ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Editorial Board

Editors–in–Chief

V.I. Burenkov, M. Otelbaev, V.A. Sadovnichy

Vice–Editors–in–Chief

K.N. Ospanov, T.V. Tararykova

Editors

Sh.A. Alimov (Uzbekistan), H. Begehr (Germany), T. Bekjan (Kazakhstan), O.V. Besov (Russia), N.K. Bliev (Kazakhstan), N.A. Bokayev (Kazakhstan), A.A. Borubaev (Kyrgyzstan), G. Bourdaud (France), A. Caetano (Portugal), M. Carro (Spain), A.D.R. Choudary (Pakistan), V.N. Chubarikov (Russia), A.S. Dzumadildaev (Kazakhstan), V.M. Filippov (Russia), H. Ghazaryan (Armenia), M.L. Goldman (Russia), V. Goldshtein (Israel), V. Guliyev (Azerbaijan), D.D. Haroske (Germany), A. Hasanoglu (Turkey), M. Huxley (Great Britain), P. Jain (India), T.Sh. Kalmenov (Kazakhstan), B.E. Kangyzhin (Kazakhstan), K.K. Kenzhibaev (Kazakhstan), S.N. Kharin (Kazakhstan), E. Kissin (Great Britain), V.I. Korzyuk (Belarus), A. Kufner (Czech Republic), L.K. Kussainova (Kazakhstan), P.D. Lamberti (Italy), M. Lanza de Cristoforis (Italy), F. Lanzara (Italy), V.G. Maz'ya (Sweden), K.T. Mynbayev (Kazakhstan), E.D. Nursultanov (Kazakhstan), R. Oinarov (Kazakhstan), I.N. Parasidis (Greece), J. Pečarić (Croatia), S.A. Plaksa (Ukraine), L.-E. Persson (Sweden), E.L. Presman (Russia), M.A. Ragusa (Italy), M.D. Ramazanov (Russia), M. Reissig (Germany), M. Ruzhansky (Great Britain), M.A. Sadybekov (Kazakhstan), S. Sagitov (Sweden), T.O. Shaposhnikova (Sweden), A.A. Shkalikov (Russia), V.A. Skvortsov (Poland), G. Sinnamon (Canada), E.S. Smailov (Kazakhstan), V.D. Stepanov (Russia), Ya.T. Sultanaev (Russia), D. Suragan (Kazakhstan), I.A. Taimanov (Russia), J.A. Tussupov (Kazakhstan), U.U. Umirbaev (Kazakhstan), Z.D. Usmanov (Tajikistan), N. Vasilevski (Mexico), Dachun Yang (China), B.T. Zhumagulov (Kazakhstan)

Managing Editor

A.M. Temirkhanova

© The L.N. Gumilyov Eurasian National University

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 3 (2022), 67 – 81

ON ESTIMATES FOR NORMS OF SOME INTEGRAL OPERATORS WITH OINAROV'S KERNEL

K. Kuliev

Communicated by R. Oinarov

Key words: integral operator, norm, weight function, Lebesgue space, integral inequality, kernel.

AMS Mathematics Subject Classification: 25D15, 47A30, 47B01, 47B37.

Abstract. In this work, we give estimates for the norm of the integral operator

$$H : L_{p,v} \to L_{q,u}, \ (Hf)(x) := \int_a^x k(x,t)f(t)dt \tag{0.1}$$

with the so-called *Oinarov's kernel* $k(x,t)$ in the weighted Lebesgue spaces

$$L_{p,v} = \{f : \|f\|_{p,v}^p := \int_a^b |f(t)|^p v(t)dt < \infty\}$$

and

$$L_{q,u} = \{f : \|f\|_{q,u}^q := \int_a^b |f(t)|^q u(t)dt < \infty\},$$

in the case $1 < q < p < \infty$.

DOI: https://doi.org/10.32523/2077-9879-2022-13-3-67-81

1 Introduction

Let $(a,b) \subset R$ and u, v be weight functions in (a,b), i.e., positive measurable functions defined a.e. in (a,b). If $k(x,t) \equiv 1$ then V.G. Mazya and A.L. Rozin [7, Theorem 5, p. 47] proved that the condition

$$B = \left(\int_a^b \left(\int_a^t v^{1-p'}(\tau)d\tau\right)^{\frac{r}{q'}} \left(\int_t^b u(\tau)d\tau\right)^{\frac{r}{q}} v^{1-p'}(t)dt\right)^{\frac{1}{r}} < \infty$$

is necessary and sufficient for the boundedness of operator (0.1) and that the following estimates hold

$$\left(\frac{p-q}{p}\right)^{\frac{1}{q'}} B \le \|H\| \le (p')^{\frac{1}{pq'}} q^{\frac{1}{q}} B,$$

where $1 < q < p < \infty$, $p' = p/(p-1)$ and $1/r = 1/q - 1/p$. This case is well studied and other conditions and estimates have also been found by various scholars, e.g., by Persson-Stepanov in [13] and by Kufner-Kuliev in [5].

The problem of boundedness of operator (0.1) with different type of kernels in the weighted Lebesgue spaces began to be studied in the last decades of the last century. Let us now give some results concerning operators of this type. For example, F.J. Martin-Reyes and E. Sawyer [10] considered the operator with the kernel $k(x,t)=\varphi\left(\frac{t}{x}\right)$, where $\varphi:(0,1)\to(0,\infty)$ is non-increasing and satisfies the following inequality: for some $C_0>0$

$$\varphi(ab)\leq C_0(\varphi(a)+\varphi(b)) \quad \text{for} \quad 0<a,b<1,$$

and V.D. Stepanov [15] considered Volterra convolution operators, i.e., operator (0.1) with the kernel $k(x,t)=\varphi(x-t)$, where φ satisfies the following conditions:

(a) $\varphi(x)\geq 0$ is non-decreasing on $(0,\infty)$,

(b) for some $C>0$, $\varphi(x+y)\leq C(\varphi(x)+\varphi(y))$ for all $x,y\in(0,\infty)$.

