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INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN MORREY SPACES

V.I. Burenkov, D.J. Joseph

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Key words: entire functions of exponential types, Morrey spaces, Bernstein's inequality, inequalities of different metrics and of different dimensions.

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Abstract. A detailed exposition of Bernstein's inequality, inequalities of different metrics and of different dimensions for entire functions of exponential type in Lebesgue spaces is given in the book of S.M. Nikol'skii [8]. In this paper, we state analogues of these inequalities in the Morrey spaces.

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1 Introduction

Definition 1. Let $\nu > 0$. A function $g : \mathbb{C}^n \to \mathbb{C}$ is called an entire function of exponential type ν, if the following properties hold:

1) it expands into a power series for any $z \in \mathbb{C}^n$

$$g(z) = \sum_{k \in \mathbb{N}_0^n} a_k z^k \equiv \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n}, \tag{1.1}$$

2) $\forall \varepsilon > 0 \quad \exists A_\varepsilon > 0$ such that for all $z \in \mathbb{C}^n$ the following inequality holds:

$$|g(z)| \le A_\varepsilon e^{(\nu+\varepsilon)(|z_1|+|z_2|+\dots+|z_n|)}. \tag{1.2}$$

Denote by $E_\nu(\mathbb{C}^n)$ the set of all entire functions of exponential type ν and let $E_\nu(\mathbb{R}^n)$ be the set of all functions g, defined on $\mathbb{R}^n$, for each of which $g(x) = G(x+iy)|_{y=0}$, $x \in \mathbb{R}^n$, for some function $G \in E_\nu(\mathbb{C}^n)$. In what follows, we always assume that $\nu > 0$, without specifying this in each statement.

Let $1 \le p \le \infty$ and

$$\mathfrak{M}_{\nu,p}(\mathbb{R}^n) = E_\nu(\mathbb{R}^n) \cap L_p(\mathbb{R}^n). \tag{1.3}$$

In book [8] the following inequalities are proved for entire functions of exponential type $g \in \mathfrak{M}_{\nu,p}(\mathbb{R}^n)$.

1. (Bernstein's inequality) Let $1 \le p \le \infty$, then for any function $g \in \mathfrak{M}_{\nu p}(\mathbb{R}^n)$

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{L_p(\mathbb{R}^n)} \le \nu \|g\|_{L_p(\mathbb{R}^n)}, \quad j = 1, \ldots, n. \tag{1.4}$$

2. (Inequality of different metrics) Let $1 \le p < q \le \infty$, then for any function $g \in \mathfrak{M}_{\nu p}(\mathbb{R}^n)$

$$\|g\|_{L_q(\mathbb{R}^n)} \le 2^n \nu^{n(\frac{1}{p} - \frac{1}{q})} \|g\|_{L_p(\mathbb{R}^n)}. \tag{1.5}$$

3. (Inequality of different dimensions) Let $1 \le p \le \infty$, $1 \le m < n$, $x = (u, v)$, $u = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $v = (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m}$, then for any function $g \in \mathfrak{M}_{\nu p}(\mathbb{R}^n)$

$$\left\| \|g(u,v)\|_{L_{\infty,v}(\mathbb{R}^{n-m})} \right\|_{L_{p,u}(\mathbb{R}^m)} \le 2^{n-m} \nu^{\frac{n-m}{p}} \|g\|_{L_p(\mathbb{R}^n)}, \tag{1.6}$$

in particular,

$$\|g(u,0)\|_{L_p(\mathbb{R}^m)} \le 2^{n-m} \nu^{\frac{n-m}{p}} \|g\|_{L_p(\mathbb{R}^n)}. \tag{1.7}$$

These inequalities play an important role in developing the theory of function spaces with fractional order of smoothness basing on the approximation theory described in detail in [8]. The aim of this paper is to state similar inequalities for the case in which the space $L_p(\mathbb{R}^n)$ is replaced by the Morrey space $M_p^\lambda(\mathbb{R}^n)$.

2 Morrey spaces $M_p^\lambda(\mathbb{R}^n)$

Definition 2. Let $0 < p \le \infty$ and $0 \le \lambda \le \frac{n}{p}$, then $f \in M_p^\lambda(\mathbb{R}^n)$, if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty. \tag{2.1}$$

We note some properties of these spaces.

