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ASYMPTOTICS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR THE EQUATION $\varepsilon y'' + xp(x)y' - q(x)y = f$

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Key words: asymptotic solution, Dirichlet boundary value problem, Neumann boundary value problem, Robin boundary-value problem, bisingularly perturbed problem, small parameter, regularly singular point.

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Abstract. Uniform asymptotic expansions of solutions of two-point boundary value problems of Dirichlet, Neumann and Robin for a linear inhomogeneous ordinary differential equation of the second order with a small parameter at the highest derivative are constructed. A feature of the considered two-point boundary value problems is that the corresponding unperturbed boundary value problems for an ordinary differential equation of the first order has a regularly singular point at the left end of the segment. Asymptotic solutions of boundary value problems are constructed by the modified Vishik-Lyusternik-Vasilyeva method of boundary functions. Asymptotic expansions of solutions of two-point boundary value problems are substantiated. We propose a simpler algorithm for constructing an asymptotic solution of bisingular boundary value problems with regular singular points, and our boundary functions constructed in a neighborhood of a regular singular point have the property of "boundary layer", that is, they disappear outside the boundary layer.

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1 Introduction

As we know, the mathematical models of many problems in science and technology are ordinary differential equations with small parameters in the highest derivatives [1], [4], [5], [7]-[18]. For example, the equation

$$\varepsilon y''_\varepsilon(x) + p(x)y'_\varepsilon(x) + q(x)y_\varepsilon(x) = f(x), \ , \ x \in (0, 1),$$

describes two closely related processes. One is the stationary distribution of heat in a moving medium, depending on one variable x. The small parameter ε is the low thermal conductivity, and the coefficient $p(x)$ is related to the velocity of the medium. Another interpretation is related to the random walk of a particle on a segment, provided that $p(x)$ determines the average speed of movement, and the parameter ε is a small variance [6].

Today, differential equations with small parameters at the highest derivative or singularly perturbed differential equations already constitute an independent area of mathematics. Singularly perturbed differential equations are of great applied interest, as evidenced by the increase in publications on this topic [1] - [22].

The results obtained can find applications in continuum mechanics, hydro- and aerodynamics, magneto-hydrodynamics, oceanology, etc.

2 Formulation of the problem

Let us investigate the boundary value problems generated by the equation

$$\varepsilon\, y_\varepsilon''(x) + xp(x)y_\varepsilon'(x) - q(x)y_\varepsilon(x) = f(x), \qquad 0 < x < 1, \tag{2.1}$$

and one of the boundary conditions of the form

$$y_\varepsilon(0) = a, \;\; y_\varepsilon(1) = b, \tag{2.2}$$

$$y_\varepsilon'(0) = a, \;\; y_\varepsilon'(1) = b, \tag{2.3}$$

$$y_\varepsilon(0) - h_1 y_\varepsilon'(0) = a, \;\;\; y_\varepsilon(1) + h_2 y_\varepsilon'(1) = b, \tag{2.4}$$

here $0 < \varepsilon << 1$, a, b, $0 < h_1$, $0 < h_2$ are given constants, $p(0) = 1, q(0) = 2$, p, q, $f \in C^\infty[0,1]$, $f''(0) \neq 0$, $0 < p, q$. $y_\varepsilon(x)$ is the required function depending on the small parameter ε and on the independent variable x.

The two-point boundary value problem (2.1), (2.2) is called the Dirichlet problem, (2.1), (2.3) is the Neumann problem and (2.1), (2.4) is the Robin problem.

Solutions to boundary value problems exist, are unique and bounded [6].

The specific features of the considered boundary value problems are: the presence of a small parameter ε at $y_\varepsilon''(x)$ and the corresponding unperturbed equation

$$xp(x)y_0'(x) - q(x)y_0(x) = f(x), \tag{2.5}$$

which has a regular singular point at $x = 0$ (see [2]).