S. Bloom and R. Kerman [2] and R. Oinarov [11, 12] gave equivalent conditions for the boundedness of operator (0.1) for kernels $k(x,t)$ which are continuous nonnegative functions increasing in the first argument, decreasing in the second argument and satisfying the condition: there exists a number $h\geq 1$ such that

$$k(x,s)\leq h(k(x,t)+k(t,s))$$

for all $a<s\leq t\leq x<b$. Functions $k(x,t)$ satisfying the above conditions are called *Oinarov's kernels*, which includes also the above kernels. In works [1] and [4] the boundedness of a certain class of integral operators (Riemann-Liouville operators) in various spaces are considered, as well as in the multidimensional case [14]. In [16] was studied properties of discrete operators with a kernel that satisfy the discrete analogue of the Oinarov-type condition.

In works on this topic, the main focus was on finding equivalent conditions for the boundedness of operator (0.1), while exact estimates for the operator norms are very rare. However, in the theory of differential equations and in other fields of mathematics it is very important to obtain exact estimates. In [6] the main purpose was to study another object, but the authors also gave estimates for the norm of the operator with a kernel of a polynomial function.

It should also be noted that recently, in 2021 A. Kalybay and A. Baiarystanov [3] gave estimates for the norm of operator (0.1) with Oinarov's kernel in the case $1<p\leq q<\infty$. In this work we consider the remaining case $1<q<p<\infty$.

The paper is organized as follows. The first section is introduction. In the second section we give the main results. The proofs of the results are given in the third section.

2 Main results

In this section, we give our main results about the lower and upper estimates for the norm of operator (0.1). Therefore, we first pay attention to the boundedness of the operator. In [11] R. Oinarov proved that the finiteness of

$$B_0:=\left(\int_a^b\left(\int_x^b k^q(t,x)u(t)dt\right)^{\frac{r}{q}}\left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{r}{q'}}v^{1-p'}(x)dx\right)^{\frac{1}{r}}$$

and

$$B_1:=\left(\int_a^b\left(\int_x^b u(t)dt\right)^{\frac{r}{p}}\left(\int_a^x k^{p'}(x,t)v^{1-p'}(t)dt\right)^{\frac{r}{p'}}v^{1-p'}(x)dx\right)^{\frac{1}{r}},$$

where $p'=\frac{p}{p-1}, q'=\frac{q}{q-1}$ and $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, is equivalent to the boundedness of operator (0.1). So, we further suppose that

$$B_0<\infty, \quad B_1<\infty. \tag{2.1}$$

Then our first result reads:

Theorem 2.1. *Let $1 < q < p < \infty$ and (2.1) hold. Then the norm of operator H satisfies the inequality*

$$\max\left\{q\left(\frac{p'}{r}\right)^{\frac{1}{q'}} B_0, p'\left(\frac{q}{r}\right)^{\frac{1}{p}} B_1\right\} \leq \|H\| \leq X, \tag{2.2}$$

where X is the unique positive solution of the equation

$$X^{q'} - hq^{\frac{1}{q-1}}(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}X = hq^{\frac{1}{q-1}}p'(p-1)^{\frac{2}{p}}B^{q'} \tag{2.3}$$

and $B = \max\{B_0, B_1\}$.

Remark 1. *Equation (2.3) has a unique positive solution, since*

$$h(x) = \frac{x^{q'}}{q^{\frac{1}{q-1}}p'(p-1)^{\frac{2}{p}}B^{q'} + q^{\frac{1}{q-1}}(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}x}$$

is a continuous and monotonically increasing function of x in $(0, \infty)$, $h(0) = 0$ and $h(\infty) = \infty$.

Example. Let $q = 2$ and $h \geq 1$. Then equation (2.3) and its positive solution take the forms

$$X^2 - 2^{\frac{p+1}{p}}h(p')^{\frac{1}{p'}}BX = 2hp'(p-1)^{\frac{2}{p}}B^2$$

and

$$X = \left(2^{\frac{1}{p}}(p')^{\frac{1}{p'}}h + \sqrt{2^{\frac{2}{p}}(p')^{\frac{2}{p'}}h^2 + 2hp'(p-1)^{\frac{2}{p}}}\right)B,$$

respectively.

Remark 2. *Similarly, the above results can be given for the following integral operator*

$$H^* : L_{p,v} \to L_{q,u}, \ (H^*f)(x) := \int_x^b k(t,x)g(t)dt. \tag{2.4}$$

For H^*

$$\max\left\{q\left(\frac{p'}{r}\right)^{\frac{1}{q'}} B_0^*, p'\left(\frac{q}{r}\right)^{\frac{1}{p}} B_1^*\right\} \leq \|H^*\| \leq X,$$

where X is the unique positive solution of (2.3) with $B = \max\{B_0^*, B_1^*\}$,

$$B_0^* = \left(\int_a^b \left(\int_a^x k^q(x,t)u(t)dt\right)^{\frac{r}{q}} \left(\int_x^b v^{1-p'}(t)dt\right)^{\frac{r}{q'}} v^{1-p'}(x)dx\right)^{\frac{1}{r}},$$

$$B_1^* = \left(\int_a^b \left(\int_a^x u(t)dt\right)^{\frac{r}{p}} \left(\int_x^b k^{p'}(t,x)v^{1-p'}(t)dt\right)^{\frac{r}{p'}} u(x)dx\right)^{\frac{1}{r}}.$$

