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EXACT AND APPROXIMATE SOLUTIONS TO THE STEFAN PROBLEM IN ELLIPSOIDAL COORDINATES

S.A. Kassabek, S.N. Kharin, D. Suragan

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Key words: softening stage, electrical contact heating, Stefan problems, Joule heating, heat balance integral method.

AMS Mathematics Subject Classification: 80A22, 40C10.

Abstract. In this paper, we present exact and approximate solutions of the Stefan problems in ellipsoidal coordinates. We consider two models of electrical contact heating for melting process. The first problem describes the contact heating for liquid and solid zones based on the two-phase Stefan problem, where time t is present as a parameter. Contact heating including softening processes are described by a mathematical model based on the three-phase Stefan problem for the ellipsoidal heat equation. Numerical results are presented and discussed.

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1 Introduction

The mathematical model of transient phenomena of contact heating in closed electrical contacts is well-known. Stationary temperature and electromagnetic fields in symmetric electrical contacts were described in [5]. Electrical contact heating during current passage is a result of many physical phenomena. One of the most important factor of contact heating is internal heat sources due to the Joule heating in current constriction areas. All consecutive stages of contact heating from pre-softening to melting processes for the spherical case are presented in [7]. However, influence of softening contact is an open question and which needs to be studied. This paper presents a mathematical model describing heating and melting processes of closed contacts taking into account the softening stage. In the case, in which the Fourier criterion is sufficiently large Fo >> 1, the quasi-stationary model of contact heating is valid.

A mathematical model describing these processes are based on the Stefan problem [3, 11]. From the theoretical point of view, these problems are among most interesting problems in the theory of non-linear parabolic equations. The Stefan problem requires to determine the temperature and the moving boundary interface.

In this paper, we present two models of contact heating. The first model illustrates the melting process based on the two-phase Stefan problem, where time t is presented as a parameter. The second model describes the heating process with a softening zone. In this case, we find an approximate solution of the Stefan problem by using the heat balance integral (HBI) method introduced by Goodman [1]. In accordance with this method, the temperature profile along dimensional coordinates has to be given, but time dependent coefficients must be found from the heat equation and boundary conditions. Therefore, the solution of the stationary problem is suitable for constructing a temperature profile for the quasi-stationary problem, as it is done in [10]. In this direction, we also refer to papers [1, 2, 8, 10, 12, 17], for example. As a rule, the simple parabolic profile of temperature gives a good approximation and it is used very often. In Section 2, we derive the one-dimensional quasi-stationary heat equation in ellipsoidal coordinates. The two phase quasi-stationary Stefan problem describing the contact heating without softening stage is presented in Section 3. The formulation of a mathematical model of electrical contact heating based on the softening zone and the main results of this paper are given in Section 4. Concluding remarks are discussed briefly in Section 5.

2 Heat conduction equation in ellipsoidal coordinates

A mathematical model based on a system of differential equations describing non-stationary heat transfer phenomena and potential fields in closed electrical contacts has to be considered taking into account the Joule heating for each electrode, that is, it can be written in the form:

$$c_i\gamma_i\frac{\partial\theta_i}{\partial t}=\operatorname{div}\left(\lambda_i\operatorname{grad}\theta_i\right)+\frac{1}{\rho_i}\operatorname{grad}^2\varphi_i, \tag{2.1}$$

$$\operatorname{div}\left(\frac{1}{\rho_i}\operatorname{grad}\varphi_i\right)=0. \tag{2.2}$$

Here subscripts $i=1$ and $i=2$ refer to cathode and anode, respectively, θ_i is the temperature, φ_i is the potential, c_i is the specific heat, γ_i is the density, λ_i is the thermal conductivity, and ρ_i is the specific resistance. If we assume that the contact spot is a circle with radius r_0 and we have axial symmetry, then in cylindrical coordinates the system takes the following form

$$c_i\gamma_i\frac{\partial\theta_i}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_i r\frac{\partial\theta_i}{\partial r}\right)+\frac{\partial}{\partial z}\left(\lambda_i\frac{\partial\theta_i}{\partial z}\right)+\frac{1}{\rho_i}\left[\left(\frac{\partial\varphi_i}{\partial r}\right)^2+\left(\frac{\partial\varphi_i}{\partial z}\right)^2\right], \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\rho_i}r\frac{\partial\varphi_i}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho_i}\frac{\partial\varphi_i}{\partial z}\right)=0. \tag{2.4}$$

The well-known Holm theorem [4] determining the analogy between electrical and thermal fields suggests an idea about a representation of its solution.

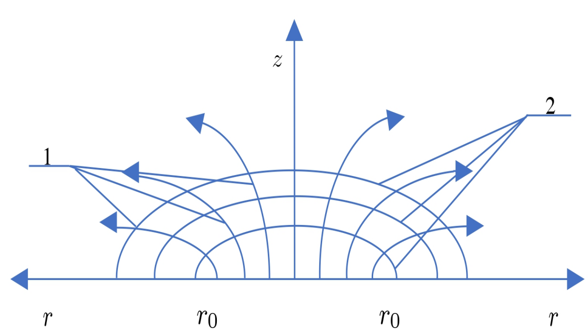

Figure 1: Electrical (temperature) field in the contact constriction region. 1 - equipotential (isothermal) surfaces; 2 - electrical (thermal) current lines.

Let us seek θ_i and φ_i as functions depending on generalised coordinates ξ and η, which are determined by

$$\frac{r^2}{\xi^2+r_0^2}+\frac{z^2}{\xi^2}=1 \quad \text{and} \quad \frac{r^2}{\eta^2}-\frac{z^2}{r_0^2-\eta^2}=1, \tag{2.5}$$

e.g.,

$$\xi(r,z)=\frac{1}{\sqrt{2}}\sqrt{s_1+\sqrt{{s_1}^2+4{r_0}^2z^2}}, \quad s_1=z^2+r^2-r_0^2, \tag{2.6}$$

$$\eta(r,z)=\frac{1}{\sqrt{2}}\sqrt{s_2+\sqrt{{s_2}^2-4{r_0}^2z^2}}, \quad s_2=z^2+r^2+r_0^2. \tag{2.7}$$

Using relations between cylindrical and elliptical coordinates, where $\frac{r^2}{\xi^2+r_0^2}+\frac{z^2}{\xi^2}=1$ are family of isothermal ellipsoids of revolution and orthogonal family of hyperboloids $\frac{r^2}{\eta^2}-\frac{z^2}{r_0^2-\eta^2}=1$ which are the surfaces of heat flow of electric current, see Figure 1, we get the non-stationary heat equation in ellipsoidal coordinates

$$\frac{\partial\theta}{\partial t}=\frac{a^2}{r_0^2-\eta^2+\xi^2}\left[\left(r_0^2-\eta^2\right)\frac{\partial^2\theta}{\partial\eta^2}+\frac{r_0^2-2\eta^2}{\eta^2}\frac{\partial\theta}{\partial\eta}\right]+ \tag{2.8}$$

$$+\frac{a^2}{r_0^2-\eta^2+\xi^2}\left[\left(r_0^2+\xi^2\right)\frac{\partial^2\theta}{\partial\xi^2}+2\xi\frac{\partial\theta}{\partial\xi}+\frac{I^2\rho}{4\pi^2c\gamma\left(r_0^2+\xi^2\right)}\right]. \tag{2.9}$$

