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$\varphi-$APPROXIMATE BIPROJECTIVE AND $\varphi-$APPROXIMATE AMENABLE BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra and φ be a continuous homomorphism on A. We define the notions of a $\varphi-$approximate biprojective and $\varphi-$approximate amenable Banach algebra A, and consider relations between them and some their properties.

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1 Introduction

An amenable Banach algebra was introduced by Johnson in [8]. He showed that A is an amenable Banach algebra if and only if A has an approximate diagonal, that is, a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha a - a m_\alpha \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$, for all $a \in A$, where π denotes the product morphism from $A \hat{\otimes} A$ to A given by $\pi(a \otimes b) = ab$ for all $a, b \in A$. The notion of a biprojective Banach algebra was introduced by Helemskii [6]. In [6], A is called biprojective, if there exists a bounded A-bimodule map $\theta : A \longrightarrow A \hat{\otimes} A$ such that $\pi \circ \theta = id_A$, where id_A is the identity operator on A.

Recently, some authors have added a type of twist to the amenability definition. Given a continuous homomorphism φ from A to A, they defined and studied $\varphi-$derivations and $\varphi-$amenability (see [10] and [13]). Suppose that A is a Banach algebra and $\varphi \in Hom(A)$, where $Hom(A)$ denotes the set of all continuous homomorphisms from A to A, and X is a Banach $A-$bimodule. A linear operator $D : A \longrightarrow X$ is called a φ-derivation if $D(ab) = D(a)\varphi(b) + \varphi(a)D(b)$, for all $a, b \in A$. A $\varphi-$derivation D is called a $\varphi-$inner derivation if there is $x \in X$ such that $D(a) = \varphi(a)x - x\varphi(a)$, for all $a \in A$. Let $\mathcal{Z}^1_\varphi(A, X)$ be the set of all continuous $\varphi-$derivations and $\mathcal{N}^1_\varphi(A, X)$ be the set of all $\varphi-$inner derivations from A into X. The first cohomology group is $\mathcal{H}^1_\varphi(A, X) = \mathcal{Z}^1_\varphi(A, X)/\mathcal{N}^1_\varphi(A, X)$. A Banach algebra A is called $\varphi-$*amenable* if $\mathcal{H}^1_\varphi(A, X^*) = \{0\}$, for all $A-$bimodules X. Note that every derivation of a Banach algebra A into an A-bimodule X is an id_A-derivation.

Motivated by these considerations, the author and M. Lashkarizadeh Bami introduced some generalizations of Helemskii´s concept such as a $\varphi-$approximately biflat and $\varphi-$pseudo amenable Banach algebra. In [4] it is said that a Banach algebra A is $\varphi-$approximately biflat if there is a net $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)^{**}$ $(\alpha \in I)$ of bounded $A-$bimodule morphisms such that $\pi^{**} \circ \theta_\alpha \circ \varphi(a) \to \varphi(a)$. Also, A is called $\varphi-$pseudo amenable if it admits a $\varphi-$approximate virtual diagonal, i.e., there exists a net $(m_\alpha) \subset A \hat{\otimes} A$ (not necessarily with bounded) such that $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$, for all $a \in A$.

In this paper, we first introduce new definition of a $\varphi-$approximate biprojective and a $\varphi-$approximate amenable Banach algebra A, where φ is a continuous homomorphism on A. It is shown that A is $\varphi-$approximately amenable if A is $\varphi-$approximate biprojective and it has a central approximate identity. We define a new concept of a $\varphi-$approximate biprojective for Banach algebras. We show difference between this new concept and the classical one. It should be emphasized that $\varphi-$approximate biprojective in the sense of definition in [5] is a slight generalization of the notion of approximate biprojective. In Example 3 we construct a $\varphi-$approximate biprojective Banach algebra which is not a $\varphi-$approximate amenable Banach algebra. If φ is an idempotent homomorphism then the definition of a $\varphi-$approximate biprojective in [5] results in a new definition.

