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PROPAGATION OF NONSMOOTH WAVES UNDER SINGULAR PERTURBATIONS OF THE WAVE EQUATION

B.E. Kanguzhin

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Key words: delta-shaped perturbations, evolution equation, singular perturbations, wave equation.

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Abstract. The method of characteristics for the wave equation can be applied not only for unbounded strings. The method of incident and reflected waves is effectively used in the case of a mixed problem for a bounded string. This method can also be modified for multipoint mixed problems for the wave equation. In this paper, the method of incident and reflected waves is adapted for multi-point problems with discontinuous derivatives. An analogue of the d'Alembert formula for discontinuous multipoint problems for the wave equation in the case of a bounded string is proved.

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1 Introduction

Hamiltonians with point singularities arise in physical problems [13]. In the papers [17, 14, 20, 18] similar operators are investigated and physical applications of such operators are given. Mathematical questions of one-dimensional operators with singularities in the form of generalized functions have been studied in detail in [19, 12, 4, 16]. Delta-like perturbations of the multidimensional Laplace operator can be found in [6, 8, 9, 1]. In these papers, delta-shaped perturbations are explicitly described in an equivalent way through functional-boundary operators. An alternative approach to studying such perturbations is to describe the domain and their actions in terms of quadratic forms.

Let us first consider the Sturm-Liouville equation with a smooth potential $q(x)$ on the union of two disjoint intervals

$$-y''(x) + q(x)\, y(x) = f(x), \; x \in (0, x_0) \cup (x_0, 1). \tag{1.1}$$

If $f \in L_2(0,1)$, then there exist limit values

$$y(0),\; y'(0),\; y(x_0-0),\; y'(x_0-0),\; y(x_0+0),\; y'(x_0+0), y(1),\; y'(1).$$

If the communication channel between the intervals $(0, x_0)$ and $(x_0, 1)$ is consistent, then both differences $\{y(x_0+0) - y(x_0-0)\}$ и $\{y'(x_0+0) - y'(x_0-0)\}$ are equal to zero. If at least one of the noted differences is nonzero, then the communication channel is said to be distorted between intervals. In this case, the following functionals play important roles

$$\gamma_1(y) := y(x_0+0) - y(x_0-0),\; \gamma_2(y) := y'(x_0+0) - y'(x_0-0).$$

If the values of the above functionals are zero on solutions of equation (1.1), then the communication channel is consistent. Otherwise, the communication channel with distortions is open and the processes on the intervals $(0, x_0)$ and $(x_0, 1)$ interact according to inconsistent laws. Therefore, at the point $\{x_0\}$ there are given equations

$$\gamma_1(y) = a, \quad \gamma_2(y) = b \tag{1.2}$$

Thus, on the intervals $(0, x_0)$ and $(x_0, 1)$ differential equation (1.1) is given, and on the singleton $\{x_0\}$ algebraic equations (1.2) are given. In general, the operator defined on $(0, x_0) \cup \{x_0\} \cup (x_0, 1)$ is a functional differential operator. The introduced operator will be denoted by B_{max}.

Note that conditions (1.2) on the singleton $\{x_0\}$ simulate the presence of delta-shaped perturbations of the differential operator.

Inhomogeneous operator equation

$$B_{max} w = f, \ \forall f \in L_2(0,1) \tag{1.3}$$

has solutions in the corresponding natural class. In order for equation (1.3) to have a unique solution, it is necessary to restrict the domain of the operator B_{max}. Usually, the domain is restricted by adding boundary conditions. Let B_1 be one of such invertible restrictions of the maximal operator B_{max}.