For stationary regime [6], it was proved in [5] that the ratio $\frac{\nabla^2\xi}{(\nabla\xi)^2}$ must be independent of the variables r and z, it should depend only on the variable ξ. Indeed, using relation (2.5) it is easy to check that

$$\frac{\nabla^2\xi}{(\nabla\xi)^2}=\frac{2\xi}{r_0^2+\xi^2}. \tag{2.10}$$

Hereby, the temperature $\theta(r,z)$ depends only on ξ and does not depend on η, i.e. if $\frac{\partial\theta}{\partial t}=0$, then $\frac{\partial\theta}{\partial\eta}=\frac{\partial^2\theta}{\partial\eta^2}=0$ and we get

$$\left(r_0^2+\xi^2\right)\frac{\partial^2\theta}{\partial\xi^2}+2\xi\frac{\partial\theta}{\partial\xi}+\frac{I^2\rho}{4\pi^2c\gamma\left(r_0^2+\xi^2\right)}=0. \tag{2.11}$$

For the quasi-stationary case, temperature also depends on η weakly, thus in (2.8) we can have $\eta=0$, $\frac{\partial\theta}{\partial\eta}=0$, $\frac{\partial^2\theta}{\partial\eta^2}=0$, hence

$$\frac{\partial\theta}{\partial t}=\frac{a^2}{r_0^2+\xi^2}\left[\left(r_0^2+\xi^2\right)\frac{\partial^2\theta}{\partial\xi^2}+2\xi\frac{\partial\theta}{\partial\xi}+\frac{I^2\rho}{4\pi^2c\gamma\left(r_0^2+\xi^2\right)}\right] \tag{2.12}$$

or

$$\frac{\partial\theta}{\partial t}=a^2\left[\frac{\partial^2\theta}{\partial\xi^2}+\frac{2\xi}{r_0^2+\xi^2}\frac{\partial\theta}{\partial\xi}+\frac{I^2\rho}{4\pi^2c\gamma\left(r_0^2+\xi^2\right)^2}\right]. \tag{2.13}$$

After substitution $\zeta=\arctan\left(\frac{\xi}{r_0}\right)$, we get

$$\frac{\partial\theta}{\partial t}=\frac{a^2{r_0}^2}{(r_0^2+r_0^2\tan^2(\zeta))^2}\left[\frac{\partial^2\theta}{\partial\zeta^2}+\frac{I^2\rho}{4\pi^2c\gamma r_0^2}\right] \tag{2.14}$$

or

$$\frac{\partial\theta}{\partial t}=\frac{a^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta}{\partial\zeta^2}+\frac{I^2\rho}{4\pi^2c\gamma r_0^2}\right]. \tag{2.15}$$

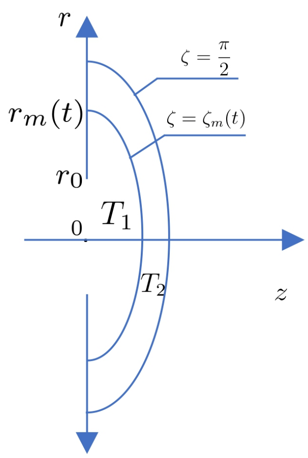

Figure 2: Temperature distribution in quasi-stationary regime

3 Quasi-stationary linear Stefan problem

Quasi-stationary heat transfer for melting process can be described by the following model:

$$\begin{cases} \operatorname{div}(\lambda_1 \operatorname{grad} T_1) + \frac{1}{\rho_1}\operatorname{grad}^2 \varphi_1 = 0 \\ \operatorname{div}\left(\frac{1}{\rho_1}\operatorname{grad}\varphi_1\right) = 0 \end{cases}, \quad (r,z) \in D_1, \tag{3.1}$$

$$\begin{cases} \operatorname{div}(\lambda_2 \operatorname{grad} T_2) + \frac{1}{\rho_2}\operatorname{grad}^2 \varphi_2 = 0 \\ \operatorname{div}\left(\frac{1}{\rho_2}\operatorname{grad}\varphi_2\right) = 0 \end{cases}, \quad (r,z) \in D_2, \tag{3.2}$$

where

$$D_1 = \{(r,z) : 0 < z < \sigma(r,t), \quad 0 < r < r_m(t)\}, \tag{3.3}$$

and

$$D_2 = \{(r,z) : \sigma(r,t) < z < \infty, \quad r_m(t) < r < \infty\}, \tag{3.4}$$

where T_i are the electrical contact temperature, φ_i are electrical potentials, $\sigma(r,t)$ is the unknown moving boundary, $\lambda(T_i)$ and $\rho(T_i)$ are heat conductance and electrical resistance, respectively, $i = 1$ for the melted zone and $i = 2$ for the solid zone.

We consider the functions T_i and φ_i depending on the variable $\xi(r,z)$, then we get following boundary conditions at $z = 0\,(\xi = 0)$:

$$\frac{dT_1}{d\xi} = 0, \tag{3.5}$$

$$\varphi_1\Big|_{0 \le r \le r_0} = 0, \quad \frac{\partial \varphi_1}{\partial z}\Big|_{r_0 \le r \le r_m(t)} = 0, \tag{3.6}$$

with $z = \sigma(r,t)\,(\xi = \xi_m(t))$:

$$T_1 = T_2 = T_m, \quad \lambda_1 \frac{dT_1}{d\xi} = \lambda_2 \frac{dT_2}{d\xi}, \tag{3.7}$$

$$\varphi_1 = \varphi_2, \quad \frac{1}{\rho_1}\frac{d\varphi_1}{d\xi} = \frac{1}{\rho_2}\frac{d\varphi_2}{d\xi}. \tag{3.8}$$

Boundary conditions $z = \infty$ or $r = \infty$ ($\xi = \infty$):

$$\frac{dT_2}{d\xi} = 0, \tag{3.9}$$

$$\varphi_2 = \frac{u_c}{2}, \tag{3.10}$$

with the Stefan condition

$$\lambda_1 \frac{dT_1}{d\xi}\Big|_{\xi=\xi_m(t)} - \lambda_2 \frac{dT_2}{d\xi}\Big|_{\xi=\xi_m(t)} = L\gamma \frac{d\xi}{dt}, \tag{3.11}$$

where $\xi_m(t)$ is the melting boundary, u_c is the potential and L is the latent heat of melting.