1. It is immediately clear from the definition that for $\lambda = 0$

$$\|f\|_{M_p^0(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)}.$$

2. For $\lambda = \frac{n}{p}$

$$\|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} = v_n^{\frac{n}{p}} \|f\|_{L_\infty(\mathbb{R}^n)},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$.

3. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then the spaces $M_p^\lambda(\mathbb{R}^n)$ consist only of functions equivalent to 0 on $\mathbb{R}^n$.

4. For any $\varepsilon > 0$

$$\|f(\varepsilon x)\|_{M_p^\lambda(\mathbb{R}^n)} = \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{M_p^\lambda(\mathbb{R}^n)}. \tag{2.2}$$

5. Let $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta(x) = 1$ for any $x \in B(0,1), 0 < p < \infty$, $0 < \lambda < \frac{n}{p}, \mu \in \mathbb{R}$. Then

$$|x|^\mu \eta(x) \in L_p(\mathbb{R}^n) \Leftrightarrow \mu > -\frac{n}{p},$$

$$|x|^\mu \eta(x) \in M_p^\lambda(\mathbb{R}^n) \Leftrightarrow \mu \ge \lambda - \frac{n}{p}$$

and

$$|x|^\mu(1-\eta(x)) \in L_p(\mathbb{R}^n) \Leftrightarrow \mu < -\frac{n}{p},$$

$$|x|^\mu(1-\eta(x)) \in M_p^\lambda(\mathbb{R}^n) \Leftrightarrow \mu \le \lambda - \frac{n}{p},$$

$$|x|^\mu \in M_p^\lambda(\mathbb{R}^n) \Leftrightarrow \mu = \lambda - \frac{n}{p}.$$

This implies, in particular, that $L_p(\mathbb{R}^n) \not\subset M_p^\lambda(\mathbb{R}^n)$ and also $M_p^\lambda(\mathbb{R}^n) \not\subset L_p(\mathbb{R}^n)$. In this connection it is useful to consider the spaces $\widehat{M}_p^\lambda(\mathbb{R}^n) = L_p(\mathbb{R}^n) \cap M_p^\lambda(\mathbb{R}^n)$ with the quasinorm

$$\|f\|_{\widehat{M}_p^\lambda(\mathbb{R}^n)} = \max\{\|f\|_{L_p(\mathbb{R}^n)}, \|f\|_{M_p^\lambda(\mathbb{R}^n)}\}. \tag{2.3}$$

For these spaces

$$|x|^\mu\eta(x) \in \widehat{M}_p^\lambda(\mathbb{R}^n) \Leftrightarrow \mu \ge \lambda - \frac{n}{p},$$

$$|x|^\mu(1-\eta(x)) \in \widehat{M}_p^\lambda(\mathbb{R}^n) \Leftrightarrow \mu < -\frac{n}{p}.$$

Note that the space $\widehat{M}_p^\lambda(\mathbb{R}^n)$ (in contrast to the space $M_p^\lambda(\mathbb{R}^n)$) has the monotonicity property with respect to the parameter λ:

$$\widehat{M}_p^\mu(\mathbb{R}^n) \subset \widehat{M}_p^\lambda(\mathbb{R}^n),\ 0 < \lambda < \mu < \infty.$$

Moreover, $0 < p < \infty$.

$$\|f\|_{\widehat{M}_p^\lambda(\mathbb{R}^n)} \le \|f\|_{\widehat{M}_p^\mu(\mathbb{R}^n)}. \tag{2.4}$$

6. Invariance with respect to translation:

$$\|f(y+h)\|_{M_{p,y}^\lambda(\mathbb{R}^n)} = \|f(y)\|_{M_p^\lambda(\mathbb{R}^n)} \quad \forall h \in \mathbb{R}^n. \tag{2.5}$$

According to (2.3) and (2.5) also

$$\|f(y+h)\|_{\widehat{M}_{p,y}^\lambda(\mathbb{R}^n)} = \|f(y)\|_{\widehat{M}_p^\lambda(\mathbb{R}^n)} \quad \forall h \in \mathbb{R}^n. \tag{2.6}$$

For further properties of the Morrey spaces, their generalizations and applications see survey papers [2], [3], [6], [7], [9], [10], [11] and references threin.

