

$$N f_G^{\mathcal{X}}(N)\xi(N) \leq \frac{N P(N)}{N^{n+\alpha}} \to 0. \tag{3.3}$$

Here, $P(n)$ is a polynomial of degree at most n, such that $f_G^{\mathcal{X}}$ bounded above by P. We need to show that the sum in the first summand is uniformly bounded too. We get

$$\left(k\xi\left(k\right) - \left(k+1\right)\xi\left(k+1\right)\right) = \frac{1}{k^{n+(\alpha-1)}} - \frac{1}{\left(k+1\right)^{n+(\alpha-1)}} \leq \left(n+\alpha-1\right)\frac{1}{k^{n+\alpha}}. \tag{3.4}$$

To obtain this inequality we apply Lagrange's theorem to the function $f(x) = \frac{1}{x^{n+\alpha}}$ on the interval $[k, k+1]$, and use that $\forall \xi \in [k, k+1] \hookrightarrow |f'(\xi)| < |f'(k)|$. Since $f_G^{\mathcal{X}}(k)$ is a polynomial of degree n, then

$$\sum_{k=1}^{N-1} f_G^{\mathcal{X}}(k)\left(k\xi\left(k\right) - \left(k+1\right)\xi\left(k+1\right)\right)$$

$$\leq \sum_{k=1}^{N-1} \frac{\left(n+\alpha-1\right) f_G^{\mathcal{X}}(k)}{k^{n+\alpha}} \leq \left(n+\alpha-1\right) \sum_{k=1}^{\infty} \frac{f_G^{\mathcal{X}}(k)}{k^{n+\alpha}} < \infty.$$

Thus, each term is uniformly bounded for $N \in \mathbb{N}$, and hence so is their sum. Since each term of the sum below is non-negative, we obtain

$$\sum_{k=0}^{\infty} k\,|S_k|\,\xi(k) < \infty. \tag{3.5}$$

□

The assumption $\alpha > 1$ is essential, as the following example shows.

Example 4. Let $G = \mathbb{Z} = \{a^k | k \in \mathbb{Z}\}$. We take $z = e = a^0$ as the central element for d_z^τ. Consider a homomorphism $\tau\colon a^k \mapsto k$. Put $\mathcal{X} = \{a\}$.

The growth function is $f(k) = 2k - 1$ at $k \geq 1$. When $\alpha = 1$, we have $\xi(k) = \frac{1}{k^2}$, $\xi(0) = 1$. Consider the elements $u_n = \sum\limits_{k=1}^{n} a^k$. $\|u_k\|_{\sup} = 1$. Whereas the image norm generated by the function ξ is

$$\|d_z^\tau u_k\| = \left\| \sum_{k=1}^{n} k a^k \right\| = \sum_{k=1}^{n} \frac{k}{k^2} \to \infty. \tag{3.6}$$

Thus, the central derivation can be unbounded when $\alpha = 1$.

4 Dependence of continuity on the choice of generating set

Let $\mathcal{X}$, $\mathcal{Y}$ be two different generating sets of group G, $|\mathcal{X}| = m_0$, $|\mathcal{Y}| = n_0$. Let us show that in the general case the identity operator between spaces with norms induced by $\mathcal{X}$ and $\mathcal{Y}$ may not be continuous. So, in this regard, the continuity of operators is not always well defined in the following sense. The norms considered above depend on a generating set $\mathcal{X}$. So the continuity of operators can depend on our choice of $\mathcal{X}$.

Example 5. The identity operator $\mathrm{id}\colon \mathbb{C}[F_2]_{\mathcal{X}} \to \mathbb{C}[F_2]_{\mathcal{Y}}$ is not continuous.

Proof. Consider, for example, a free group $F_2 = \langle a, b\rangle$. Consider the following generating sets: $\mathcal{X} = \{x_1 = a, x_2 = b\}$, $\mathcal{Y} = \{y_1 = ab^{-1}, y_2 = b\}$. There is no relations between y_1, y_2 because ($a \mapsto ab^{-1}$, $b \mapsto b$) provides the automorphism of the free group. It is easy to verify that ($a \mapsto ab, b \mapsto b$) defines the inverse map.