By [5] and boundary conditions (3.6), (3.8) and (3.10) the solutions for potentials are

$$\varphi_1'(\xi) = \frac{I^2 \rho_1(T_1)}{2\pi(r_0^2 + \xi^2)}, \quad \varphi_2'(\xi) = \frac{I^2 \rho_2(T_2)}{2\pi(r_0^2 + \xi^2)}. \tag{3.12}$$

We consider the case in which $\rho(T_1) = \rho_1(1+\alpha_1[T_1 - T_m])$, $\rho(T_2) = \rho_2(1+\alpha_2 T_2)$ and $\lambda(T_i) = \lambda_i = const$ for $i = 1, 2$.

Heat conduction equation in ellipsoidal coordinates after introducing the substitution $\zeta = \arctan\left(\frac{\xi}{r_0}\right)$ in equations (3.1), (3.2) with potential solutions (3.12) will be written in the form:

$$\frac{d^2 T_1}{d\zeta^2} + \frac{w_1^2}{\alpha_1}[1 + \lambda_1(T_1 - T_m)] = 0, \quad 0 < \zeta < \zeta_m(t), \tag{3.13}$$

$$\frac{d^2 T_2}{d\zeta^2} + \frac{w_2^2}{\alpha_2}[1 + \lambda_2 T_2)] = 0, \quad \zeta_m(t) < \zeta < \frac{\pi}{2}, \tag{3.14}$$

with the boundary conditions:

$$\frac{dT_1}{d\zeta}\Big|_{\zeta=0} = 0, \tag{3.15}$$

$$T_1(\zeta_m(t)) = T_2(\zeta_m(t)) = T_m, \tag{3.16}$$

$$T_2\left(\frac{\pi}{2}\right) = 0. \tag{3.17}$$

The solutions of problem (3.13), (3.14), (3.15), (3.16) and (3.17) for the ellipsoidal domain, see Figure 2, are

$$T_1(\zeta) = \frac{1}{\lambda_1}\left(\frac{\cos w_1 \zeta}{\cos w_1 \zeta_m(t)} + \alpha_1 T_m - 1\right), \tag{3.18}$$

$$T_2(\zeta) = \frac{(1+\alpha_2 T_m)\sin\left[w_2\left(\frac{\pi}{2} - \zeta\right)\right] - \sin\left[w_2\left(\zeta_m(t) - \zeta\right)\right]}{\alpha_2 \sin\left[w_2\left(\frac{\pi}{2} - \zeta_m(t)\right)\right]} - \frac{1}{\alpha_2}, \tag{3.19}$$

where $w_i^2 = \frac{I^2 \rho_i \alpha_i}{4\pi^i r_0^2 \lambda_i}$, $\zeta_m(t)$ is the unknown moving interface, T_m is the melting temperature, α_i are the temperature coefficients of resistance, λ_i are the heat conductances for $i = 1, 2$.

Using Stefan condition (3.11) and the following formulas

$$\frac{dT_1}{d\xi}\Big|_{\xi=\xi_m(t)} = \frac{dT_1}{d\zeta}\frac{d\zeta}{d\xi}\Big|_{\xi=\xi_m(t)} = -\frac{w_1 \sin(w_1 \zeta_m(t))}{\alpha_1 \cos(w_1 \zeta_m(t))} \cdot \frac{r_0}{r_0^2 + \xi_m^2(t)}, \tag{3.20}$$

$$\left.\frac{dT_2}{d\xi}\right|_{\xi=\xi_m(t)} = \left.\frac{dT_2}{d\zeta}\frac{d\zeta}{d\xi}\right|_{\xi=\xi_m(t)} = \frac{w_2 r_0\left[1-(1+\alpha_2 T_m)\cos(w_2\left(\frac{\pi}{2}-\zeta_m(t)\right)\right]}{\alpha_2\sin w_2\left(\frac{\pi}{2}-\zeta_m(t)\right)(r_0^2+\xi_m^2(t))}, \tag{3.21}$$

and

$$\frac{d\xi}{dt} = \frac{d\xi}{d\zeta}\frac{d\zeta}{dt} = \frac{r_0}{\cos^2(\zeta)}\cdot\frac{d\zeta}{dt}, \tag{3.22}$$

we get

$$-\frac{\lambda_1 w_1}{\alpha_1}\tan(w_1\zeta_m(t)) + \frac{\lambda_2 w_2(1+\alpha_2 T_m)}{\alpha_2}\cot w_2\left(\frac{\pi}{2}-\zeta_m(t)\right) - \tag{3.23}$$

$$-\frac{\lambda_2 w_2}{\alpha_2}\csc w_2\left(\frac{\pi}{2}-\zeta_m(t)\right) = \frac{L\gamma r_0}{\cos^4(\zeta_m(t))}\cdot\frac{d\zeta_m(t)}{dt}. \tag{3.24}$$

From expression (3.24) we can find the unknown moving boundary $\zeta_m(t)$. Figure 3 shows the numerical results for the copper electrical contact Cu with current $I = 200\,kA$ and arc radius $r_0 = 1.23\cdot 10^{-6}m$.

4 Heating of electrical contact with softening stage

The process of closed electrical contact heating can be divided in a chain of consecutive stages. The first stage $S_1\,(0\le t\le t_{soft})$ corresponds to the initial period of contact heating by the Joule heating. The temperature of electrode $\theta_1(\zeta,t)$ during this stage increases to the softening temperature θ_{soft}, thus the termination time t_{soft} of the first stage is defined from the equation

$$\theta_1\left(0,t_{soft}\right) = \theta_{soft}. \tag{4.1}$$

The electrode at this stage consists of only one zone $D_3(0<\zeta<\pi/2)$.

The second stage $S_2\,(t_{soft}\le t\le t_{melt})$ lasts from t_{soft} to t_{melt}, when the electrode begins to melt. The electrode now consists of two zones: the soft zone $D_3(0<\zeta<\beta(t))$ and the solid zone $D_2(\beta(t)<\zeta<\pi/2)$. Here $\zeta=\beta(t)$ is the free moving boundary interface of phase transformation. The time t_{melt} can be found from the equation

$$\theta_2\left(0,t_{melt}\right) = \theta_{melt}, \tag{4.2}$$

where $\theta_2(\zeta,t)$ is the temperature distribution inside the zone D_2, and θ_{melt} is the melting of electrode.

The third stage $S_3\,(t_{melt}\le t\le t_{arc})$ is characterized by appearance of a new region $D_1(0<\zeta<\alpha(t))$ occupied by melted region of electrode. The region $D_2(\alpha(t)<\zeta<\beta(t))$ with two moving boundaries is occupied by softened material, while the region $D_3(\beta(t)<\zeta<\pi/2)$ remains solid. It is the last stage of arcing of total duration t_{arc}.

The temperature field in electrical contacts with heating $\theta_1(\zeta,t)$, softening $\theta_2(\zeta,t)$, and melting $\theta_3(\zeta,t)$ inside corresponding zones D_1, D_2, and D_3 are described by the heat equations in ellipsoidal symmetry:

$$\frac{\partial\theta_i}{\partial t} = \frac{a_i^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_i}{\partial\zeta^2}+\omega_i^2\right], \qquad i=1,2,3, \tag{4.3}$$

where $\omega_i = \frac{I}{2\pi r_0}\sqrt{\frac{\rho_i}{c_i\gamma_i}}$ and r_0 is the radius of contact spot, I is the current, a_i is the thermal diffusivity of the zone D_i.