Then it is of interest to study the one-dimensional evolution equation of the form

$$u_{tt} = B_1 u + g(t, x), \ t > 0, \ x \neq x_0 \tag{1.4}$$

with the initial conditions

$$u(0, \ x) = u_0(x), \quad u_t(0, \ x) = u_1(x), \quad x \neq x_0.$$

2 On a class of functions

Let k_1, k_2, k_3 be non-negative numbers. For further purposes, it is convenient to introduce the operator PC of odd periodic continuation, which maps each function $\omega(x)$ continuous on the segment $[0, 1]$ with the condition $\omega(0) = 0$ to the function $\varphi(x)$, defined in the following way

$$\varphi(x) = \omega(x), \ 0 < x < \frac{1}{8},$$

$$\varphi(x) = (1 + k_3)\omega(x) - k_3 \omega\left(\frac{1}{4} - x\right), \ \frac{1}{8} < x < \frac{1}{4},$$

$$\varphi(x) = (1 + k_2)(1 + k_3)\omega(x) + k_3(1 + 2k_2)\omega\left(x - \frac{1}{4}\right) - k_2(1 + k_3)\omega\left(\frac{1}{2} - x\right), \ \frac{1}{4} < x < \frac{1}{2},$$

$$\varphi(x) = (1 + k_1)(1 + k_2)(1 + k_3)\omega(x) + (1 + k_1)k_3(1 + 2k_2)\omega\left(x - \frac{1}{4}\right)$$
$$+ k_2(1 + k_3)(1 + 2k_1)\omega\left(x - \frac{1}{2}\right) - k_1 k_3(1 + 2k_2)\omega\left(\frac{3}{4} - x\right)$$
$$- k_1(1 + k_3)(1 + k_2)\omega(1 - x), \ \frac{1}{2} < x < \frac{3}{4},$$

$$\varphi(x) = (1 + k_1)(1 + k_2)(1 + k_3)\omega(x) + (1 + k_1)k_3(1 + 2k_2)\omega\left(x - \frac{1}{4}\right)$$
$$+ k_2(1 + k_3)(1 + 2k_1)\omega\left(x - \frac{1}{2}\right) + k_1 k_3(1 + 2k_2)\omega\left(x - \frac{3}{4}\right),$$

$$-k_1\left(1+k_3\right)\left(1+k_2\right)\omega\left(1-x\right), \ \frac{3}{4}\leq x<1.$$

The resulting function $\varphi(x)$ is denoted by $PC\omega(x)$. First, we extend the function $\varphi(x)$ in an odd way to the segment $[-1,0]$, and then we extend the function $\varphi(x)$ from the segment $[-1,1]$ with a period equal to two to the whole real axis. The function extended in this way will be denoted by $\widetilde{\varphi}(x)$. The set of all functions $\widetilde{\varphi}(x)$ constructed in this way is denoted by $D(k_1,k_2,k_3)$. Note that the function $\widetilde{\varphi}(x)$ is continuous on the real axis whenever $\varphi(1)=0$.

3 Mixed problem for the wave equation

Throughout this section we use the notation

$$\Omega=\left(0,\ \frac{1}{8}\right)\cup\left(\frac{1}{8},\ \frac{1}{4}\right)\cup\left(\frac{1}{4},\ \frac{1}{2}\right)\cup\left(\frac{1}{2},\ 1\right).$$

Consider the following mixed problem for the wave equation

$$u_{tt}\left(x,t\right)=u_{xx}\left(x,t\right),\ t>0,\ x\in\Omega \tag{3.1}$$

with the initial conditions

$$u\left(x,0\right)=\varphi\left(x\right),\quad u_t\left(x,0\right)=\psi\left(x\right),\ x\in\Omega, \tag{3.2}$$

and the homogeneous boundary conditions $u\left(0,t\right)=0,\quad u\left(1,t\right)=0$, as well as the initial-boundary conditions

$$u\left(\frac{1}{2^s}+0,\ t\right)=u\left(\frac{1}{2^s}-0,\ t\right),\ u_x\left(\frac{1}{2^s}+0,t\right)=\left(1+2k_s\right)u_x\left(\frac{1}{2^s}-0,t\right),\ s=1,2,3 \tag{3.3}$$

Conditions (3.3) can be interpreted as matching conditions at the internal vertices of the graph. A similar (only stationary) problem on graphs was studied in [7]. Further, we will assume that the initial data $\varphi(x),\quad \psi(x),\ x\in\Omega$ are selected from the set $D(k_1,k_2,k_3)$.