Denote by $\mathrm{id}\colon \mathbb{C}[F_2]_{\mathcal{X}} \to \mathbb{C}[F_2]_{\mathcal{Y}}$ the identity operator. Consider elements of the form a^n. We have

$$\frac{\|a^n\|_{\mathcal{X}}}{\|a^n\|_{\mathcal{Y}}} = \frac{\|x_1^n\|_{\mathcal{X}}}{\|(y_1 y_2)^n\|_{\mathcal{Y}}} = \frac{e^{-n}}{e^{-2n}} = e^n \to \infty. \tag{4.1}$$

That is, in general, the continuity of the operators may depend on the choice of the generating set.

However, the mapping induced by ($x_1 \mapsto y_1, x_2 \mapsto y_2$) is an automorphism as well as an isometry. Indeed, in $\mathbb{C}[F_2]_{\mathcal{X}}$ we look at the length of elements expressing them through x_i, in $\mathbb{C}[F_2]_{\mathcal{Y}}$ through y_i, and this mapping changes x_i to y_i in each word. □

Proposition 4.1. *For a group with the norm induced by a polynomial function (as in Theorem* 3.1*) it is true that* $\mathrm{id}\colon \mathbb{C}[G]_{\mathcal{X}} \to \mathbb{C}[G]_{\mathcal{Y}}$ *is continuous.*

Proof. Denote by $w(x_1, \dots, x_n)$ a word in the alphabet $\mathcal{X}$. Let us express all the elements of $\mathcal{Y}$ through the elements of $\mathcal{X}$. We have

$$\begin{aligned} y_1 &= w_1(x_1, \dots, x_n) \\ &\dots \\ y_m &= w_m(x_1, \dots, x_n) \end{aligned} . \tag{4.2}$$

Denote $C_1 = \max(|w_i|)$. Then $\forall g \in G$ we get $|g|_{\mathcal{Y}} \leq C_1 |g|_{\mathcal{X}}$, and similarly there exists C_2 such that $|g|_{\mathcal{X}} \leq C_2 |g|_{\mathcal{Y}}$. We can assume that C_1, $C_2 > 1$. Then for an element $u = \sum a_g g$ we have

$$\frac{\|u\|_{\mathcal{Y}}}{\|u\|_{\mathcal{X}}} = \frac{\sum_{g\in G} |a_g| \xi(|g|_{\mathcal{Y}})}{\sum_{g\in G} |a_g| \xi(|g|_{\mathcal{X}})} \leq \frac{\sum_{g\in G} |a_g| \xi(|g|_{\mathcal{X}}/C_2)}{\sum_{g\in G} |a_g| \xi(|g|_{\mathcal{X}})}. \tag{4.3}$$

Since norm is induced by the function of the form $\xi(k) = \frac{1}{k^\alpha}$, $\xi(0) = 1$, where $\alpha > 1$, we get

$$\frac{\sum_{g\in G} |a_g|\xi(|g|_{\mathcal{X}}/C_2)}{\sum_{g\in G} |a_g|\xi(|g|_{\mathcal{X}})} \leq \frac{C_2^\alpha \sum_{g\neq e} |a_g|\xi(|g|_{\mathcal{X}}) + |a_e|}{\sum_{g\neq e} |a_g|\xi(|g|_{\mathcal{X}}) + |a_e|} \leq C_2^\alpha. \tag{4.4}$$

Thus we obtain that the norm of the identity mapping is finite. □

Corollary 4.1. *Let X be a normed space. The continuity of any linear operator with values in the group algebra $A\colon X \to \mathbb{C}[G]_\xi$, where the norm on the group algebra is induced by the function $\xi(k) = \frac{1}{k^\alpha}$, $\xi(0) = 1$ and a generating system $\mathcal{X}$, does not depend on the choice of $\mathcal{X}$.*

Proof. Let X be an arbitrary normed space, $A_{\mathcal{X}}\colon X \to \mathbb{C}[G]_{\mathcal{X}}$, $A_{\mathcal{Y}}\colon X \to \mathbb{C}[G]_{\mathcal{Y}}$ be the same maps as maps between the sets, but with values in group algebra with different norms. We claim that the first operator is continuous iff the second operator is continuous.

Let $A_{\mathcal{Y}}$ be continuous. According to the theorem $I_{\mathcal{Y}\to\mathcal{X}}\colon \mathbb{C}[G]_{\mathcal{Y}} \to \mathbb{C}[G]_{\mathcal{X}}$ is continuous. Since $A_{\mathcal{X}} = A_{\mathcal{Y}} \circ I_{\mathcal{Y}\to\mathcal{X}}$, we obtain the continuity of $A_{\mathcal{X}}$ as a composition of continuous operators. □

Example 6. In general, the continuity of the identity operator $\mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_{\alpha,\exp}$ depends on a generating set inducing the norm in $\mathbb{C}[G]_{\alpha,\exp}$.