3 Bernstein's inequality for Morrey spaces

In the one-dimensional case, the interpolation formula for the derivative of an entire function g of exponential type $\nu > 0$ has the form

$$g'(x) = \frac{\nu}{\pi^2}\sum_{-\infty}^{\infty}\frac{(-1)^{k-1}}{(k-\frac{1}{2})^2}g\Big(x+\frac{\pi}{\nu}(k-\frac{1}{2})\Big), \quad x \in \mathbb{R}, \tag{3.1}$$

where the series converges uniformly (see, for example, book [8]).

Theorem 3.1. *Let $Z(\mathbb{R}^n)$ be a normed space of functions defined on $\mathbb{R}^n$ and the norm $\|\cdot\|_{Z(\mathbb{R}^n)}$ be invariant with respect to translation, that is for any function $f \in Z(\mathbb{R}^n)$*

$$\|f(\cdot + h)\|_{Z(\mathbb{R}^n)} = \|f\|_{Z(\mathbb{R}^n)} \quad \forall h \in \mathbb{R}^n. \tag{3.2}$$

Then for any function $g \in E_\nu(\mathbb{R}^n) \cap Z(\mathbb{R}^n)$

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{Z(\mathbb{R}^n)} \le \nu \|g(x)\|_{Z(\mathbb{R}^n)}, \; j = 1, \dots, n. \tag{3.3}$$

The proof is based on representation (3.1).

Corollary 3.1. *Let $1 \le p \le \infty$, $0 \le \lambda \le \frac{n}{p}$, then for any function $g \in E_\nu(\mathbb{R}^n) \cap M_p^\lambda(\mathbb{R}^n)$*

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{M_p^\lambda(\mathbb{R}^n)} \le \nu \|g\|_{M_p^\lambda(\mathbb{R}^n)}, \quad j = 1, \dots, n. \tag{3.4}$$

This inequality also holds if $M_p^\lambda(\mathbb{R}^n)$ is replaced by $\widehat{M}_p^\lambda(\mathbb{R}^n)$.

4 Inequality of different metrics for Morrey spaces

First of all, we give definitions and facts related to the theory of Fourier transforms necessary for what follows.

Definition 3. Let functions $f, g \in L_1(\mathbb{R}^n)$, then the convolution is the function $f * g : \mathbb{R}^n \to \mathbb{R}^n$, defined by the equality

$$(f * g)(t) = \int_{\mathbb{R}^n} f(t - \tau) g(\tau) d\tau, \; t \in \mathbb{R}^n. \tag{4.1}$$

Lemma 4.1. *Let $1 \le p \le q \le \infty$, $f \in L_{q'}(\mathbb{R}^n)$ and $g \in \mathfrak{M}_{\nu,p}(\mathbb{R}^n)$. Then for any $x, y \in \mathbb{R}^n$*

$$|(f * g)(x) - (f * g)(y)| \le M \|f\|_{L_{q'}(\mathbb{R}^n)} \|g\|_{L_p(\mathbb{R}^n)} |x - y|, \tag{4.2}$$

where $M = 2^n n \nu^{1 + \frac{1}{q} - \frac{1}{p}}$.

Definition 4. The Fourier transform of a function $f \in L_1(\mathbb{R}^n)$ is given by the following formula:

$$(Ff)(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx, \; \xi \in \mathbb{R}^n, \tag{4.3}$$

where $\xi \cdot x = \xi_1 x_1 + \dots + \xi_n x_n$.

Definition 5. If $f \in L_p(\mathbb{R}^n)$, where $1 < p \le 2$, then the Fourier transform is given by the equality

$$(Ff)(\xi) = \lim_{r \to \infty} \big(F(f \chi_{B(0,r)}) \big)(\xi) \text{ in } L_{p'}(\mathbb{R}^n), \tag{4.4}$$

where $p' = \frac{p}{p-1}$ $\big(\frac{1}{p} + \frac{1}{p'} = 1\big)$.