Theorem 3.1. *Let $\varphi\left(x\right),\psi\left(x\right)\in\ D(k_1,k_2,\ k_3)$. Assume that $\varphi\left(1\right)=\ \psi\left(1\right)=0$. Then the solution of the problem* (3.1)-(3.3) *with the indicated boundary and initial-boundary conditions has a unique solution that has the representation*

$$u\left(x,t\right)=\frac{1}{2}\widetilde{\varphi}\left(x+t\right)+\frac{1}{2}\widetilde{\varphi}\left(x-t\right)+\frac{1}{2}\int_{x-t}^{x+t}\widetilde{\psi}\left(\xi\right)d\xi$$

where $\widetilde{\varphi}$ is a 2-periodic extension to the whole real axis of the function φ and $\widetilde{\psi}$ is a 2-periodic extension to the whole real axis of the function ψ.

A similar mixed multipoint problem for the wave equation with only smooth initial data was studied in [5]. The main result of [5] is the extension of the d'Alembert formula for multipoint problems with smooth data. Theorem 3.1 generalized the d'Alembert formula for mixed problems for the wave equation with nonsmooth solutions. Note that in Theorem 3.1 a new class of initial data $D(k_1,k_2,\ k_3)$ is introduced and described.

Let us give some examples to clarify the meaning of Theorem 3.1. Discontinuous solutions of the wave equation in gas dynamics are called shock waves. It is interesting to trace the propagation of shock waves.

Example 1. In the case $k_1 = k_2 = k_3 = 0$ the set $D(0,0,0)$ consists of continuously differentiable functions on the interval $[0,1]$. If $\varphi(1) = \psi(1) = 0$, then a smooth solution of mixed problem (3.4)-(3.5) with the conditions $u(0,t) = 0, \quad u(1,t) = 0$ has the form

$$u(x,t) = \frac{1}{2}\widetilde{\varphi}(x+t) + \frac{1}{2}\widetilde{\varphi}(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{\psi}(\xi) d\xi.$$

The geometric interpretation of the obtained solution can be found in [10].

Example 2. Let $k_2 = k_3 = 0, \; k_1 \neq 0$. In this case, the class of initial data $D(k_1,0,0)$ consists of functions of the form

$$\varphi(x) = \omega(x) \;, \; 0 < x < \frac{1}{2},$$
$$\varphi(x) = (1+k_1)\,\omega(x) - k_1\omega(1-x), \; \frac{1}{2} < x < 1,$$

where $\omega(x)$ is an arbitrary continuously differentiable function on $[0,1]$ with the condition $\omega(0) = 0$. Then the solution of the mixed problem in Theorem 3.1 will be continuous whenever $\varphi(1) = \psi(1) = 0$. However, its derivative with respect to the variable x will have discontinuities due to a similar discontinuity of the function $\varphi(x)$ at the point $x = \frac{1}{2}$. The initial discontinuity extends along the characteristics.

Proof of Theorem 3.1. Let us prove Theorem 3.1 for $\psi(x) \equiv 0$. Since the standard procedure in [11] allows us to formulate and prove the theorem for other nontrivial $\psi(x)$.

In the proof of Theorem 3.1, we essentially use the spectral properties of the following eigenvalue problem.

$$-y''(x) = \lambda y(x) \quad x \in \left(0, \; \frac{1}{8}\right) \cup \left(\frac{1}{8}, \; \frac{1}{4}\right) \cup \left(\frac{1}{4}, \; \frac{1}{2}\right) \cup \left(\frac{1}{2}, \; 1\right) \tag{3.4}$$

with the conditions $y(0) = 0, \quad y(1) = 0,$

$$y\left(\frac{1}{2^s}+0\right) = y\left(\frac{1}{2^s}-0\right), \; y'\left(\frac{1}{2^s}+0\right) = (1+2k_s)\,y'\left(\frac{1}{2^s}-0\right), \; s = 1,2,3. \tag{3.5}$$

Notation and facts about the problem required for the proof of (3.4)-(3.5) are given in Appendix 1 below.