Theorem 2.2. *Let $1 < q < p < \infty$ and (2.1) hold. Then the norm of operator H satisfies the inequality*

$$\min\left\{q\left(\frac{p'}{r}\right)^{\frac{1}{q'}}, p'\left(\frac{q}{r}\right)^{\frac{1}{p}}\right\} B \leq \|H\| \leq q\left(hp' + h^q(p')^{\frac{q'}{p'}}(q')^{\frac{q'}{p}}\right)^{\frac{1}{q'}} B.$$

3 Proofs

In this section we give proofs of the main results. The assumptions in (2.1) imply the boundedness of operator (0.1), i.e., the following Hardy type inequality: for some $C > 0$

$$\left(\int_a^b \left(\int_a^x k(x,t)f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b f^p(t)v(t)dt\right)^{\frac{1}{p}} \tag{3.1}$$

for all $f \in L_p(v)$.

Proof of Theorem 2.1. (The lower estimate.) If we choose

$$\bar{f}(t) = \left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{r}{pq}} \left(\int_a^t v^{1-p'}(x)dx\right)^{\frac{r}{pq'}} v^{1-p'}(t),$$

then $B_0^{\frac{r}{p}} = \|\bar{f}\|_{p,v}$ and substituting $\bar{f}$ into (3.1) we have

$$CB_0^{\frac{r}{p}} = C\|\bar{f}\|_{p,v} \ge \|H\bar{f}\|_{q,v} = \left(\int_a^b \left(\int_a^x k(x,t)\bar{f}(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}$$

$$= \left(q\int_a^b \left(\int_a^x k(x,t)\bar{f}(t)\left(\int_a^t k(x,s)\bar{f}(s)ds\right)^{q-1} dt\right) u(x)dx\right)^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}}\left(\int_a^b \bar{f}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)\bar{f}(s)ds\right)^{q-1} dx\right) dt\right)^{\frac{1}{q}},$$

where we have used Fubini's theorem. Since $k(x,t)$ is monotonically decreasing with respect to the second argument we get

$$CB_0^{\frac{r}{p}} \ge q^{\frac{1}{q}}\left(\int_a^b \bar{f}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,t)\bar{f}(s)ds\right)^{q-1} dx\right) dt\right)^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}}\left(\int_a^b \bar{f}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)\left(\int_a^t \bar{f}(s)ds\right)^{q-1} dt\right)^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}}\left(\int_a^b \bar{f}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)\right.$$

$$\left.\times\left(\int_a^t \left(\int_s^b k^q(x,s)u(x)\right)^{\frac{r}{pq}}\left(\int_a^s v^{1-p'}(x)dx\right)^{\frac{r}{pq'}} v^{1-p'}(s)ds\right)^{q-1} dt\right)^{\frac{1}{q}}$$

$$\ge q^{\frac{1}{q}}\left(\int_a^b \bar{f}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{1+\frac{r(q-1)}{pq}}\right.$$

$$\left.\times\left(\int_a^t \left(\int_a^s v^{1-p'}(x)dx\right)^{\frac{r}{pq'}} v^{1-p'}(s)ds\right)^{q-1} dt\right)^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left(\frac{1}{\frac{r}{pq'} + 1} \right)^{\frac{1}{q'}} \left(\int_a^b \bar{f}(t) \left(\int_t^b k^q(x,t)u(x)dx \right)^{1+\frac{r(q-1)}{pq}} \right.$$

$$\left. \times \left(\int_a^t v^{1-p'}(s)ds \right)^{\left(1+\frac{r(q-1)}{pq}\right)(q-1)} dt \right)^{\frac{1}{q}}$$

$$= q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} \left(\int_a^b \left(\int_t^b k^q(x,t)u(x)dx \right)^{\frac{r}{q}} \left(\int_a^t v^{1-p'}(s)ds \right)^{\frac{r}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{q}}$$

$$= q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} B_0^{\frac{r}{q}}.$$

Therefore,

$$C \geq q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} B_0.$$

To prove an analogous assertion for B_1, we use the equivalent inequality (the so-called conjugate inequality, see [8, pp. 78-79]) to (3.1) with the same constant C:

$$\left(\int_a^b \left(\int_x^b k(t,x)g(t)dt \right)^{p'} v^{1-p'}(x)dx \right)^{\frac{1}{p'}} \leq C \left(\int_a^b g^{q'}(x)u^{1-q'}(x)dx \right)^{\frac{1}{q'}}, \tag{3.2}$$

for all $g \in L_{q'}(u^{1-q'})$. Let

$$\bar{g}(t) = \left(\int_t^b u(x)dx \right)^{\frac{r}{pq'}} \left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)dx \right)^{\frac{r}{p'q'}} u(t)$$

then $B_1^{\frac{r}{q'}} = \|\bar{g}\|_{q',u^{1-q'}}$ and substituting $\bar{g}$ into (3.2), we have

$$CB_1^{\frac{r}{q'}} = C\|\bar{g}\|_{q',u^{1-q'}} \geq \left(\int_a^b \left(\int_x^b k(t,x)\bar{g}(t)dt \right)^{p'} v^{1-p'}(x)dx \right)^{\frac{1}{p'}}$$

$$= \left(p' \int_a^b \left(\int_x^b k(t,x)\bar{g}(t) \left(\int_t^b k(\tau,x)\bar{g}(\tau)d\tau \right)^{p'-1} dt \right) v^{1-p'}(x)dx \right)^{\frac{1}{p'}},$$