For numerical calculations we consider the copper electrical contact Cu with current $I = 200\,kA$ and arc radius $r_0 = 1.23\cdot 10^{-6}m$

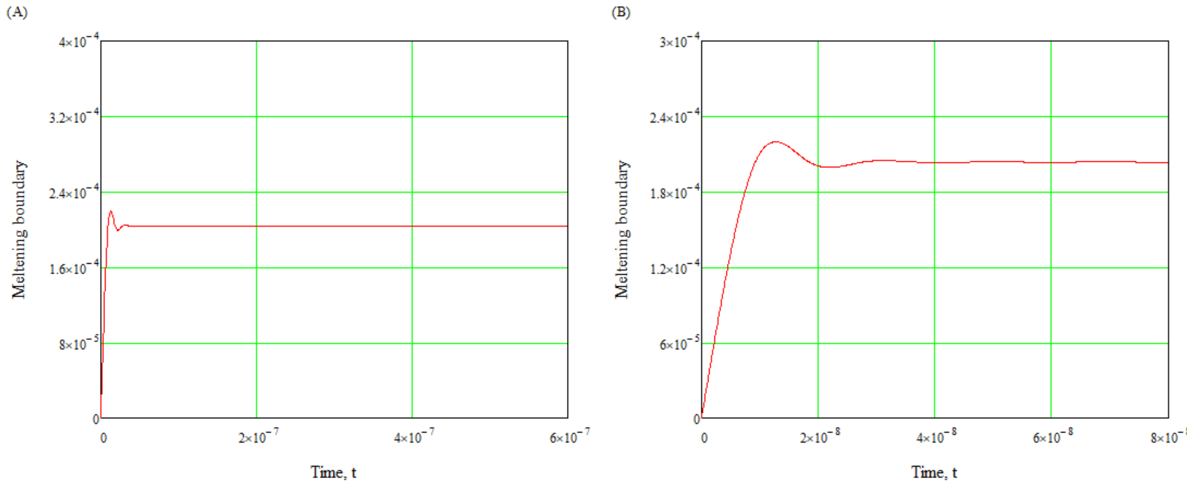

Figure 3: Temperature distribution at $\zeta = 0$ for Quadratic profile.

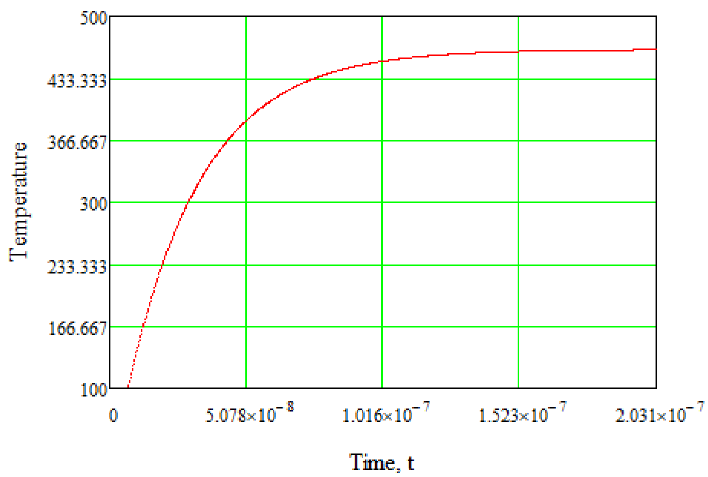

Figure 4: Temperature distribution at $\zeta = 0$ for Quadratic profile.

4.1 Stage 1

Let us consider the first period ($0 \leq t \leq t_{soft}$) of nonstationary heating with the temperature field $\theta_1(\zeta, t)$. In this stage we have only one region D_3, where contact material is solid and temperature attains the softening point. Thus, we consider the heat equation

$$\frac{\partial \theta_1}{\partial t} = \frac{a_1^2}{r_0^2} \cos^4(\zeta) \left[\frac{\partial^2 \theta_1}{\partial \zeta^2} + \omega_{11}^2 \right], \quad 0 < \zeta < \frac{\pi}{2}, \tag{4.4}$$

with the boundary conditions

$$\frac{\partial \theta_1(0, t)}{\partial \zeta} = 0, \tag{4.5}$$

and

$$\theta_1(\pi/2, t) = 0, \tag{4.6}$$

and the initial condition

$$\theta_1(\zeta, 0) = 0. \tag{4.7}$$

The exact solution of the parabolic equation (4.4) given in [9] has the form

$$\theta_1(\zeta,t) = \sum_{n=0}^{\infty} C_n \theta^n(\zeta,t), \tag{4.8}$$

where

$$\theta^n(\zeta,t) = \sum_{i=0}^{n} t^i \phi_{n,i}(\zeta), \tag{4.9}$$

$$\phi_{n,n}(\zeta) = A + B\zeta, \tag{4.10}$$

$$\phi_{n,i}(\zeta) = n(n-1)\dots(i+1)L_f^{n-i}[\phi_{n,n}(\zeta)]; \quad i = 0,1,\dots,n-1. \tag{4.11}$$

The integral operator L_f is introduced as follows

$$L_f[y(\zeta)] = \frac{r_0^2}{a_1^2}\int \cos^2\zeta \left(\int \frac{y(\zeta)d\zeta}{\cos^2\zeta}\right) d\zeta, \tag{4.12}$$

where the degree of the operator is defined as

$$L_f^i[y(\zeta)] = L_f[L_f^{i-1}[y(\zeta)]]. \tag{4.13}$$

The integral heat power balance method is applied to problem (4.4)-(4.7). Goodman [2] used the HBI method to solve the one-dimensional heat problems with fixed and moving boundaries. The idea of this method is to substitute the exact solution by an approximate solution by choosing the temperature profile for the temperature distribution in the whole domain. The classical quadratic profile used for describing the temperature distribution has the form

$$T(\zeta,t) = A(t)\zeta^2 + B(t)\zeta + C(t), \tag{4.14}$$

where $A(t), B(t)$, and $C(t)$ are functions of time.

For the temperature distribution $\theta_1(\zeta,t)$, we assume that the temperature profile given in the form

$$T_1(\zeta,t) = A_1(t)\left(\zeta^2 - \frac{\pi^2}{4}\right) \quad \text{in } 0 \le \zeta \le \frac{\pi}{2}, \tag{4.15}$$

satisfies boundary conditions (4.5) and (4.6), where the coefficient $A_1(t)$ is a general function of time. Integrating equation (4.4) with respect to the space variable from $\zeta = 0$ to $\zeta = \frac{\pi}{2}$ we have

$$\frac{r_0^2}{a_1^2}\int_0^{\pi/2} \frac{\partial\theta_1}{\partial t} d\zeta = \int_0^{\pi/2} \cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2} + \omega_{11}^2\right] d\zeta. \tag{4.16}$$

By using Leibniz integral formula, we obtain

$$\frac{r_0^2}{a_1^2}\frac{d}{dt}\left[\int_0^{\pi/2} \theta_1 d\zeta\right] = \int_0^{\pi/2} \cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2} + \omega_{11}^2\right] d\zeta, \tag{4.17}$$

which is called the energy integral equation.