We expand the initial function $\varphi(x)$ along the system of eigenfunctions $\left\{s\left(x,\rho_n^{(k)}\right), \; k = 1,2,3,4, \quad n \geq 0\right\}$.

$$\varphi(x) = \sum_{k=1}^{4}\sum_{n=0}^{\infty} c_n^{(k)} \; s\left(x,\rho_n^{(k)}\right), \tag{3.6}$$

where $c_0^{(k)} = 0, \; c_n^{(k)} = \frac{A}{B}, \; A = \int_0^1 \varphi(x) s\left(x,\rho_n^{(k)}\right) dx, \quad B = \int_0^1 s\left(x,\rho_n^{(k)}\right) s\left(x,\rho_n^{(k)}\right) dx.$

Note that series (3.6) converges uniformly on Ω.

We seek the solution of the problem in the form

$$u(x,t) = \sum_{k=1}^{4}\sum_{n=0}^{\infty} d_n^{(k)}(t) \; s\left(x,\rho_n^{(k)}\right). \tag{3.7}$$

The standard procedure in [11] allows us to write out the coefficients in the following form

$$d_n^{(k)}(t) = c_n^{(k)} cos \rho_n^{(k)} t. \tag{3.8}$$

We need the following obvious lemma.

Lemma 3.1. *For $x \in \Omega$ the following formula is valid*

$$2\cos\left(\rho_n^{(k)} t\right) \quad s\left(x, \rho_n^{(k)}\right) = \tilde{s}\left(x+t, \rho_n^{(k)}\right) + \tilde{s}\left(x-t, \rho_n^{(k)}\right),$$

where $\tilde{s}$ is a 2-periodic extension to the whole real axis of the function s.

Lemma 3.1 follows from the formula $2\cos(t) \quad sin\,(x) = sin\,(x+t) + sin\,(x-t)$. Putting relations (3.8) into equality (3.7) and taking into account Lemma 2, we have

$$u(x,t) = \frac{1}{2}\sum_{k=1}^{4}\sum_{n=0}^{\infty} \mathrm{c}_n^{(k)} \widetilde{s}\left(x+t, \rho_n^{(k)}\right) + \frac{1}{2}\sum_{k=1}^{4}\sum_{n=0}^{\infty} \mathrm{c}_n^{(k)} \widetilde{s}\left(x-t, \rho_n^{(k)}\right).$$

This implies the following representation

$$u(x,t) = \frac{1}{2}\widetilde{\varphi}(x+t) + \frac{1}{2}\widetilde{\varphi}(x-t),$$

where $\widetilde{\varphi}$ is a 2-periodic extension to the whole real axis of the function φ. The uniqueness of the solution to problem (3.1)-(3.3) is proved by the method given in the work of V.A. Il'in [5]. This completes the proof of Theorem 3.1 in the case $\psi(x) \equiv 0$. $\square$

4 Appendix. Spectral properties of problem (3.4)-(3.5)

It is convenient to introduce the solution of equation (3.4) with $\lambda = \rho^2$ defined by the formula

$$s(x,\rho) = \frac{\sin(\rho x)}{\rho}, \quad 0 < x < \frac{1}{8},$$

$$s(x,\rho) = (1+k_3)\frac{\sin(\rho x)}{\rho} + k_3 \frac{\sin(\rho\left(x-\frac{1}{4}\right))}{\rho}, \quad \frac{1}{8} < x < \frac{1}{4},$$

$$s(x,\rho) = (1+k_2)(1+k_3)\frac{\sin(\rho x)}{\rho} + k_3(1+2k_2)\frac{\sin(\rho\left(x-\frac{1}{4}\right))}{\rho} + k_2(1+k_3)\frac{\sin(\rho\left(x-\frac{1}{2}\right))}{\rho},$$
$$\frac{1}{4} < x < \frac{1}{2},$$

$$s(x,\rho) = (1+k_1)(1+k_2)(1+k_3)\frac{\sin(\rho x)}{\rho} + (1+k_1)k_3(1+2k_2)\frac{\sin(\rho\left(x-\frac{1}{4}\right))}{\rho}$$
$$+k_2(1+k_3)(1+2k_1)\frac{\sin(\rho\left(x-\frac{1}{2}\right))}{\rho} + k_1k_3(1+2k_2)\frac{\sin(\rho\left(x-\frac{3}{4}\right))}{\rho}$$
$$+k_1(1+k_3)(1+k_2)\frac{\sin(\rho(x-1))}{\rho}, \quad \frac{1}{2} < x < 1,$$