where we have used Fubini's theorem. Since the function $k(x,t)$ is monotonically increasing with the first argument we get

$$CB_1^{\frac{r}{q'}} \geq (p')^{\frac{1}{p'}} \left(\int_a^b \bar{g}(t) \left(\int_a^t k(t,x)v^{1-p'}(x) \left(\int_t^b k(t,x)\bar{g}(\tau)d\tau \right)^{p'-1} dx \right) dt \right)^{\frac{1}{p'}}$$

$$= (p')^{\frac{1}{p'}} \left(\int_a^b \bar{g}(t) \left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)dx \right) \left(\int_t^b \bar{g}(\tau)d\tau \right)^{p'-1} dt \right)^{\frac{1}{p'}}$$

$$= (p')^{\frac{1}{p'}} \left(\int_a^b \bar{g}(t) \left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)dx \right) \right.$$

$$\times \left(\int_t^a \left(\int_\tau^b u(x)dx \right)^{\frac{r}{pq'}} \left(\int_a^\tau k^{p'}(\tau,x)v^{1-p'}(x)dx \right)^{\frac{r}{p'q'}} u(\tau)d\tau \right)^{p'-1} dt \Bigg)^{\frac{1}{p'}}$$

$$\geq (p')^{\frac{1}{p'}} \left(\int_a^b \bar{g}(t) \left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)dx \right)^{1+\frac{r(p'-1)}{p'q'}} \right.$$

$$\left. \times \left(\int_t^b \left(\int_\tau^b u(x)dx \right)^{\frac{r}{pq'}} u(\tau)d\tau \right)^{p'-1} dt \right)^{\frac{1}{p'}}$$

$$= (p')^{\frac{1}{p'}} \left(\frac{1}{\frac{r}{pq'}+1} \right)^{\frac{1}{p}} \left(\int_a^b \bar{g}(t) \left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)dx \right)^{1+\frac{r(p'-1)}{p'q'}} \right.$$

$$\left. \times \left(\int_t^b u(\tau)d\tau \right)^{\left(1+\frac{r}{pq'}\right)(p'-1)} dt \right)^{\frac{1}{p'}}$$

$$= p' \left(\frac{q}{r} \right)^{\frac{1}{p}} \left(\int_a^b \left(\int_t^b u(\tau)d\tau \right)^{\frac{r}{p}} \left(\int_a^t k^{1-p'}(t,x)v^{1-p'}(x)dx \right)^{\frac{r}{p'}} u(t)dt \right)^{\frac{1}{p'}}$$

$$= p' \left(\frac{q}{r} \right)^{\frac{1}{p}} B_1^{\frac{r}{p'}}.$$

Therefore,

$$C \geq p' \left(\frac{q}{r} \right)^{\frac{1}{p}} B_1.$$

So, we finally obtain that

$$C \geq \max \left\{ q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} B_0, p' \left(\frac{q}{r} \right)^{\frac{1}{p}} B_1 \right\},$$

which implies the lower estimate for the norm, i.e.,

$$\max \left\{ q \left(\frac{p'}{r} \right)^{\frac{1}{q'}}, p' \left(\frac{q}{r} \right)^{\frac{1}{p}} \right\} B \leq \max \left\{ q \left(\frac{p'}{r} \right)^{\frac{1}{q'}} B_0, p' \left(\frac{q}{r} \right)^{\frac{1}{p}} B_1 \right\}$$

$$\leq \|H\| = \inf\{C \geq 0 : \|Hf\|_{q,u} \leq C\|f\|_{p,v} \quad \text{for all } f \in L_{p,v}\}.$$

(The upper estimate.) Let us denote

$$I = \int_a^b \left(\int_a^x k(x,t)f(t)dt \right)^q u(x)dx,$$

then we get

$$I = \int_a^b \left(\int_a^x \frac{d}{dt} \left(\int_a^t k(x,s)f(s)ds \right)^q dt \right) u(x)dx$$

$$= q \int_a^b \left(\int_a^x k(x,t)f(t) \left(\int_a^t k(x,s)f(s)ds \right)^{q-1} dt \right) u(x)dx$$

$$= q\int_a^b f(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)dt$$

$$\leq q\|f\|_{p,v}\left(\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)^{p'}dt\right)^{1/p'}$$

$$= q\|f\|_{p,v}J^{1/p'}. \tag{3.3}$$

We now proceed by estimating J. To do this, we estimate its inner integral separately. Using Hölder's inequality with the exponents $\frac{[q]}{q-1}$ and $\frac{[q]}{1-\{q\}}$ we have

$$\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx$$

$$= \int_t^b \left[k^{\{q\}}(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{[q]}\right]^{\frac{q-1}{[q]}} \times \left[k^q(x,t)u(x)\right]^{\frac{1-\{q\}}{[q]}}dx$$

$$\leq \left(\int_t^b k^{\{q\}}(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{[q]}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}.$$

We derive the following estimate by using the properties of Oinarov's kernel and then Newton's binomial formula

$$\leq h^{q-1}\left(\int_t^b k^{\{q\}}(x,t)u(x)\left(k(x,t)\int_a^t f(s)ds + \int_a^t k(t,s)f(s)ds\right)^{[q]}dx\right)^{\frac{q-1}{[q]}}$$

$$\times\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$= h^{q-1}\left(\int_t^b k^{\{q\}}(x,t)u(x)\left(\sum_{n=0}^{[q]} C_{[q]}^n k^n(x,t)\left(\int_a^t f(s)ds\right)^n\right.\right.$$

$$\left.\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}\right)dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$= h^{q-1}\left(\sum_{n=0}^{[q]} C_{[q]}^n\left(\int_t^b k^{\{q\}+n}(x,t)u(x)dx\right)\left(\int_a^t f(s)ds\right)^n\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}. \tag{3.4}$$