Substituting the temperature profile (4.15) into equation (4.17), we obtain the ordinary differential equation for $A_1(t)$:

$$\frac{dA_1(t)}{dt} + K_1 A_1(t) = -K_2, \tag{4.18}$$

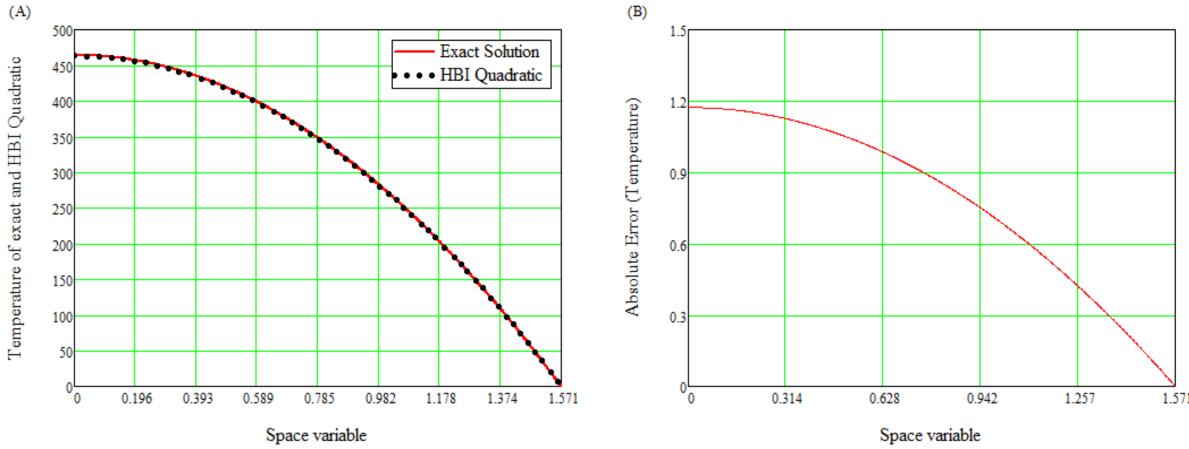

Figure 5: (A)Temperature distribution of exact and approximate solution at $t = t_{soft}$. (B) Absolute error in the quadratic profile at $t = 1.683 \cdot 10^{-7} sec$.

where

$$K_1 = \frac{9a_1^2}{2\pi^2 r_0^2}, \quad K_2 = \frac{9a_1^2\omega_{11}^2}{4\pi^2 r_0^2}, \tag{4.19}$$

with the condition $A_1(0) = 0$. Finally, we have the temperature profile

$$T_1(\zeta, t) = \frac{K_2}{K_1} \cdot \left(e^{-K_1 \cdot t} - 1\right) \cdot \left(\zeta^2 - \frac{\pi^2}{4}\right). \tag{4.20}$$

The calculation at the first stage of contact heating for the copper electrical contacts is shown in Figure 4. The temperature T_1 reaches the softening point $\theta_{soft} = 463\, K$ when $t_{soft} = 1.683 \cdot 10^{-7} sec$. The temperature distribution for the quadratic temperature profile at $t = t_{soft}$ and the spatial distribution of the absolute error are shown in Figure 5. Hence, we can see that error of approximation is less than 0.3 %.

4.2 Stage 2

In the second stage, a new region D_2 occurs and now we have two regions D_2 and D_3. Mathematical formulation for these zones are given as

$$\frac{\partial \theta_{21}}{\partial t} = \frac{a_{21}^2}{r_0^2} \cos^4(\zeta) \left[\frac{\partial^2 \theta_{21}}{\partial \zeta^2} + \omega_{21}^2\right], \quad 0 < \zeta < \beta(t), \tag{4.21}$$

$$\frac{\partial \theta_{22}}{\partial t} = \frac{a_{22}^2}{r_0^2} \cos^4(\zeta) \left[\frac{\partial^2 \theta_{22}}{\partial \zeta^2} + \omega_{22}^2\right], \quad \beta(t) < \zeta < \frac{\pi}{2}, \tag{4.22}$$

with boundary and initial conditions

$$\theta_{22}(0, t_{soft}) = \theta_{soft}, \quad \theta_{21}(\zeta, t_{soft}) = f(\zeta), \tag{4.23}$$

$$\left.\frac{\partial \theta_{21}}{\partial \zeta}\right|_{\zeta=0} = 0, \quad \theta_{22}|_{\zeta=\frac{\pi}{2}} = 0, \tag{4.24}$$

$$\theta_{21}|_{\zeta=\beta(t)} = \theta_{22}|_{\zeta=\beta(t)} = \theta_{soft}, \tag{4.25}$$

$$f(0) = \theta_{soft}, \quad \beta(t_{soft}) = 0, \quad f\left(\frac{\pi}{2}\right) = 0, \tag{4.26}$$

and Stefan's condition

$$-\lambda_{21}\frac{\partial\theta_{21}}{\partial\zeta}\bigg|_{\zeta=\beta(t)} = -\lambda_{22}\frac{\partial\theta_{22}}{\partial\zeta}\bigg|_{\zeta=\beta(t)}, \tag{4.27}$$

where

$$f(\zeta) = \theta_1(\zeta, t_{soft}), \quad \omega_{21} = \frac{I}{2\pi r_0}\sqrt{\frac{\rho_{11}}{c_{11}\gamma_{11}}}, \quad \omega_{22} = \frac{I}{2\pi r_0}\sqrt{\frac{\rho_{12}}{c_{12}\gamma_{12}}}. \tag{4.28}$$

The quadratic profile used by Goodman [2] has the form $T(\zeta, t) = a(\zeta - s) + b(\zeta - s)^2$, where s is the melt front and the coefficients a and b are functions of time. We now choose temperature profile for equations (4.21) and (4.22) in the form:

$$T_{21}(\zeta, t) = A_{21}(t)(\beta^2(t) - \zeta^2) + \theta_{soft} \quad \text{in } 0 \le \zeta \le \beta(t), \tag{4.29}$$

$$T_{22}(\zeta, t) = 2\theta_{soft}(\beta(t) - \zeta)(\zeta - \pi/2) + \frac{\zeta - \pi/2}{\beta(t) - \pi/2}\theta_{soft} \quad \text{in } \beta(t) \le \zeta \le \frac{\pi}{2}, \tag{4.30}$$

which satisfy boundary conditions (4.24) and (4.25). From Stefan's condition (4.27) one finds the coefficient

$$A_{21}(t) = \frac{\lambda_{22}}{2\lambda_{21}\beta(t)}\left(\beta(t) - \frac{\pi}{2} + \frac{\theta_{soft}}{\beta(t) - \frac{\pi}{2}}\right). \tag{4.31}$$