Earlier, in [6], a similar equation with Dirichlet boundary conditions was investigated by the matching method. In [3], the Dirichlet problem for equation (2.1) is investigated by the method of structural matching. In these works, the asymptotic solution consists of three or four components and their construction and justification, in our opinion, is relatively complicated. In paper [2] the Dirichlet problem for differential equation (2.1) was investigated by the generalized method of boundary functions. However, the functions $\pi_k(t)$, $k = 0, 1, 2, \ldots$ constructed in the neighborhood of the regular singular point $x = 0$, are bounded, but do not possess the "layerwise" property, that is, they do not disappear outside the boundary layer, which is essential in the theory of the boundary layer. We propose a simpler algorithm for constructing an asymptotic solution of boundary value problems (2.1) - (2.4), which consists of two composite functions and our boundary functions $\pi_k(t)$, $k = 0, 1, 2, \ldots$, constructed in the vicinity of the point $x = 0$, have the property of "boundary layer", that is, they have a power-type decay outside the boundary layer.

3 Main result

First, we prove an auxiliary lemma.

Lemma 3.1. *Problems generated by equation* (2.5) *and one of the boundary conditions of the form*

$$y_0(1) = b, \tag{3.1}$$

$$y_0'(1) = b, \tag{3.2}$$

$$y_0(1) + h_2 y_0'(1) = b, \tag{3.3}$$

have unique solutions representable in the form, respectively

$$y_{0,i}(x) = e^{\int_1^x \frac{q(s)}{sp(s)}ds}\left(c_i + \int_1^x \frac{f(\tau)}{\tau p(\tau)} e^{-\int_1^\tau \frac{q(s)}{sp(s)}ds} d\tau\right), \quad i = 1, 2, 3; \tag{3.4}$$

where $c_1 = b$; $c_2 = \dfrac{p(1)bf(1)}{q(1)}$; $c_3 = \dfrac{p(1)bf(1)h_2}{p(1) + q(1)h_2}$.

Proof. First, we construct a general solution, for this we write equation (2.5) in the form:

$$y_0'(x) - \frac{q(x)}{xp(x)} y_0(x) = \frac{f(x)}{xp(x)} \Rightarrow \left(y_0(x)e^{-\int_1^x \frac{q(s)}{sp(s)}ds}\right)' = \frac{f(x)}{xp(x)} e^{-\int_1^x \frac{q(s)}{sp(s)}ds}.$$

After integration, we get the general solution:

$$y_0(x) = cE(x) + E(x)\int_1^x \frac{f(\tau)}{\tau p(\tau)} E^{-1}(\tau) d\tau,$$

where $E(x) = e^{\int_1^x \frac{q(s)}{sp(s)}ds}$, c is an arbitrary constant.

Taking into account boundary condition (3.1), we have: $c = b$.

Calculating the derivative of the general solution and taking into account boundary condition (3.2), we get:

$$c_2 = \frac{p(1)b - f(1)}{q(1)}.$$

From the general solution and from the derivative of the general solution, for $x = 1$, we have:

$$y_0(1) = c, \quad y_0'(1) = \frac{f(1)}{p(1)} + \frac{q(1)}{p(1)} y_0(1) = \frac{f(1)}{p(1)} + \frac{q(1)}{p(1)} c.$$

Taking into account condition (3.3), we get the following:

$$y_0(1) + h_2 y_0'(1) = c + h_2\left(\frac{f(1)}{p(1)} + \frac{q(1)}{p(1)} c\right) = b \Rightarrow$$

$$c\Big(p(1) + h_2 q(1)\Big) = p(1)bf(1)h_2 \Rightarrow c = \frac{p(1)b - f(1)h_2}{p(1) + q(1)h_2}.$$

□

Corollary 3.1. *The* (3.9) *solution can be represented as:*

$$y_0(x) = c_0 x^2 \ln x + Q(x),$$

where $Q \in C^\infty[0, 1]$, *and* c_0 *is a constant.*

Proof. By the conditions of the problem, the functions $q(x)$ and $p(x)$ can be represented as:

$$q(x) = 2 + q_1 x + q_2 x^2 + \ldots + q_n x^n + \ldots; \quad p(x) = 1 + p_1 x + p_2 x^2 + \ldots + p_n x^n + \ldots,$$

therefore, the following relations are valid:

$$\int_1^\tau \frac{q(s)}{sp(s)} ds = 2\ln\tau + Q_0(\tau),$$

and

$$e^{\int_1^\tau \frac{q(s)}{sp(s)}ds} = e^{2\ln\tau + Q_0(\tau)} = \tau^2 Q_1(\tau),$$

here $Q_0 \in C^\infty[0,1],\ Q_1(\tau) = e^{Q_0(\tau)} \in C^\infty[0,1]$.