The solution $s(x,\rho)$ is chosen so that conditions (3.5) and $s(0,\rho) = 0$ are satisfied. Therefore, the zeros of the function

$$\Delta(\lambda) = (1+k_1)(1+k_2)(1+k_3)\frac{\sin(\rho)}{\rho} + (1+k_1)k_3(1+2k_2)\frac{\sin(\frac{3\rho}{4})}{\rho}$$
$$+k_2(1+k_3)(1+2k_1)\frac{\sin(\frac{\rho}{2})}{\rho} + k_1k_3(1+2k_2)\frac{\sin(\frac{\rho}{4})}{\rho},$$

where $\lambda = \rho^2$ are the eigenvalues of the initial problem. Note that $\Delta(0) > 0$. We transform the function $\Delta(\lambda)$ to the form

$$\Delta(\lambda) = \frac{\sin(\frac{\rho}{4})}{\rho}\Big\{k_1k_3(1+2k_2) + 2k_2(1+k_3)(1+2k_1)(\cos\left(\frac{\rho}{4}\right)$$
$$+ (1+k_1)k_3(1+2k_2)\left(4\cos^2\left(\frac{\rho}{4}\right) - 1\right)$$
$$+ 4(1+k_1)(1+k_2)(1+k_3)\cos\left(\frac{\rho}{4}\right)\left(2\cos^2\left(\frac{\rho}{4}\right) - 1\right)\Big\}.$$

Hence it follows that it is necessary to investigate the cubic equation

$$\begin{aligned} \mathrm{P(z)}\equiv 8\,(1+k_1)\,(1+k_2)\,(1+k_3)\,z^3 + 4\,(1+k_1)\,k_3\,(1+2k_2)\,z^2 \\ -2\,(1+k_3)\,(2+k_1)\,z - k_3\,(1+2k_2) = 0 \end{aligned} \tag{4.1}$$

Note that

$$\mathrm{P}\,(1) = 16k_1k_2k_3 + 8k_1k_2 + 10k_1k_3 + 14k_3k_2 + 6k_1 + 8k_2 + 7k_3 + 4 > 0,$$

$$\mathrm{P}\,(-1) = -8k_1k_2 - 2k_1k_3 - 2k_3k_2 - 6k_1 - 8k_2 - k_3 - 4 < 0.$$

The roots of cubic equation (4.1) cannot be multiple, since the quadratic equation for its derivative

$$24z^2 + \frac{8\,(1+k_1)\,k_3\,(1+2k_2)}{(1+k_1)\,(1+k_2)\,(1+k_3)}z^1 - \frac{2\,(1+k_3)\,(2+k_1)}{(1+k_1)\,(1+k_2)\,(1+k_3)} = 0$$

has two different real roots. Let us prove that the indicated cubic equation has three real roots and they all lie in the interval $(-1,1)$. Let z_1 be a root of the cubic equation (4.1), which is either complex or lies outside the interval $(-1,1)$. Then it follows from the relation

$$\cos\left(\frac{\rho}{4}\right) = z_1$$

that the initial eigenvalue problem has complex eigenvalues. But this contradicts the fact that the original problem is self-adjoint (see [21]).