Proof. Consider the free group $F_2 = \langle a, b\rangle$. The growth function of a free group of rank $n \geq 2$ with respect to a freely generating set is

$$k \mapsto 1 + \frac{n}{n-1} \cdot \left((2\cdot n - 1)^k - 1\right). \tag{4.5}$$

We have the following below and above estimates

$$n^k \leq 1 + \frac{n}{n-1} \cdot \left((2\cdot n - 1)^k - 1\right) \leq 2(2n)^k. \tag{4.6}$$

Consider the norm $\|g\| = e^{-\ln(2n+1)|g|} = e^{-\ln(5)|g|}$ with respect to $\mathcal{X}$. In this norm the operator $L\colon g \mapsto |g|\, g$ is continuous, as we saw in Theorem 2.1, hence, the identity operator is continuous.

It is known that in a free group of rank 2 one can find a subgroup of infinite rank ([11], p. 29), hence, subgroups of an arbitrarily large finite rank as well. Let $F_6 \subset F_2$ be a free group of rank 6. Let F_6 be freely generated by $y_1 = w_1(a, b), \dots y_6 = w_6(a, b)$.

Now consider the following generating set $\mathcal{Y} = \{a, b, y_1, \dots y_6\}$ for group F_2. Furthermore among the words of length no more than k, there are all words of subgroup F_6 of length no more than k. That is, the growth rate of the group $f_G^{\mathcal{X}}(k) \geq 6^k$. Actually one can see that even the number of elements of length exactly k is not less than 6^k.

Consider the next element of group algebra $\sum\limits_{|g|=1} g$. Its supremum norm is 1. The norm of the image $\|g\|_{\alpha,\exp} \geq 6^k e^{-\ln(5)k} \to \infty$ as k increases. That is, the identity operator is not continuous in this case.

Statement 4.1. *For any norm of form $\|g\| = e^{-\alpha k}$ we can choose a generating set in F_2 such that the identity operator will be unbounded.*

It suffices to find a free subgroup of rank greater than α. After that the statement can be proved in the same fashion. □

Example 7. In general, the continuity of central derivation $d_z^\tau\colon \mathbb{C}[G]_{\sup} \to \mathbb{C}[G]_{\alpha,\exp}$ depends on a generating set inducing the norm in $\mathbb{C}[G]_{\alpha,\exp}$.

Proof. Consider F_2 with respect to free generating set $\mathcal{X}$. Take $z = e$. We define the homomorphism τ by putting it equal to 1 on all generators. Also, consider a generating set $\mathcal{Y}$ in which we added all words from G that have a length no more than 7 with respect to $\mathcal{X}$.

Let us show that if we choose constant $\alpha = \ln 5$ the central derivation is continuous in the norm induced by $\mathcal{X}$, but not by $\mathcal{Y}$.

Since $\ln 5 > \alpha$, continuity for $\mathcal{X}$ follows by Theorem 1. Let us show that with respect to $\mathcal{Y}$ the central derivation is not bounded. Denote by S^+_{7k} the set of elements $g \in G$ of length $|g|_{\mathcal{X}} = 7k$ such that they can be written in the shortest form without using inverse elements to generators. Note that for $g \in S^+_{7k}$ the length $|g|_{\mathcal{Y}} \leq k$.

Consider elements with unit supremum norm of the form $u_k = \sum_{g \in S^+_{7k}} g$. We have the following estimate of the norm of the image u_k with respect to $\mathcal{Y}$

$$\|u_k\|_{\mathcal{Y}} = \sum_{g \in S^+_{7k}} \tau(g) \, \|g\|_{\mathcal{Y}} = \sum_{g \in S^+_{7k}} |g|_{\mathcal{X}} \, \|g\|_{\mathcal{Y}} \geq 7k2^{7k} e^{-\ln(5)k} \to \infty. \tag{4.7}$$

In the last inequality, we use that the number of elements in S^+_{7k} is 2^{7k}. Thus, it is shown that d^τ_z is not continuous in the norm induced by $\mathcal{Y}$. □

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