2 Main results

Let A be a Banach algebra and X, Y be Banach $A-$bimodules. Then a $A-$bimodule morphism from X to Y is a morphism $\varphi : X \longrightarrow Y$ satisfying

$$\varphi(a\cdot x) = a\cdot\varphi(x), \quad \varphi(x\cdot a) = \varphi(x)\cdot a \ \ (\forall a\in A,\ \forall x\in X).$$

Definition 1. Let A be a Banach algebra and $\varphi \in Hom(A)$. We say that A is $\varphi-$approximate biprojective if there exists a continuous $A-$bimodule homomorphism $\theta_\alpha : A \longrightarrow (A\hat{\otimes} A)$ such that $\pi\circ\theta_\alpha\circ\varphi(a)\rightarrow\varphi(a)$, $\forall a\in A$.

Proposition 2.1. *Let A be a Banach algebra and $\varphi\in Hom(A)$. If A has an approximate diagonal, then A is $\varphi-$approximate biprojective.*

Proof. Suppose that a net (m_α) in $(A\hat{\otimes} A)$ is an approximate diagonal. Define $\theta_\alpha : A \longrightarrow (A\hat{\otimes} A)$ by $a\mapsto a\cdot m_\alpha$ $(a\in A)$. Then for every $a\in A$, we have

$$\begin{aligned}\pi\circ\theta_\alpha\circ\varphi(a) &= \pi\circ\theta_\alpha(\varphi(a))\\ &= \pi(\varphi(a)\cdot m_\alpha)\\ &= \varphi(a)\cdot\pi(m_\alpha)\longrightarrow\varphi(a).\end{aligned}$$

□

Since the proof of the following result is similar to the proofs of theorems in [5], we omit it.

Theorem 2.1. *Suppose that A is a $\varphi-$approximate biprojective Banach algebra. If I is a closed ideal of A with one sided bounded approximate identity such that $\varphi(I)\subset I$, then I is $\varphi|_I-$approximate biprojective, where $\varphi|_I$ is the restriction of φ to I.*

In the next result, $\varphi : A\longrightarrow A$ is a homomorphism and I is a closed ideal of A which is φ-invariant, that is, $\varphi(I)\subset I$, and also we consider the map $\tilde{\varphi} : A/I \longrightarrow A/I$ defined by $\tilde{\varphi}(a+I) = \varphi(a)+I$.

Theorem 2.2. *Suppose that A is a $\varphi-$approximate biprojective Banach algebra with one sided bounded approximate identity. If I is a closed ideal of A, then A/I is $\tilde{\varphi}-$approximately biprojective.*

We quote the following result from [12].

Lemma 2.1. *Let A be a Banach algebra. Then there exists an $A-$bimodule homomorphism γ : $(A\hat{\otimes} A)^* \longrightarrow (A^{**}\hat{\otimes} A^{**})^*$ such that for any functional $f\in(A\hat{\otimes} A)^*$, all elements $\varphi,\psi\in A^{**}$ and nets $(a_\alpha),(b_\beta)$ in A with $w^*-lim_\alpha a_\alpha = \varphi$ and $w^*-lim_\beta b_\beta=\psi$, we have*

$$\gamma(f)(\varphi\otimes\psi) = lim_\alpha lim_\beta f(a_\alpha\otimes b_\beta).$$

Theorem 2.3. *Suppose that A is a Banach algebra and $\varphi\in Hom(A)$. If A^{**} is $\varphi^{**}-$approximate biflat, then A is $\varphi-$approximate biflat.*

For the case of $\varphi-$approximate biprojective, we have the following partial result which is an easy implication of Theorem 2.2.