Hence we have

$$e^{\int_1^x \frac{q(s)}{sp(s)}ds} \int_1^x \frac{f(\tau)}{\tau p(\tau)} e^{-\int_1^\tau \frac{q(s)}{sp(s)}ds} d\tau = x^2 Q_1(x) \int_1^x \frac{1}{\tau^3} Q_2(\tau) d\tau = c_0 x^2 \ln x + Q_3(x),$$

where $Q_i \in C^\infty[0,1]$.

Considering these properties of integrals, solution (3.4) can be written as

$$y_0(x) = c_0 x^2 \ln x + Q(x).$$

□

It is easy to see that $y_0 \in C^1[0,1]$, but $y_0 \notin C^2[0,1]$.

In what follows, the series used in the article are asymptotic expansions of relevant functions.

External asymptotic solutions of boundary value problems (2.1), (3.14); (2.1), (3.2) and (2.1), (3.3) can be written as:

$$V_\varepsilon(x) = f_1 x^2 \ln x + \varepsilon(\ln x)\tilde{v}_1(x) + \varepsilon\Big(\frac{\varepsilon}{x^2}\Big)\tilde{v}_2(x) + \ldots + \varepsilon\left(\frac{\varepsilon}{x^2}\right)^n \tilde{v}_{n+1}(x) + \ldots,\ x \to 0,$$

here $\tilde{v}_k \in C^\infty[0,1],\ k \in N$.

This means that the studied boundary value problems of Dirichlet, Neumann and Robin are bisingular [6].

Formal asymptotic solutions of all boundary value problems of Dirichlet, Neumann and Robin will be sought in the form:

$$y_\varepsilon(x) = \sum_{k=0}^\infty \varepsilon^k w_k(x) + \sum_{k=0}^\infty \mu^k \pi_k(t), \tag{3.5}$$

where $\mu = \sqrt{\varepsilon},\ x = t\mu$.

We write differential equation (2.1) in the form

$$\varepsilon\, y''_\varepsilon(x) + xp(x)y'_\varepsilon(x) - q(x)y_\varepsilon(x) = f(x) - g_\varepsilon(x) + g_\varepsilon(x), \quad 0 \le x \le 1, \tag{3.6}$$

where $g_\varepsilon(x) = \sum_{k=1}^\infty \varepsilon^k g_k \ln x$, still unknown constants g_k are elaborated below.

Substituting (3.5) into (3.6) we get:

$$lw_0 \equiv xp(x)w'_0(x) - q(x)w_0(x) = f(x), \tag{3.7}$$

$$lw_k = g_k \ln x - w''_{k-1}(x),\ k \in N; \tag{3.8}$$

$$\sum_{k=0}^\infty \mu^k \Big(\pi''_k(t) + p(\mu t)t\pi'_k(t) - q(\mu t)\pi_k(t)\Big) = -\sum_{k=1}^\infty \mu^{2k} g_k \ln(\mu t). \tag{3.9}$$

We require the following conditions to be met, respectively:

1) in the case of Dirichlet boundary condition (2.2):

$$w_0(1) = b,\ \ w_k(1) = 0,\ k \in N. \tag{3.10}$$

$$\pi_0(0) = a - w_0(0);\ \pi_{2k-1}(0) = 0;\quad \pi_{2k}(0) = -w_k(0);\ \pi_{k-1}(\mu^{-1}) = 0,\ \ k \in N; \tag{3.11}$$

2) in the case of Neumann boundary condition (2.3):

$$w_0'(1) = b,\ \ w_k'(1) = 0,\ \ k \in N. \tag{3.12}$$

$$\pi_1'(0) = a - w_0'(0);\ \pi_{2k-2}'(0) = 0;\ \pi_{2k+1}'(0) = -w_k'(0);\ \pi_{k-1}'(\mu^{-1}) = 0,\ \ k \in N; \tag{3.13}$$

3) in the case of Robin boundary condition (2.4):

$$w_0(1) + h_2 w_0'(1) = b,\ \ w_k(1) + h_2 w_k'(1) = 0,\ \ k \in N. \tag{3.14}$$

$$\pi_0'(0) = 0,\ \ \pi_1'(0) = \frac{1}{h_1}(w_0(0) + \pi_0(0) - a) - w_0'(0),\quad \pi_{2k+1}'(0) = \frac{1}{h_1}(w_k(0) + \pi_{2k}(0)) - w_k'(0),$$
$$\pi_{2k}'(0) = \frac{1}{h_1}\pi_{2k-1}(0),\quad \pi_0'(\mu^{-1}) = 0,\ h_1\ pi_k'(\mu^{-1}) - \pi_{k-1}(\mu^{-1}) = 0,\ \ k \in N. \tag{3.15}$$