Thus, the cubic equation has three different roots z_1, z_2, z_3 from the interval $(-1,1)$. Therefore, there are three series of eigenvalues

$$\lambda_n^{(k)} = 16\ \left(2\pi n \mp \arccos\ z_k\right)^2,\ k = 1,2,3.$$

The fourth series of eigenvalues follows from the equation $\sin(\frac{\rho}{4}) = 0$ and has the form

$$\lambda_n^{(4)} = 16\ \left(\pi n\right)^2,\ n > 0.$$

For each series of eigenvalues, one can write down the eigenfunctions $s(x,\rho_n^{(k)}), k = 1,2,3,4$, where $\rho_n^{(k)} = 4(2\pi n \pm \arccos z_k), k = 1,2,3, \rho_n^{(4)} = 4\pi n$. Let us present useful properties of the system of eigenfunctions $\left\{s\left(x,\rho_n^{(k)}\right),\ k = 1,2,3,4,\quad n \geq 0\right\}$.

Lemma 4.1. *The system of eigenfunctions* $\left\{s\left(x,\rho_n^{(k)}\right),\ k = 1,2,3,4,\quad n \geq 0\right\}$ *forms an orthogonal basis in the space* $L_2(0,1)$.

Proof. Take two arbitrary eigenvalues $\lambda_n^{(k)}$ and $\{\lambda\}_m^{(t)}$. Consider the product

$$(\lambda_n^{(k)} - \lambda_m^{(t)}) \int_0^1 s\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right) dx$$

$$= \int_0^1 s\left(x, \rho_n^{(k)}\right) s''\left(x, \rho_m^{(t)}\right) dx - \int_0^1 s''\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right) dx$$

$$= \left(s\left(x, \rho_n^{(k)}\right) s'\left(x, \rho_m^{(t)}\right) - s'\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right)\right)\Big|_0^{\frac{1}{8}}$$

$$+ \left(s\left(x, \rho_n^{(k)}\right) s'\left(x, \rho_m^{(t)}\right) - s'\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right)\right)\Big|_{\frac{1}{8}}^{\frac{1}{4}}$$

$$+ \left(s\left(x, \rho_n^{(k)}\right) s'\left(x, \rho_m^{(t)}\right) - s'\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right)\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$+ \left(s\left(x, \rho_n^{(k)}\right) s'\left(x, \rho_m^{(t)}\right) - s'\left(x, \rho_n^{(k)}\right) s\left(x, \rho_m^{(t)}\right)\right)\Big|_{\frac{1}{2}}^{1} = 0.$$

Here, we take into account conditions (3.5) and the fact that

$$s\left(0, \rho_n^{(k)}\right) = 0, \quad s\left(1, \rho_n^{(k)}\right) = 0, \quad s\left(0, \rho_m^{(t)}\right) = 0, \quad s\left(1, \rho_m^{(t)}\right) = 0.$$

The obtained relation implies the orthogonality of the system. The basicity of the system follows from the fact that the initial problem is self-adjoint.

The system of eigenfunctions $\left\{s\left(x, \rho_n^{(k)}\right), k = 1, 2, 3, 4, \quad n \geq 0\right\}$ is actually defined on the whole axis, not just on the union of intervals $\left(0, \ \frac{1}{8}\right) \cup \left(\frac{1}{8}, \ \frac{1}{4}\right) \cup \left(\frac{1}{4}, \ \frac{1}{2}\right) \cup \left(\frac{1}{2}, \ 1\right)$. Let us explain how the function s defined on $\left(0, \ \frac{1}{8}\right)\cup\left(\frac{1}{8}, \ \frac{1}{4}\right)\cup\left(\frac{1}{4}, \ \frac{1}{2}\right)\cup\left(\frac{1}{2}, \ 1\right)$ is continued to $\left(-\frac{1}{8}, 0\right)\cup\left(-\frac{1}{4}, -\frac{1}{8}\right)\cup\left(-\frac{1}{2}, -\frac{1}{4}\right)\cup\left(-1, -\frac{1}{2}\right)$. For

$$x \in \left(-\frac{1}{8}, 0\right) \cup \left(-\frac{1}{4}, -\frac{1}{8}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-1, -\frac{1}{2}\right)$$