Note that finding of the softening interface $\zeta = \beta(t)$ is identical to determining of the thermal layer. Hence, for $\theta_{21}(\zeta, t)$ we choose the region $0 \le \zeta \le \beta(t)$ as the thermal appropriate for this problem and for $\theta_{22}(\zeta, t)$ we integrate heat equation (4.22) from $\zeta = \beta(t)$ to $\zeta = \pi/2$ by substitution the quadratic profile $T_{22}(\zeta, t)$ into the equation

$$\int_{\beta(t)}^{\pi/2} \cos^4(\zeta)\left[\frac{\partial^2\theta_{22}}{\partial\zeta^2} + \omega_{22}^2\right] d\zeta = F(\beta(t)), \tag{4.32}$$

where

$$F(\beta(t)) = \frac{a_{22}^2(w_{22}^2 + 4\theta_{soft})\left[12\beta(t) - 6\pi + 8\sin(2\beta(t)) + \sin(4\beta(t))\right]}{32r_0^2}. \tag{4.33}$$

In view of the interface boundary conditions (4.25) and Stefan's condition (4.27) we get the ordinary differential equation

$$\left[\frac{(4\beta^2(t) - 4\pi\beta(t) + \pi^2 + 2)\theta_{soft}}{4}\right]\frac{d\beta(t)}{dt} = F(\beta(t)), \tag{4.34}$$

with the condition $\beta(t_{soft}) = 0$.

Figure 6 shows the temperature in the soft (red solid line) and the solid phases. In Figure 6(A) the plot shows the distribution of the temperature at the left boundary for $\theta_1(0, t)$ and $\theta_1(\beta(t_{melt}), t)$. Figure 6(B) shows the temperature at $\zeta = \beta(t_{melt})$ and the Figure 6(C) shows the space distribution of the temperature at the melting time t_{melt}. The temperature allocation at $t = t_{melt}$ presented in Figure 8(A) and the dynamics of the free boundary shown in Figure 8(B). The time t_{melt} required for the softening of filament in electrical contacts given in (4.21), (4.22), (4.23), (4.24), (4.25), (4.26) and (4.27) is equal to $6.913 \cdot 10^{-7} sec$.

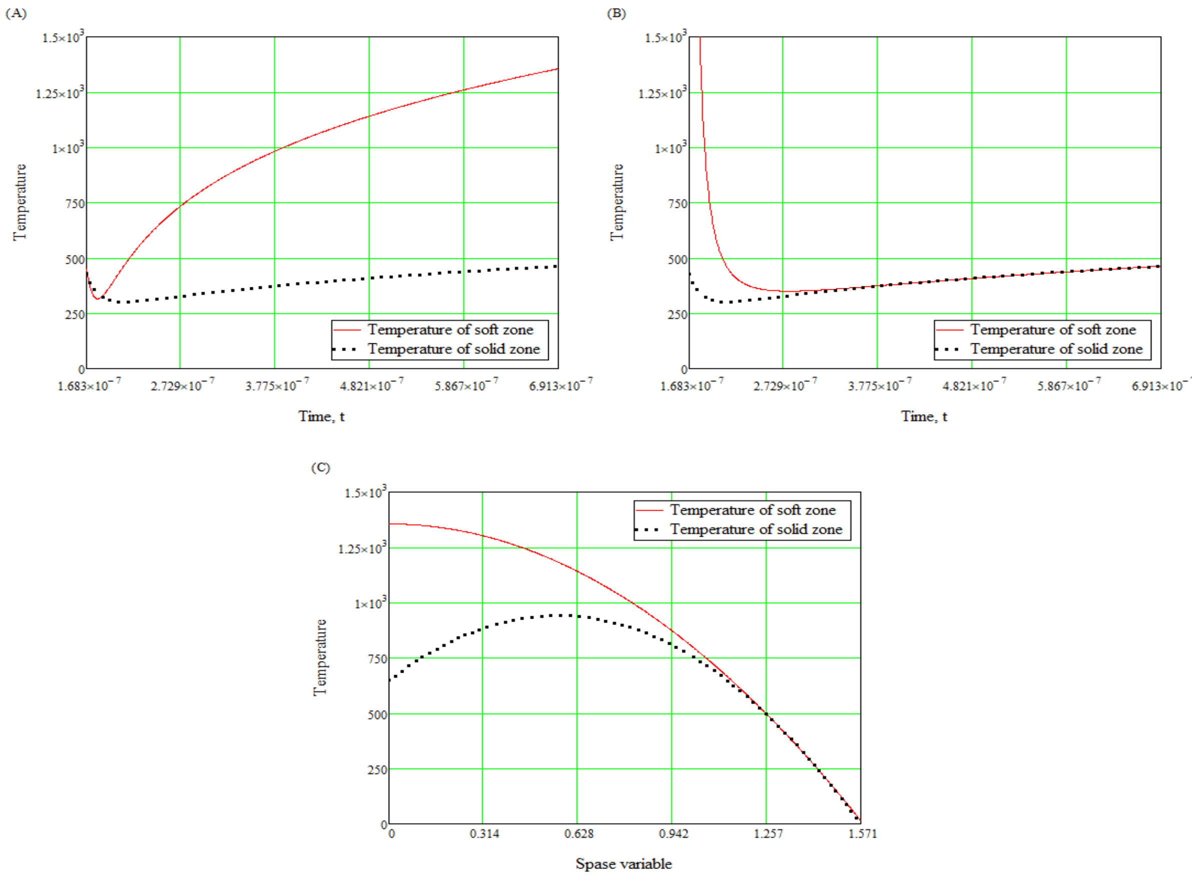

Figure 6: Dynamics of temperature in softening stage where the (solid red line) is the temperature of soft phase and the other is the temperature of solid phase, (A) $\theta_1(0,t)$, $\theta_2(\beta(t_{melt}),t)$, (B) $\theta_1(\beta(t_{melt}),t)$, $\theta_2(\beta(t_{melt}),t)$ and (C) $\theta_1(\zeta,t_{melt})$, $\theta_2(\zeta,t_{melt})$