(Equality (4.4) also holds if $p = 1$ and Ff is defined by (4.3).) Moreover, $Ff \in L_{p'}(\mathbb{R}^n)$,

$$\|Ff\|_{L_2(\mathbb{R}^n)} = \|f\|_{L_2(\mathbb{R}^n)} \tag{4.5}$$

(Parseval's equality), for $1 \le p < 2$

$$\|Ff\|_{L_{p'}(\mathbb{R}^n)} = (2\pi)^{n(\frac{1}{2} - \frac{1}{p})} \left(\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}} \right)^{\frac{n}{2}} \|f\|_{L_p(\mathbb{R}^n)} \le (2\pi)^{n(\frac{1}{2} - \frac{1}{p})} \|f\|_{L_p(\mathbb{R}^n)} \tag{4.6}$$

(Hausdorff-Joung-Beckner inequality, the constant $(2\pi)^{n(\frac{1}{2} - \frac{1}{p})} \left(\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}} \right)^{\frac{n}{2}}$ is sharp).

Definition 6. If $f \in L_p(\mathbb{R}^n)$, where $1 \le p \le \infty$, then the Fourier transform $\mathcal{F}f$ is defined in the Schwartz space of tempered distributions $S'(\mathbb{R}^n)$ as a continuous linear functional on $S(\mathbb{R}^n)$, given by the equality

$$(\mathcal{F}f, \varphi) = (f, F\varphi) = \int_{\mathbb{R}^n} f(x)(F\varphi)(x)dx \quad \forall\varphi \in S(\mathbb{R}^n). \tag{4.7}$$

If $1 \le p \le 2$, then

$$(\mathcal{F}f, \varphi) = (f, F\varphi) = \int_{\mathbb{R}^n} (Ff)(\xi)\varphi(\xi)d\xi \quad \forall\varphi \in S(\mathbb{R}^n),$$

that is, $\mathcal{F}f$ is a regular distribution generated by the Fourier transform $Ff \in L_{p'}(\mathbb{R}^n)$, given by equality (4.3) for $p = 1$ and equality (4.4) for $1 < p \le 2$.

Theorem 4.1. *(Theorem of L. Schwartz, see, for example, book [8]) If $1 \le p \le \infty$, and $g \in \mathfrak{M}_{\nu,p}(\mathbb{R}^n)$ then the Fourier transform Fg, understood in the sense of the Schwartz space $S'(\mathbb{R}^n)$ of tempered distributions, is equal to zero outside the closure of the cube*

$$\Delta_\nu = \{|x_j| < \nu, \ j = 1, \dots, n\}. \tag{4.8}$$

Recall that for $\varphi, g \in L_1(\mathbb{R}^n)$

$$(F(\varphi * g))(\xi) = (2\pi)^{\frac{n}{2}}(F\varphi)(\xi)(Fg)(\xi), \ \xi \in \mathbb{R}^n. \tag{4.9}$$

It immediately follows from (4.9) that, if $(F\varphi)(\xi) = (2\pi)^{-\frac{n}{2}}$ for any $\xi \in$ *supp* Fg, then $F(\varphi * g) = Fg$ and

$$g(x) = (\varphi * g)(x) \tag{4.10}$$

for almost all $x \in \mathbb{R}^n$. If $\varphi \in L_1(\mathbb{R}^n, g \in \mathfrak{M}_{\nu,1}(\mathbb{R}^n)$ and $(F\varphi)(\xi) = (2\pi)^{-\frac{n}{2}}$ for any $\xi \in \Delta_\nu$, then both functions g and, according to Lemma 4.1 with $f = \varphi$, $p = 1$, $q = \infty$, the convolution $\varphi * g$ are continuous on $\mathbb{R}^n$, so inequality (4.10) holds for any $x \in \mathbb{R}^n$.

Lemma 4.2. *Let $1 \le p \le \infty$, $\varphi \in L_{p'}(\mathbb{R}^n)$, $g \in \mathfrak{M}_{\nu,p}(\mathbb{R}^n)$, and the Fourier transform $F\varphi$, understood in general in the sense of the Schwartz space $S'(\mathbb{R}^n)$ of tempered distributions, is equal to $(2\pi)^{-\frac{n}{2}}$ on Δ_μ for some $\mu > \nu$. Then equality (4.10) is holds for all $x \in \mathbb{R}^n$.*

Under other assumptions on the function φ this assertion was proved in book [8], Lemma 8.5.2 and in paper [5], Section 3, Lemma 1.