Based on the Lemma 3.1, solutions of boundary value problems (3.7), (3.10); (3.7), (3.12); (3.7), (3.14) exist and all of them can be represented as:

$$w_0(x) = \alpha_0 x^2 \ln x + \tilde{w}_0(x),$$

where $\tilde{w}_0 \in C^\infty[0, 1]$.

Calculate $w_0''(x)$:

$$w_0'(x) = 2\alpha_0 x \ln x + \alpha_0 x + \tilde{w}'(x),\quad w_0''(x) = 2\alpha_0 \ln x + 3\alpha_0 + \tilde{w}''(x).$$

Taking into account Lemma 3.1, solutions of boundary value problems (3.8), (3.10); (3.2), (3.17) and (3.2), (3.24) can be written as:

$$w_1(x) = c_i\, E(x) + E(x) \int_1^x \frac{g_1 \ln \tau - w_0''(\tau)}{\tau p(\tau)} E^{-1}(\tau) d\tau,$$

where $E(x) = e^{\int_1^x \frac{q(s)}{sp(s)} ds}$, $c_1 = 0$; $c_2 = -\dfrac{f(1)}{q(1)}$; $c_3 = -\dfrac{f(1)h_2}{p(1) + h_2 q(1)}$.

Let $g_1 = 2\alpha_0$, then

$$w_1(x) = \alpha_1 x^2 \ln x + \tilde{w}_1(x),$$

where $\tilde{w}_1 \in C^\infty[0, 1]$.

Similarly, for $g_2 = 2\alpha_1$, we have

$$w_2(x) = \alpha_2 x^2 \ln x + \tilde{w}_2(x),$$

where $\tilde{w}_2 \in C^\infty[0, 1]$.

Continuing this process, we sequentially determine $w_k(x)$, for $g_k(x) = 2\alpha_{k-1}$:

$$w_k(x) = \alpha_k x^2 \ln x + \tilde{w}_k(x),$$

where $\tilde{w}_k \in C^\infty[0,1]$.

Thus, we have defined all functions $w_k(x)$ and $g_k(x)$.

The function $g_\varepsilon(x)$ can be represented as follows:

$$g_\varepsilon(x) = \left(\sum_{k=1}^{\infty} \varepsilon^k g_k\right) \ln x \ \Rightarrow \ g_\mu(\mu t) = \left(\sum_{k=1}^{\infty} \varepsilon^k g_k\right) \ln(\mu t).$$

We write differential equation (3.9) in the form

$$\sum_{k=0}^{\infty} \mu^k \left(\pi_k''(t) + t\pi_k'(t) - 2\pi_k(t) + \mu t^2 \tilde{p}(\mu t)\pi_k'(t) - \mu t \tilde{q}(\mu t)\pi_k(t)\right) = -\sum_{k=1}^{\infty} \mu^{2k} g_k \ln(\mu t),$$

where $p(x) = p(0) + x\tilde{p}(x)$, $q(x) = q(0) + x\tilde{q}(x)$, $\tilde{p}$, $\tilde{q} \in C^\infty[0,1]$.

Hence, we write down the following differential equations

$$L\pi_0 \equiv \pi_0''(t) + t\pi_0'(t) - 2\pi_0(t) = 0, \tag{3.16}$$

$$L\pi_1 = \Phi_0(\mu t, t), \tag{3.17}$$

$$L\pi_2 = \Phi_1(\mu t, t) - g_1 \ln(\mu t), \tag{3.18}$$

$$L\pi_{2k+1} = \Phi_{2k}(\mu t, t), \tag{3.19}$$

$$L\pi_{2k} = \Phi_{2k-1}(\mu t, t) - g_k \ln(\mu t), \tag{3.20}$$

where $\Phi_k(\mu t, t) = t\tilde{q}(\mu t)\pi_k(t) - t^2 \tilde{p}(\mu t)\pi_k'(t)$.