Using the Hölder inequality to the first integral of the sum for $0 < n < [q]$ we have

$$\int_t^b k^{\{q\}+n}(x,t)u(x)dx = \int_t^b \left(k^q(x,t)u(x)\right)^{\frac{n-1+\{q\}}{q-1}}\left(k(x,t)u(x)\right)^{\frac{[q]-n}{q-1}}dx$$

$$\leq \left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n-1+\{q\}}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}.$$

Then (3.4) is estimated as follows:

$$\leq h^{q-1}\left(\sum_{n=0}^{[q]} C_{[q]}^{n}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n-1+\{q\}}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}\left(\int_a^t f(s)ds\right)^{n}\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$= h^{q-1}\left(\sum_{n=0}^{[q]} C_{[q]}^{n}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}\right.$$

$$\left.\times\left(\int_a^t f(s)ds\right)^{n}\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}$$

$$= h^{q-1}\left(\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.$$

$$\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right)^{q-1}.$$

Therefore, we obtain

$$\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx$$

$$\leq h^{q-1}\left[\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.$$

$$\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right]^{q-1}.$$

From this we get

$$J=\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)^{p'}dt$$

$$\leq h^{(q-1)p'}\int_a^b v^{1-p'}(t)\left[\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.$$

$$\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right]^{(q-1)p'}dt.$$

Using the Minkowski inequality, we obtain

$$\leq h^{(q-1)p'}\left(\left[\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt\right]^{\frac{1}{(q-1)p'}}\right.$$

$$+\left[\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)dx\right)^{p'}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}dt\right]^{\frac{1}{(q-1)p'}}\Bigg)^{(q-1)p'}$$

$$=h^{(q-1)p'}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right)^{(q-1)p'},$$

where

$$I_1=\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt,$$

$$I_2=\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)dx\right)^{p'}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}dt.$$

From this we have that

$$I\le q\|f\|_{p,v}J^{1/p'}\le q\|f\|_{p,v}h^{q-1}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right)^{q-1},$$

$$I^{\frac{1}{q-1}}\le q^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{1}{q-1}}h\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right).\tag{3.5}$$

Next we estimate I_1 and I_2, separately. Let us begin with I_1, for which we use the following Hardy inequality

$$I_1^{\frac{1}{(q-1)p'}}=\left(\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt\right)^{\frac{1}{(q-1)p'}}$$

$$\le C_{p,(q-1)p'}\left(\int_a^b f^p(t)v(t)dt\right)^{\frac{1}{p}},\tag{3.6}$$

since $(q-1)p'<p$ if the corresponding equivalent condition is satisfied:

$$A_1=\left(\int_a^b\left(\int_t^b v^{1-p'}(s)\left(\int_s^b k^q(x,s)u(x)dx\right)^{p'}ds\right)^{\frac{\tilde r}{(q-1)p'}}\right.$$

$$\left.\times\left(\int_a^t v^{1-p'}(s)ds\right)^{\frac{\tilde r}{((q-1)p')'}}v^{1-p'}(t)dt\right)^{\frac{1}{\tilde r}}<\infty,$$

where $\frac{1}{\tilde r}=\frac{1}{(q-1)p'}-\frac{1}{p}$. Now we show that $A_1<\infty$. Let denote $\bar p=\frac{\tilde r}{(q-1)p'}$,

$$\bar u(t)=\left(\int_a^t v^{1-p'}(s)ds\right)^{\frac{\tilde r}{((q-1)p')'}}v^{1-p'}(t)$$

and

$$\bar v(t)=v^{\frac{rp'}{p^2q'}}(t)\left(\int_a^t v^{1-p'}(t)dt\right)^{\frac{r}{q'}}.$$

Then using the Hardy inequality for the function $\bar f(t)=v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}$ we obtain

$$A_1^{(q-1)p'}=\left(\int_a^b\left(\int_t^b \bar f(s)ds\right)^{\bar p}\bar u(t)dt\right)^{\frac{1}{\bar p}}\le\bar C\left(\int_a^b \bar f^{\bar p}(t)\bar v(t)dt\right)^{\frac{1}{\bar p}}\tag{3.7}$$

$$= \bar{C}\left(\int_a^b \left(\int_x^b k^q(t,x)\,u(t)\,dt\right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t)\,dt\right)^{\frac{r}{q'}} v^{1-p'}(x)\,dx\right)^{\frac{qp'}{r}}$$

$$= \bar{C} B_0^{qp'} < \infty.$$

An estimate for the constant $\bar{C}$ follows from the following calculations:

$$\bar{A} = \sup_{a<s<b} \left(\int_a^s \bar{u}(t)dt\right)^{\frac{1}{\bar{p}}} \left(\int_s^b \bar{v}^{1-\bar{p}'}(t)dt\right)^{\frac{1}{\bar{p}'}}$$

$$= \sup_{a<s<b} \left(\int_a^s \left(\int_a^t v^{1-p'}(s)ds\right)^{\frac{\tilde{r}}{((q-1)p')'}} v^{1-p'}(t)dt\right)^{\frac{qp'}{r}}$$

$$\times \left(\int_s^b \left(v^{\frac{rp'}{p^2q'}}(t) \left(\int_a^t v^{1-p'}(t)dt\right)^{\frac{r}{q'}}\right)^{1-(\frac{r}{qp'})'} dt\right)^{\frac{1}{(\frac{r}{qp'})'}}$$