Theorem 4.2. *(Young-type inequality for Morrey spaces, see paper [4]) Let*

$$1 \le p \le q \le \infty, \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p},$$

$f_1 \in L_r(\mathbb{R}^n)$ and $f_2 \in \widehat{M}_p^\lambda(\mathbb{R}^n)$. Then

$$\|f_1 * f_2\|_{M_q^{\frac{p\lambda}{q}}(\mathbb{R}^n)} \le \|f_1\|_{L_r(\mathbb{R}^n)} \|f_2\|^{\frac{p}{q}}_{M_p^\lambda(\mathbb{R}^n)} \|f_2\|^{1-\frac{p}{q}}_{L_p(\mathbb{R}^n)}. \tag{4.11}$$

Theorem 4.3. *Let $1 \le p \le q \le \infty, 0 \le \lambda \le \frac{n}{p}$, then there exists $c > 0$, such that*

$$\|g\|_{M_q^{\frac{p\lambda}{q}}(\mathbb{R}^n)} \le c\nu^{n(\frac{1}{p}-\frac{1}{q})} \|g\|^{\frac{p}{q}}_{M_p^\lambda(\mathbb{R}^n)} \|g\|^{1-\frac{p}{q}}_{L_p(\mathbb{R}^n)} \tag{4.12}$$

for any $\nu > 0$ and $g \in E_\nu(\mathbb{R}^n) \cap \widehat{M}_p^\lambda(\mathbb{R}^n)$.

Corollary 4.1. *Under the assumptions of Theorem* 4.3 *there exists* $\hat{c} > 0$*, such that*

$$\|g\|_{\widehat{M}_q^{\frac{p\lambda}{q}}(\mathbb{R}^n)} \leq \hat{c}\nu^{n(\frac{1}{p}-\frac{1}{q})}\|g\|_{\widehat{M}_p^{\lambda}(\mathbb{R}^n)} \tag{4.13}$$

for any $\nu > 0$ *and* $g \in E_\nu(\mathbb{R}^n) \cap \widehat{M}_p^{\lambda}(\mathbb{R}^n)$.

Remark 1. (Unimprovability of the inequality of different metrics in $M_p^\lambda(\mathbb{R}^n)$.) Suppose that for some $\mu \geq 0$ and $c > 0$, for any $\nu > 0$ and $g \in E_\nu(\mathbb{R}^n) \cap \widehat{M}_p^{\lambda}(\mathbb{R}^n)$ the following inequality holds:

$$\|g\|_{M_q^\mu(\mathbb{R}^n)} \leq c\nu^{n(\frac{1}{p}-\frac{1}{q})}\|g\|^{\frac{p}{q}}_{M_p^\lambda(\mathbb{R}^n)}\|g\|^{1-\frac{p}{q}}_{L_p(\mathbb{R}^n)}.$$

Then

$$\mu = \frac{\lambda p}{q}.$$

5 Inequality of different dimensions for Morrey spaces

Definition 7. Let

$$0 < p_1, p_2 \leq \infty, \quad m_1, m_2 \in \mathbb{N}$$

$$0 \leq \lambda_1 \leq \frac{m_1}{p_1}, \quad 0 \leq \lambda_2 \leq \frac{m_2}{p_2}.$$

Define the space

$$M_{p_1}^{\lambda_1}(\mathbb{R}^{m_1}) \times M_{p_2}^{\lambda_2}(\mathbb{R}^{m_2}) \tag{5.1}$$

with mixed quasinorm as the set of all measurable functions f on $\mathbb{R}^{m_1+m_2}$, for which

$$\|f\|_{M_{p_1}^{\lambda_1}(\mathbb{R}^{m_1})\times M_{p_2}^{\lambda_2}(\mathbb{R}^{m_2})} = \|\|f(u_1,u_2)\|_{M_{p_1,u_1}^{\lambda_1}(\mathbb{R}^{m_1})}\|_{M_{p_2,u_2}^{\lambda_2}(\mathbb{R}^{m_2})} < \infty. \tag{5.2}$$

We note some properties of these spaces.