□

Lemma 3.2. *Boundary value problems generated by the equation*

$$Lz = r(t), \quad 0 < t < \mu^{-1}, \tag{3.21}$$

and one of the boundary conditions of form (3.22) *or* (3.23)*:*

$$z(0) = z^0, \ \ z(\mu^{-1}) = 0, \tag{3.22}$$

$$z'(0) = z^0, \ \ z'(\mu^{-1}) = 0, \tag{3.23}$$

have unique solutions, representable in the form, respectively:

$$z(t) = \frac{z_2(t)}{c} \int_0^t e^{t^2/2} z_1(s) r(s) ds + \frac{z_1(t)}{c} \int_t^{\mu^{-1}} e^{t^2/2} z_2(s) r(s) ds + z_2(t) \left(z^0 - \frac{1}{c} \int_0^{\mu^{-1}} e^{t^2/2} z_2(s) r(s) ds\right),$$

$$z(t) = \frac{z_2(t)}{c} \int_0^t e^{t^2/2} z_1(s) r(s) ds + \frac{z_1(t)}{c} \int_t^{\mu^{-1}} e^{t^2/2} z_2(s) r(s) ds + z_2(t) \left(\frac{z^0}{z_2'(0)} - \frac{1}{c} \int_0^{\mu^{-1}} e^{t^2/2} z_1(s) r(s) ds\right).$$

where $r \in C[0,\mu^{-1}], z_1(t)$ *and* $z_2(t)$ *is the fundamental system of solutions of the homogeneous equation* $Lz=0:$

$$z_1(t)=t^2+1,\ z_2(t)=-(t^2+1)c\int_t^{\mu^{-1}}\frac{1}{(s^2+1)^2}e^{-s^2/2}ds,$$

here $c=-\left(\int_0^{\mu^{-1}}\frac{1}{(s^2+1)^2}e^{-s^2/2}ds\right)^{-1}$, *if* $\mu\to 0$ *then* $c\to -\frac{\sqrt{2\pi}}{4}$.

Proof. The $z_2(t)$ function has the following properties:

a) $z_2(0)=1,\ z_2'(0)=c$;

b) $z_2(t)\sim t^{-3}e^{-t^2/2},\ t\to\infty,\ \mu\to 0$;

c) $z_2\in C^\infty[0,\mu^{-1}]$;

d) the function $z_2(t)$ decreases monotonically as $t\in[0,\mu^{-1}]$, $(z_2'(t)<0)$.

The proof of Lemma 3.2 is not difficult. Using the functions $z_1(t), z_2(t)$ and the Wronskian $W(z_1,\ z_2)=ce^{-t^2/2}$ one can construct a general solution to differential equation (3.21):

$$z(t)=z_2(t)\int\frac{z_1(t)r(t)}{W}dt-z_1(t)\int\frac{z_2(t)r(t)}{W}dt+c_1z_1(t)+c_2z_2(t),$$

where c_1 and c_2 are arbitrary constants.

Then, we select c_1 and c_2 so that either (3.22) or (3.23) are satisfied. □

Note that the homogeneous boundary value problem

$$L\pi_0\equiv\pi_0''(t)+t\pi_0'(t)-2\pi_0(t)=0,\ 0<t<\mu^{-1},\ \pi_0'(0)=0,\ \pi_0'(\mu^{-1})=0$$

has the only trivial solution $\pi_0(t)\equiv 0$.

Using Lemma 3.2, we prove the existence and uniqueness of the solution of the equations (3.16) - (3.20) with one of the boundary conditions (3.11), or (3.13), or (3.15).

Let us turn to the estimation of the remainder. Let

$$y_\varepsilon(x)=\sum_{k=0}^{n}\varepsilon^k w_k(x)+\sum_{k=0}^{2n+1}\mu^k\pi_k(t)+R_{n+1,\varepsilon}(x), \tag{3.24}$$

where $R_{n+1,\varepsilon}(x)$ is the remainder of the expansion.