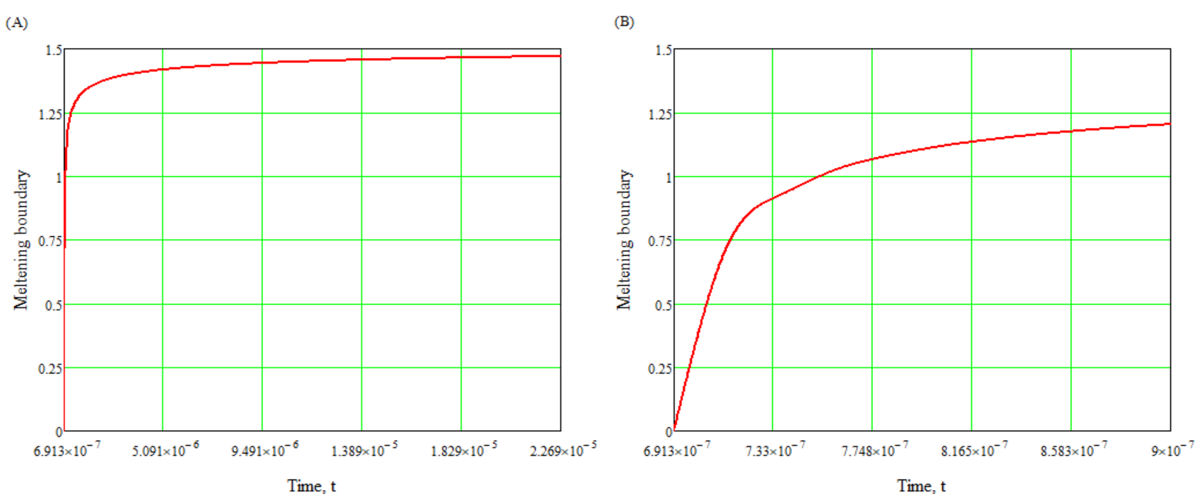

Figure 7: Distribution of the temperature $\theta(\zeta,t_{melt})$ (A) for all domain $\zeta \in \left[0,\frac{\pi}{2}\right]$ and (B) position of the melt-solid interface.

4.3 Stage 3

The third stage is characterized by formation of the third and last region D_1, where electrode begins to melt. The following model is proposed to describe this phenomena

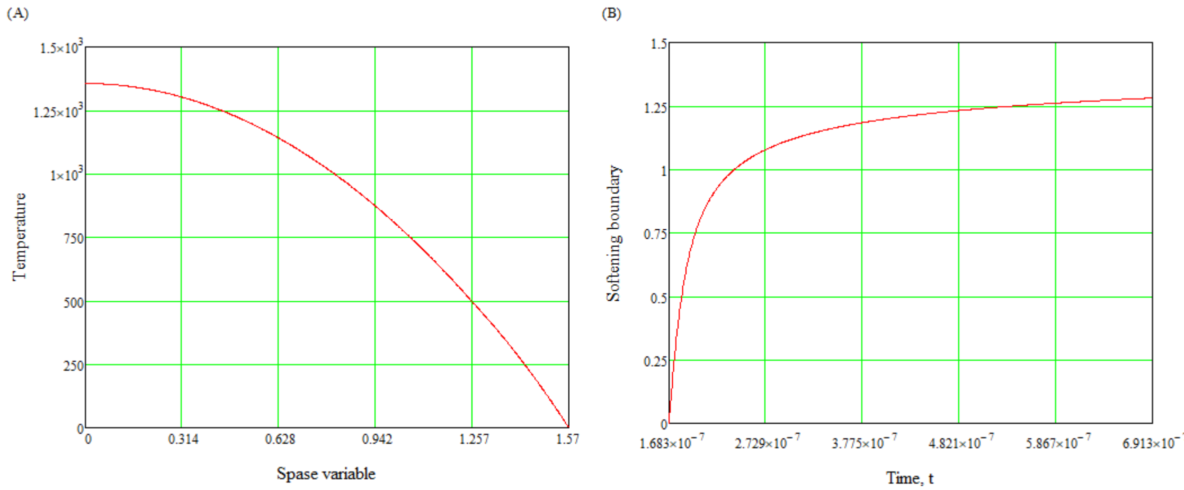

Figure 8: Distribution of the temperature $\theta(\zeta, t_{melt})$ (A) for all domain $\zeta \in \left[0, \frac{\pi}{2}\right]$ and (B) position of the soft-solid interface.

$$\frac{\partial \theta_{31}}{\partial t} = \frac{a_{31}^2}{r_0^2} \cos^4(\zeta) \left[\frac{\partial^2 \theta_{31}}{\partial \zeta^2} + \omega_{31}^2 \right], \quad 0 < \zeta < \alpha(t), \tag{4.35}$$

$$\frac{\partial \theta_{32}}{\partial t} = \frac{a_{32}^2}{r_0^2} \cos^4(\zeta) \left[\frac{\partial^2 \theta_{32}}{\partial \zeta^2} + \omega_{32}^2 \right], \quad \alpha(t) < \zeta < \frac{\pi}{2}, \tag{4.36}$$

$$\theta_{32}(0, t_{melt}) = \theta_{melt}, \quad \theta_{31}(\zeta, t_{melt}) = g(\zeta), \tag{4.37}$$

$$\left. \frac{\partial \theta_{31}}{\partial \zeta} \right|_{\zeta=0} = 0, \quad \theta_{32}|_{\zeta=\frac{\pi}{2}} = 0, \tag{4.38}$$

$$\theta_{31}|_{\zeta=\alpha(t)} = \theta_{32}|_{\zeta=\alpha(t)} = \theta_{melt}, \tag{4.39}$$

$$g(0) = \theta_{melt}, \quad \alpha(t_{melt}) = 0, \quad g\left(\frac{\pi}{2}\right) = 0, \tag{4.40}$$

and Stefan's condition

$$-\lambda_{31} \left. \frac{\partial \theta_{31}}{\partial \zeta} \right|_{\zeta=\alpha(t)} = -\lambda_{32} \left. \frac{\partial \theta_{32}}{\partial \zeta} \right|_{\zeta=\alpha(t)} + L\gamma \frac{d\alpha(t)}{dt}, \tag{4.41}$$

where

$$g(\zeta) = \theta_{21}(\zeta, t_{melt}), \quad \omega_{31} = \frac{I}{2\pi r_0} \sqrt{\frac{\rho_{21}}{c_{21}\gamma_{21}}}, \quad \omega_{32} = \frac{I}{2\pi r_0} \sqrt{\frac{\rho_{22}}{c_{22}\gamma_{22}}}. \tag{4.42}$$

For the last stage of contact heating we choose the temperature profile for equations (4.35) and (4.36) in the form:

$$T_{31}(\zeta, t) = A_{31}(t)(\alpha^2(t) - \zeta^2) + \theta_{melt} \quad \text{in } 0 \le \zeta \le \alpha(t), \tag{4.43}$$

$$T_{32}(\zeta, t) = 2\theta_{melt}(\alpha(t) - \zeta)(\zeta - \pi/2) + \frac{\zeta - \pi/2}{\alpha(t) - \pi/2}\theta_{melt} \quad \text{in } \alpha(t) \le \zeta \le \frac{\pi}{2}, \tag{4.44}$$

which satisfy boundary conditions (4.38) and (4.39). Goodman [2] used the alternative form of equation (4.41). Equation (4.39) is differentiated with respect to t. By using (4.35) and (4.41) we obtain

$$\lambda_{31} \left(\frac{\partial \theta_{31}}{\partial \zeta} \right)^2 = \lambda_{32} \frac{\partial \theta_{32}}{\partial \zeta} \frac{\partial \theta_{31}}{\partial \zeta} + L\gamma_{21} \frac{a_{31}^2}{r_0^2} \cos^4 \zeta \left[\frac{\partial^2 \theta_{31}}{\partial \zeta^2} + \omega_{31}^2 \right], \quad \zeta = \alpha(t), \tag{4.45}$$