$$= \sup_{a<s<b} \left(\frac{p-q}{(p-1)(q-1)}\right)^{\frac{qp'}{r}} \left(\int_a^s v^{1-p'}(t)dt\right)^{q-1}$$

$$\times \left(\int_s^b v^{1-p'}(t) \left(\int_a^t v^{1-p'}(t)dt\right)^{-p} dt\right)^{\frac{1}{(\frac{r}{qp'})'}}$$

$$= \sup_{a<s<b} \left(\frac{p-q}{(p-1)(q-1)}\right)^{\frac{qp'}{r}} (p-1)^{\frac{q-1}{p-1}} \left(\int_a^s v^{1-p'}(t)dt\right)^{q-1} \left(\int_a^s v^{1-p'}(t)dt\right)^{1-q}$$

$$= \left(\frac{p-q}{(p-1)(q-1)}\right)^{\frac{p-q}{p-1}} (p-1)^{\frac{1-q}{p-1}} = \left(\frac{p-q}{q-1}\right)^{\frac{p-q}{p-1}} (p-1)^{\frac{1}{p-1}} < \infty.$$

Hence for the constant $\bar{C}$ in (3.7) the following estimate holds

$$\bar{C} \le \bar{p}^{\frac{1}{\bar{p}}} (\bar{p})'^{\frac{1}{(\bar{p})'}} \bar{A}$$

$$= \left(\frac{p-1}{p-q}\right)^{\frac{p-q}{p-1}} \left(\frac{p-1}{q-1}\right)^{\frac{q-1}{p-1}} \left(\frac{p-q}{q-1}\right)^{\frac{p-q}{p-1}} (p-1)^{\frac{1}{p-1}} = (q-1)^{\frac{q-p}{p-1}}.$$

Hence,

$$A_1 \le (\bar{C})^{\frac{1}{(q-1)p'}} B_0^{\frac{q}{q-1}} \le (q-1)^{\frac{q-p}{(q-1)p}} B_0^{q'}$$

and the best constant $C_{p,(q-1)p'}$ in (3.6) satisfies the inequality

$$C_{p,(q-1)p'} \le ((q-1)p')^{\frac{1}{(q-1)p'}} (p')^{\frac{1}{((q-1)p')'}} A_1$$

$$\le ((q-1)p')^{\frac{1}{(q-1)p'}} (p')^{\frac{1}{((q-1)p')'}} (q-1)^{\frac{q-p}{(q-1)p}} B_0^{q'}$$

$$= p'(q-1)^{\frac{1}{p}} B_0^{q'}.$$

Thus, we have that

$$I_1^{\frac{1}{(q-1)p'}} \le p'(q-1)^{\frac{1}{p}} B_0^{q'} \|f\|_{p,v}. \tag{3.8}$$

Let us estimate the second integral I_2. Using integrating by parts we have

$$I_2=\int_a^b v^{1-p'}(t)\left(\int_a^t k(t,s)f(s)\,ds\right)^{(q-1)p'}\left(\int_t^b k(x,t)u(x)\,dx\right)^{p'}dt$$

$$=\int_a^b\left(\int_a^t k(t,s)f(s)\,ds\right)^{(q-1)p'}d\left(-\int_t^b\left(\int_s^b k(x,s)u(x)\,dx\right)^{p'}v^{1-p'}(s)\,ds\right)$$

$$=\int_a^b\left(\int_t^b\left(\int_s^b k(x,s)u(x)\,dx\right)^{p'}v^{1-p'}(s)\,ds\right)d\left(\int_a^t k(t,s)f(s)\,ds\right)^{(q-1)p'}.$$

For the inner integral applying the Minkowski inequality we obtain

$$\leq\int_a^b\left(\int_t^b u(x)\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}}dx\right)^{p'}d\left(\int_a^t k(t,s)f(s)\,ds\right)^{(q-1)p'}.$$

We write the inner integral of the last inequality in the form:

$$\int_a^b\left(\int_t^b\left[\left(\int_x^b u(s)\,ds\right)^{\frac{1}{p}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}}u^{1-\frac{q}{p}}(x)\right]\right.$$

$$\left.\times\left[\left(\int_x^b u(s)\,ds\right)^{-\frac{1}{p}}u^{\frac{q}{p}}(x)\right]dx\right)^{p'}d\left(\int_a^t k(t,s)f(s)\,ds\right)^{(q-1)p'}$$

and then using the Hölder inequality with the exponents $\frac{p}{p-q}$ and $\frac{p}{q}$ we get

$$\leq\int_a^b\left[\int_t^b\left(\int_x^b u(s)ds\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p}{p'(p-q)}}u(x)\,dx\right]^{\frac{p'(p-q)}{p}}$$

$$\times\left[\int_t^b\left(\int_x^b u(s)ds\right)^{-\frac{1}{q}}u(x)dx\right]^{\frac{q}{p-1}}d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}$$

$$=(q')^{\frac{q}{p-1}}(q-1)p'$$

$$\times\int_a^b\left(\int_t^b\left(\int_x^b u(s)dt\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p}{p'(p-q)}}u(x)\,dx\right)^{\frac{p'(p-q)}{p}}$$

$$\times\left(\int_t^b u(s)ds\right)^{\frac{q-1}{p-1}}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'-1}d\left(\int_a^t k(t,s)f(s)\,ds\right)$$

$$=p'(q-1)(q')^{\frac{q}{p-1}}\times$$

$$\int_a^b\left[\left(\int_t^b\left(\int_x^b u(s)\,ds\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p-1}{p-q}}u(x)dx\right)^{\frac{p-q)}{p-1}}\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{\frac{(p-q)(q-2)}{p-1}}\right]$$

$$\times\left[\left(\int_t^b u(s)ds\right)^{\frac{q-1}{p-1}}\left(\int_a^t k(t,s)f(s)ds\right)^{\frac{(q-1)^2}{p-1}}\right]d\left(\int_a^t k(t,s)f(s)ds\right).$$