Lemma 5.1. *Let* $0 < p \leq \infty$, $m_1, m_2 \in \mathbb{N}$, $0 \leq \lambda_1 \leq \frac{m_1}{p}$, $0 \leq \lambda_2 \leq \frac{m_2}{p}$, $f_1 \in M_p^{\lambda_1}(\mathbb{R}^{m_1})$ $f_2 \in M_p^{\lambda_2}(\mathbb{R}^{m_2})$. *Then* $f_1 f_2 \in M_p^{\lambda_1}(\mathbb{R}^{m_1}) \times M_p^{\lambda_2}(\mathbb{R}^{m_2})$ *and*

$$\|f_1f_2\|_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})} = \|f_1\|_{M_{p_1}^{\lambda_1}(\mathbb{R}^{m_1})}\|f_2\|_{M_{p_2}^{\lambda_2}(\mathbb{R}^{m_2})}.$$

Lemma 5.2. *Let* $0 < p \leq \infty$, $m_1, m_2 \in \mathbb{N}$, $0 \leq \lambda_1 \leq \frac{m_1}{p}$, $0 \leq \lambda_2 \leq \frac{m_2}{p}$. *Then*

$$M_p^{\lambda_1}(\mathbb{R}^{m_1}) \times M_p^{\lambda_2}(\mathbb{R}^{m_2}) \subset M_p^{\lambda_1+\lambda_2}(\mathbb{R}^{m_1+m_2}), \tag{5.3}$$

and

$$\|f\|_{M_p^{\lambda_1+\lambda_2}(\mathbb{R}^{m_1+m_2})} \leq \|f\|_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})}$$

for any $f \in M_p^{\lambda_1}(\mathbb{R}^{m_1}) \times M_p^{\lambda_2}(\mathbb{R}^{m_2})$.

If $0 < \lambda_1 + \lambda_2 < \frac{m_1+m_2}{p}$ *, then inclusion* (5.3) *is strict.*

Using Definition 7 with $\lambda_1 = \lambda_2 = 0$, inequality (1.6) can be rewritten as

$$\|g\|_{L_\infty(\mathbb{R}^{n-m})\times L_p(\mathbb{R}^m)} \leq 2^{n-m}\nu^{\frac{n-m}{p}}\|g\|_{L_p(\mathbb{R}^n)}. \tag{5.4}$$

Theorem 5.1. *Let* $1 \le p < \infty,\ m, n \in \mathbb{N},\ m < n,\ 0 \le \lambda \le \frac{n}{p}$, *then*

$$\|g\|_{L_\infty(\mathbb{R}^{n-m}) \times M_p^\lambda(\mathbb{R}^m)} \le 2^{n-m} \nu^{\frac{n-m}{p}} \|g\|_{L_p(\mathbb{R}^n) \times M_p^\lambda(\mathbb{R}^m)}, \tag{5.5}$$

in particular, if $x = (u, v),\ u = (x_1 \dots x_m),\ v = (x_{m+1}, \dots, x_n)$, *then*

$$\|g(u, 0)\|_{M_p^\lambda(\mathbb{R}^m)} \le 2^{n-m} \nu^{\frac{n-m}{p}} \|g\|_{L_p(\mathbb{R}^{n-m}) \times M_p^\lambda(\mathbb{R}^m)}. \tag{5.6}$$

Inequalities (5.5) *and* (5.6) *also hold if the space* $M_p^\lambda(\mathbb{R}^m)$ *is replaced by the space* $\hat{M}_p^\lambda(\mathbb{R}^m)$.

Remark 2. If $\lambda = 0$ then it is obvious that

$$L_p(\mathbb{R}^{n-m}) \times M_p^0(\mathbb{R}^m) = L_p(\mathbb{R}^{n-m}) \times L_p(\mathbb{R}^m) = L_p(\mathbb{R}^n) = M_p^0(\mathbb{R}^n), \tag{5.7}$$

however, for $0 < \lambda \le \frac{m}{p}$ according to Lemma 5.2

$$L_p(\mathbb{R}^{n-m}) \times M_p^\lambda(\mathbb{R}^m) \ne M_p^\lambda(\mathbb{R}^n). \tag{5.8}$$

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