The coefficient $A_{31}(t)$ can be determined by the above condition:

$$A_{31}(t) = \frac{2\lambda_{32}\alpha(t)\theta_{melt}\left(2\alpha(t) - \pi - \frac{1}{\alpha(t)-\pi}\right) - \psi(\alpha(t))}{8\lambda_{31}\alpha^2(t)} - \tag{4.46}$$

$$-\frac{\sqrt{\left[2\,\lambda_{32}\alpha(t)\theta_{melt}\left(2\alpha(t) - \pi - \frac{1}{\alpha(t)-\pi}\right) - \psi(\alpha(t))\right]^2 + 8\psi(\alpha(t))\lambda_{31}\alpha^2(t)w_{31}^2}}{8\lambda_{31}\alpha^2(t)}, \tag{4.47}$$

where $\psi(\alpha(t)) = 2L\gamma_{21}\frac{a_{31}^2}{r_0^2}\cos^4\alpha(t)$.

Integrating heat equation (4.36) from $\zeta = \alpha(t)$ to $\zeta = \frac{\pi}{2}$ we obtain the differential equation

$$\frac{d\alpha(t)}{dt} = \frac{a_{32}^2\left[w_{32}^2 - 4\theta_{melt}\right]\left(12\alpha(t) - 6\pi + 8\sin(2\alpha(t)) + \sin(4\alpha(t))\right)}{8\,r_0^2\,\theta_{melt}\left[4\alpha^2(t) - 4\pi\alpha(t) + \pi^2 + 2\right]}, \tag{4.48}$$

with the condition $\alpha(t_{melt}) = 0$.

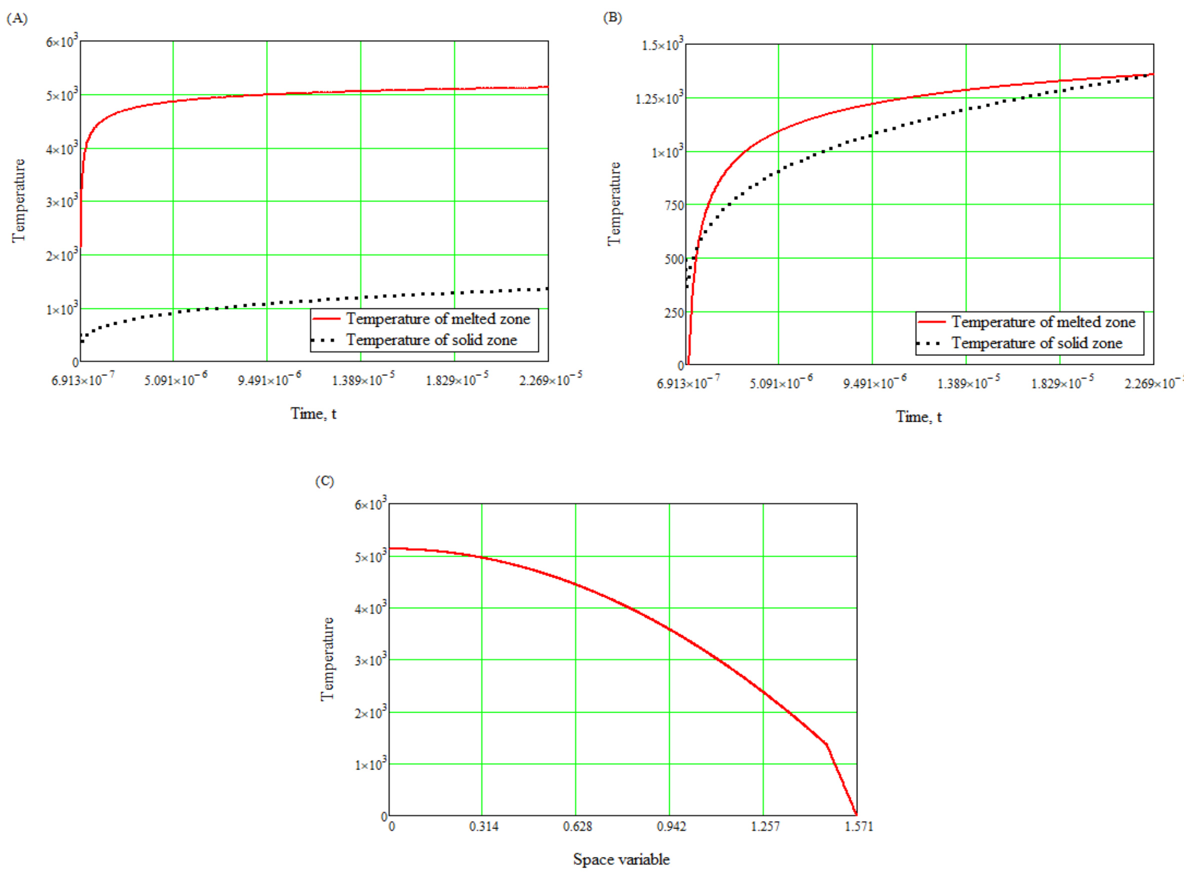

Figure 9: Dynamics of temperature in melting stage where the (solid red line) is the temperature of solid phase and the other is the temperature of solid phase, (A) $\theta_1(0,t)$, $\theta_2(\beta(t_{arc}),t)$, (B) $\theta_1(\beta(t_{arc}),t)$, $\theta_2(\beta(t_{arc}),t)$ and (C) $\theta_1(\zeta,t_{arc})$, $\theta_2(\zeta,t_{arc})$.

The temperature distribution on the melting stage and the free boundary are shown in Figure 7 and Figure 9. Figure 9 shows the temperature in the soft and solid phases. The dynamics of the free boundary is presented in Figure 7(A) and Figure 7(B). In Figure 9(A) the plot shows the distribution of the temperature at the left boundary for $\theta_1(0,t)$ and $\theta_1(\alpha(t_{arc}),t)$. Figure 9(B) shows the temperature at $\zeta = \alpha(t_{arc})$ and the Figure 9(C) shows the space distribution of the temperature at the melting time $t_{arc} = 2.269 \cdot 10^{-5} sec$.

5 Conclusion

In this paper, we consider the Stefan problem in ellipsoidal coordinates. The heating of closed electrical contacts in a quasi-stationary mode is described in ellipsoidal coordinates, which is shown in Section 3. The heat balance integral method is applied to the one-dimensional ellipsoidal Stefan problem with the Joule heat source. The results on the first stage of heating agreed well with the obtained exact solution. The error of approximation is less than 0.3 % for the classical second-degree polynomial temperature profile. The second and final stage results are also discussed in details. In fact, the HBI method can be effectively applied to the two-dimensional Stefan problem and also to the inverse Stefan problem.

Future work will concern extending the HPM presented in [13, 14, 15] to the evaporation process in electrical contacts in ellipsoidal coordinates.

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