Applying the Hölder inequality with the exponents $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$ yields

$$\leq p'\left(q-1\right)\left(q'\right)^{\frac{q}{p-1}}$$

$$\times\left[\int_a^b\left(\int_t^b\left(\int_x^b u(s)ds\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p-1}{p-q}}u(x)dx\right)\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{q-2}d\left(\int_a^t k(t,s)f(s)ds\right)\right]^{\frac{p-q}{p-1}}$$

$$\times\left[\int_a^b\left(\int_t^b u(s)ds\right)\left(\int_a^t k(t,s)f(s)ds\right)^{q-1}d\left(\int_a^t k(t,s)f(s)ds\right)\right]^{\frac{q-1}{p-1}}.$$

Using Fubini's theorem, we have

$$=q^{\frac{1-q}{p-1}}p'\left(q-1\right)\left(q'\right)^{\frac{q}{p-1}}$$

$$\times\left[\int_a^b\left(\int_t^b\left(\int_x^b u\left(s\right)ds\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p-1}{p-q}}u(x)dx\right)\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{q-2}d\left(\int_a^t k(t,s)f(s)ds\right)\right]^{\frac{p-q}{p-1}}$$

$$\times\left(\int_a^b u(t)\left(\int_a^t k(t,s)f(s)ds\right)^{q}dt\right)^{\frac{q-1}{p-1}}$$

$$=q^{\frac{1-q}{p-q}}\left(q-1\right)p'(q')^{\frac{q}{p-1}}(q-1)^{\frac{q-p}{p-1}}I^{\frac{q-1}{p-1}}$$

$$\times\left(\int_a^b\left(\int_t^b\left(\int_x^b u(s)ds\right)^{\frac{1}{p-q}}\left(\int_a^x k^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{p-1}{p-q}}u(x)dx\right)\right.$$

$$\left.\times\, d\left(\int_a^t k(t,s)f(s)ds\right)^{q-1}\right)^{\frac{p-q}{p-1}}$$

$$=(q')^{p'-1}p'I^{\frac{q-1}{p-1}}\left[\int_a^b\left(\int_a^t k\left(t,s\right)f\left(s\right)ds\right)^{q-1}\right.$$

$$\left.\times\left(\int_t^b u(s)ds\right)^{\frac{1}{p-q}}\left(\int_a^t k^{p'}\left(t,s\right)v^{1-p'}\left(s\right)ds\right)^{\frac{p-1}{p-q}}u(t)dt\right]^{\frac{p-q}{p-1}}$$

$$=p'(q')^{p'-1}\;I^{\frac{q-1}{p-1}}\times$$

$$\times\left(\int_a^b\left[\left(\int_t^b u\left(s\right)ds\right)^{\frac{1}{p-q}}\left(\int_a^t k^{p'}\left(t,s\right)v^{1-p'}(s)ds\right)^{\frac{p}{p'(p-q)}}u^{\frac{1}{q}}(t)\right]\right.$$

$$\times\left[\left(\int_a^t k(t,s)f(s)ds\right)^{q-1}u^{\frac{1}{q'}}(t)\right]dt\right)^{\frac{p-q}{p-1}}.$$

Here we apply the Hölder inequality with the exponents q and q'

$$\leq p'(q')^{p'-1}I^{\frac{q-1}{p-1}}\left(\left[\int_a^b\left(\int_t^b u(s)ds\right)^{\frac{r}{p}}\left(\int_a^t k^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{r}{p'}}u(t)dt\right]^{\frac{1}{q}}\right.$$

$$\left.\times\left[\int_a^b\left(\int_a^t k(t,s)f(s)ds\right)^{q}u(t)dt\right]^{\frac{1}{q'}}\right)^{\frac{p-q}{p-1}}=p'(q')^{p'-1}I^{\frac{q-1}{p-1}}I^{\frac{p-q}{q'(p-1)}}B_1^{p'}$$

$$\leq p'(q')^{p'-1}B_1^{p'}I^{\frac{p'}{q'}}.$$

Therefore,

$$I_2\leq p'(q')^{p'-1}B_1^{p'}I^{\frac{p'}{q'}}.$$

According to the estimate in (3.5) we obtain

$$I^{\frac{1}{q-1}}\leq hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{1}{q-1}}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right)$$

$$\leq hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{1}{q-1}}\left(p'(q-1)^{\frac{1}{p}}B_0^{q'}\|f\|_{p,v}+(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B_1^{\frac{1}{q-1}}I^{\frac{1}{q}}\right).$$

Taking into account that $I^{\frac{1}{q}}\leq\|H\|\|f\|_{p,v}$ we get

$$I^{\frac{1}{q-1}}\leq hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{q}{q-1}}\left(p'(q-1)^{\frac{1}{p}}B_0^{q'}+(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B_1^{\frac{1}{q-1}}\|H\|\right).$$

So, we have the following upper estimate for the operator norm

$$\left(\sup_{f\neq 0}\frac{I^{\frac{1}{q}}}{\|f\|_{p,v}}\right)^{q'}=\|H\|^{q'}\leq hq^{\frac{1}{q-1}}\left(p'(q-1)^{\frac{1}{p}}B^{q'}+(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}\|H\|\right),$$

i.e.