Substituting (3.24) into (2.1) - (2.4) we obtain the differential equation for the residual functions

$$\varepsilon R_{n+1,\varepsilon}''(x)+xp(x)R_{n+1,\varepsilon}'(x)-q(x)R_{n+1,\varepsilon}(x)=G(x,t,\varepsilon),\quad 0\le x\le 1 \tag{3.25}$$

and the boundary conditions take the form:

$$R_{n+1,\varepsilon}(0)=0,\quad R_{n+1,\varepsilon}(1)=0, \tag{3.26}$$

$$R_{n+1,\varepsilon}'(0)=0,\ R_{n+1,\varepsilon}'(1)=0, \tag{3.27}$$

$$R_{n+1,\varepsilon}(0)-h_1R_{n+1,\varepsilon}'(0)=\varepsilon^{n+1/2}\pi_{2n+1}(0),\quad R_{n+1,\varepsilon}(1)+h_2R_{n+1,\varepsilon}'(1)=0, \tag{3.28}$$

where $G(x,t,\varepsilon)=\mu^{2n+1}\left(\mu t\tilde{q}(\mu t)\pi_{2n+1}(t)-\mu t^2\tilde{p}(\mu t)\pi_{2n+1}'(t)\right)-\varepsilon^{n+1}w_n''(x)$.

Considering the properties of the functions $\tilde{q}(\mu t)$, $\pi_{2n+1}(t)$, $\tilde{p}(\mu t)$, $\pi'_{2n+1}(t)$, $w''_n(x)$ we obtain asymptotic estimates:

$$G(x,t,\varepsilon) = O\left(\varepsilon^{n+1/2}\right),\ 0 \le x \le 1,\ 0 \le t \le 1/\sqrt{\varepsilon},$$

$$R_{n+1,\varepsilon}(0) - h_1 R'_{n+1,\varepsilon}(0) = O(\varepsilon^{n+1/2}),\ \ \varepsilon \to 0.$$

For each problem (3.25), (3.26); (3.25), (3.27) and (3.25), (3.28) applying theorems in [6, p. 116], respectively, we obtain an estimate for $R_{n+1,\varepsilon}(x)$:

$$R_{n+1,\varepsilon}(x) = O(\varepsilon^{n+1/2}), \quad \varepsilon \to 0, \quad 0 \le x \le 1.$$

We have proved the following statement.

Theorem. *Two-point boundary value problems* (2.1), (2.2); (2.1), (2.3) *and* (2.1), (2.4) *on the segment* $0 \le x \le 1$ *for* $\varepsilon \to 0$ *have the uniform asymptotic expansion*

$$y_\varepsilon(x) = \sum_{k=0}^{n} \varepsilon^k w_k(x) + \sum_{k=0}^{2n+1} \sqrt{\varepsilon}^k \pi_k(x\mu^{-1}) + O\left(\varepsilon^{n+1/2}\right),$$

where the functions $w_k(x)$ *and* $\pi_k(x\mu^{-1})$ *are defined above.*

Conclusion. Uniform asymptotic expansions of solutions of the two point boundary value problems of Dirichlet, Neumann and Robin for a linear inhomogeneous ordinary differential equation of the second order with a small parameter at the highest derivative are constructed on the segment $[0,1]$. A specific feature of the considered two-point boundary value problems is that the corresponding unperturbed boundary value problems for an ordinary differential equation of the first order has a regular singular point at the left end of the segment, that is, for $x = 0$. Asymptotic solutions of boundary value problems are constructed by the modified Vishik-Lyusternik-Vasilyeva method of boundary functions. Asymptotic expansions of solutions of two-point boundary value problems are substantiated. Note that earlier in the [6], a similar equation with Dirichlet boundary conditions was investigated by the matching method. In [3], the Dirichlet problem for equation (2.1) is investigated by the method of structural matching. In papers, [2] and [3], the asymptotic solution consists of four components and the construction, in our opinion, is too complicated, if we compare the algorithm for constructing the asymptotic solution with ours, our algorithm is simpler. Also in paper [2] the Dirichlet problem for differential equation (2.1) was investigated by the generalized method of boundary functions. However, the functions $\pi_k(t)$, $k = 0, 1, 2, \ldots$ constructed in the neighborhood of the regular singular point $x = 0$, are bounded, but do not possess the "layerwise" property, they do not disappear outside the boundary layer, which is essential in the theory of the boundary layer. We propose a simpler algorithm for constructing the asymptotic solution of boundary value problems (2.1) - (2.4), which consists of two composite functions and our boundary functions $\pi_k(t)$, $k = 0, 1, 2, \ldots$ constructed in the vicinity of $x = 0$, have the property of "boundary layer", that is, they have a power-type decay outside the boundary layer.

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