we assume that $\tilde{s}\left(x, \rho_n^{(k)}\right) = -s\left(-x, \rho_n^{(k)}\right)$, that is, through the point $x = 0$ we have an odd continuation. Then $s\left(-1 + 0, \rho_n^{(k)}\right) = 0$, $s'\left(-1 + 0, \rho_n^{(k)}\right) = s'\left(1 - 0, \rho_n^{(k)}\right)$. Further from the set $\left(-\frac{1}{8}, 0\right) \cup \left(-\frac{1}{4}, -\frac{1}{8}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-1, -\frac{1}{2}\right) \cup \left(0, \ \frac{1}{8}\right) \cup \left(\frac{1}{8}, \ \frac{1}{4}\right) \cup \left(\frac{1}{4}, \ \frac{1}{2}\right) \cup \left(\frac{1}{2}, \ 1\right)$ on the whole real axis we continue periodically with a period is equal to two. The continuation will be a smooth function at integer points, that is, the continued function is continuous at integer points together with the first derivative. As is known from [2], an arbitrary function $\varphi(x)$ satisfying conditions (3.2) and $\varphi(0) = 0$, $\varphi(1) = 0$, expands into a uniformly convergent series along the system $\left\{s\left(x, \rho_n^{(k)}\right), \ k = 1, 2, 3, 4, \quad n \geq 0\right\}$. Therefore, the function $\varphi(x)$ will be first continued oddly to $(-\frac{1}{8}, 0) \cup (-\frac{1}{4}, -\frac{1}{8}) \cup (-\frac{1}{2}, -\frac{1}{4}) \cup (-1, -\frac{1}{2})$ and then 2-periodically to the whole real axis. The extended function is denoted by $\widetilde{\varphi}(x)$. Moreover, the expansion of $\varphi(x)$ along the system $\left\{s\left(x, \rho_n^{(k)}\right), \ k = 1, 2, 3, 4, \quad n \geq 0\right\}$ is preserved for the function $\widetilde{\varphi}(x)$. □

In Lemma (4.1), we study the properties of the root functions of the operator of double differentiation with conditions (3.3). The properties of root functions with more general integral conditions were studied in [3]. Let us give some examples when the above reasoning leads to explicit formulas.

Example 3. If $k_1 = k_2 = k_3 = 0$, then $s(x, \rho) = \frac{\sin(\rho x)}{\rho}$ $0 < x < 1$. In this case, all four series of eigenvalues can be combined into one $\rho_n^{(0)} = \pi n$, $n > 0$. The system of eigenfunctions will take the form $\{\sin(\pi n x)\ , \ n > 0\}$.

Example 4. Let $k_1 = k_3 = 0,\ k_2 \neq 0$. Then

$$s(x,\rho) = \frac{\sin(\rho x)}{\rho}\ ,\ 0 < x < \frac{1}{4},$$

$$s(x,\rho) = (1+k_2)\,\frac{\sin(\rho x)}{\rho} + k_2\frac{\sin(\rho\left(x-\frac{1}{2}\right))}{\rho},\ \frac{1}{4} < x < 1.$$

In this case, there are two series of eigenvalues

$$\rho_n^{(0)} = 2\pi n,\ n > 0,\ \rho_n^{(1)} = 2\pi(2n+1) \mp 2arccos\frac{k_2}{2(\ 1+k_2)},\ n \geq 0.$$

Example 5. Let $k_2 = k_3 = 0,\ k_1 \neq 0$. Then

$$s(x,\rho) = \frac{\sin(\rho x)}{\rho}\ ,\ 0 < x < \frac{1}{2},$$

$$s(x,\rho) = (1+k_1)\,\frac{\sin(\rho x)}{\rho} + k_1\frac{\sin(\rho\,(x-1))}{\rho},\ \frac{1}{2} < x < 1.$$

In this case, all four series of eigenvalues can be combined into one $\rho_n^{(0)} = \pi n,\ n > 0$. The system of eigenfunctions will take the form $\{c_n(x)\sin(\pi n x)\ ,\ n > 0\}$, where $c_n(x)$ is a piecewise constant function for fixed n.

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