$$\|H\|^{q'}-hq^{\frac{1}{q-1}}(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}\|H\|\leq hq^{\frac{1}{q-1}}p'(p-1)^{\frac{2}{p}}B^{q'}. \tag{3.9}$$

Consequently, we obtain

$$\frac{\|H\|^{q'}}{q^{\frac{1}{q-1}}p'(q-1)^{\frac{1}{p}}B^{q'}+q^{\frac{1}{q-1}}(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}\|H\|}\leq h.$$

By Remark 1 we have that the corresponding equation $h(x)=h$ has exactly one positive solution. If X is the solution of the equation, i.e.,

$$\frac{X^{q'}}{q^{\frac{1}{q-1}}p'(q-1)^{\frac{1}{p}}B^{q'}+q^{\frac{1}{q-1}}(p')^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p(q-1)}}B^{\frac{1}{q-1}}X}=h,$$

then

$$\|H\|\leq X.$$

Proof of Theorem 2.2. Using the Young inequality to the second term in the left hand side of (3.9) we have:

$$\|H\|^{q'} - \frac{h^q q^{\frac{q}{q-1}} (p')^{\frac{q}{p'(q-1)}} (q')^{\frac{q}{p(q-1)}} B^{\frac{q}{q-1}}}{q} - \frac{\|H\|^{q'}}{q'} \leq hq^{\frac{1}{q-1}} p' B^{q'}.$$

This implies that

$$\frac{\|H\|^{q'}}{q} \leq hq^{\frac{1}{q-1}} p' B^{q'} + h^q q^{q'-1} (p')^{\frac{q'}{p'}} (q')^{\frac{q'}{p}} B^{q'},$$

i.e.,

$$\|H\|^{q'} \leq hq^{q'} p' B^{q'} + h^q q^{q'} (p')^{\frac{q'}{p'}} (q')^{\frac{q'}{p}} B^{q'}.$$

Therefore, we get the following estimate for the norm:

$$\|H\| \leq q \left(hp' + h^q (p')^{\frac{q'}{p'}} (q')^{\frac{q'}{p}} \right)^{\frac{1}{q'}} B.$$

References

[1] Kh. Almohammad, *On modular inequalities for generalized Hardy operators on weighted Orlicz spaces.* Eurasian Math. J., 12 (2021), no. 3, 19–28.

[2] S. Bloom, R. Kerman, *Weighted norm inequalities for operators of Hardy type*, Proc. Amer. Math. Soc., 113 (1991), no. 1, 135–141.

[3] A. Kalybay, A. Baiarystanov, *Exact estimate of norm of integral operator with Oinarov's condition*, Kazah. Math. J., 21 (2021) 6–14.

[4] A. Kalybay, R. Oinarov, *Boundedness of Riemann-Liouville operator from weighted Sobolev space to weighted Lebesgue space.* Eurasian Math. J., 1 (2021), no. 12, 39–48.

[5] A. Kufner, K. Kuliev, *Some characterizing conditions for the Hardy inequality*, Eurasian Math. J., 2 (2010), no. 2, 86–98.

[6] A. Kufner, K. Kuliev, L-E. Persson, *Some higher order Hardy inequalities*, Jour. Ineq & Appl, 1 (2012), no. 69, 1–14.

[7] A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy inequality-about its history and some related results*, Pilsen, 2007.

[8] A. Kufner, L.-E. Persson, *Weighted inequalities of Hardy type*, World scientific: New Jersey, London, Singapore, Hong Kong, 2003.

[9] F.J. Martin-Reyes, *Weights, one-sided operators, singular integrals and ergodic theorems in:* Nonlinear Analysis, Function Spaces and Applications, Vol. 5 (Eds: M. Krbec, A. Kufner, B. Opic and J. Rakosnik), Prometheus, Prague, 1994, 103-137.

[10] F.J. Martin-Reyes, E. Sawyer, *Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater*, Proc. Amer. Math. Soc. 3 (1989), no. 106, 727–733.

[11] R. Oinarov, *Weighted inequalities for a class of integral operators* [Russian], Dokl. Akad. Nauk. SSSR 5 (1991) no. 319, 1076-1078 (in Russian); English translation in Soviet Math. Dokl. 1 (1992), no. 44, 291-298.

[12] R. Oinarov, *Two-sided estimates for the norm of some classes of integral operators*, Trudy Mat. Inst. Steklov 204 (1993), 240-250 (in Russian); English translation in Proc. Steklov Inst. Math. 3 (1994), no. 204, 205-214.

[13] L.-E. Persson, V.D. Stepanov, *Weighted integral inequalities with the geometric mean operator,* J. Inequal. Appl. 7 (2002), no. 5, 727–746.

[14] M.A. Senouci, *Boundedness of Riemann–Liouville fractional integral operator in Morrey spaces.* Eurasian Math. J., 1 (2021), no. 12, 82–91.

[15] V.D. Stepanov. *Weighted inequalities for a class of Volterra convolution operators*, J. London Math. Soc. 45 (1992), no. 2, 232-242.

[16] A.M. Temirkhanova, A.T. Beszhanova, *Boundedness and compactness of a certain class of matrix operators with variable limits of summation.* Eurasian Math. J., 11 (2020), no. 4, 66-75.

Komil Kuliev,
Samarkand State University,
15 University Boulevard,
Samarkand 140104, Uzbekistan;
and
Institute of Mathematics named after V.I. Romanovsky
of the Academy of Sciences of the Republic of Uzbekistan,
9 University St, Olmazor district,
Tashkent 100174, Uzbekistan.
E-mail: komilkuliev@gmail.com.

Received: 19.11.2021