Corollary 2.1. *Suppose that A is a Banach algebra and $\varphi \in Hom(A)$. If A^{**} is $\varphi^{**}-$approximate biprojective such that A is an ideal in A^{**} and A has a one sided bounded approximate identity, then A is $\varphi-$approximate biprojective.*

Proposition 2.2. *Let A be a $\varphi-$approximate biprojective Banach algebra. Let B be a $\psi-$approximate biprojective Banach algebra with $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \hat{\otimes} B$ is $\varphi\otimes\psi-$approximate biprojective.*

Proof. There exists an $A-$bimodule map $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)$ $(\alpha \in \Delta)$ with $\lim_\alpha \pi \circ \theta_1 \circ \varphi(a) = \varphi(a)$ and $B-$bimodule map $\theta_\beta : B \longrightarrow (B \hat{\otimes} B)$ $(\beta \in I)$ with $\lim_\beta \pi \circ \theta_2 \circ \varphi(b) = \varphi(b)$. Let $\theta_0 : (A \hat{\otimes} A)\hat{\otimes}(B \hat{\otimes} B) \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)$ be the isometric isomorphism given by $(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$ $(a_1, a_2 \in A,\ b_1, b_2 \in B)$. Let $E = I\times \Delta^I$ be directed by the product ordering and for each $\lambda = (\beta, \alpha) \in E$, define $\theta_\lambda = \theta_0 \circ (\theta_\alpha \otimes \theta_\beta)$. Using the iterated limit theorem [14, Theorem 2.4] the above calculation gives for $a \otimes b \in A \hat{\otimes} B$

$$\begin{aligned}
\pi \circ \theta_\lambda \circ (\varphi \otimes \psi)(a \otimes b) &= \pi \circ \theta_0 \circ (\theta_\alpha \otimes \theta_\beta) \circ (\varphi \otimes \psi)(a \otimes b) \\
&= \pi \circ \theta_0 \circ (\theta_\alpha \otimes \theta_\beta)(\varphi(a) \otimes \psi(b)) \\
&= \pi \circ \theta_0(\theta_\alpha(\varphi(a)) \otimes \theta_\beta(\psi(b))) \\
&= \pi \circ \theta_\alpha \circ \varphi(a) \otimes \pi \circ \theta_\beta \circ (\psi(b)) \\
&\longrightarrow a \otimes b.
\end{aligned}$$

Therefore, $A \hat{\otimes} B$ is $\varphi \otimes \psi-$approximate biprojective. □

The proof of the following result is similar to that of Proposition 2.2.

Proposition 2.3. *Let A be a $\varphi-$approximate biprojective Banach algebra. Let B be a $\psi-$approximate biprojective Banach algebra with $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \oplus B$ is $\varphi\oplus\psi-$approximate biprojective.*

Definition 2. *Let A be a Banach algebra and $\varphi \in Hom(A)$. We say that A is $\varphi-$contractible if it has a $\varphi-$diagonal, i.e., there is an element $m \in A \hat{\otimes} A$ for which $\varphi(a) \cdot m = m \cdot \varphi(a)$ and $\pi(m) \cdot \varphi(a) = \varphi(a)$, for all $a \in A$.*

Recall that in [4], A is $\varphi-$pseudo contractible if it has a central $\varphi-$approximate diagonal, i.e., there is a $\varphi-$approximate diagonal (m_α) satisfying $\varphi(a)m_\alpha = m_\alpha\varphi(a)$ for all $a \in A$ and all α.

A net $(e_\alpha)_{\alpha\in I}$ in A is central if $ae_\alpha = e_\alpha a$ for all $a \in A$, $\alpha \in I$.

Theorem 2.4. *Let A be $\varphi-$pseudo contractible and unital. Then A is $\varphi-$contractible.*

Proof. Since A is $\varphi-$pseudo contractible, so there is a central $\varphi-$approximate diagonal $m_\alpha \in A \hat{\otimes} A$. Let e_A be the unit element, then define $a_\alpha = \pi(m_\alpha) \otimes e_A - m_\alpha$. Thus

$$\pi(a_\alpha) = \pi(\pi(m_\alpha) \otimes e_A - m_\alpha) = \pi(m_\alpha) - \pi(m_\alpha) = 0,$$

and $a_\alpha \in \ker(\pi)$. Since $\pi(m_\alpha)\varphi(a) \longrightarrow \varphi(a)$, the right multiplication operator R_{a_α} converges to the identity operator id in $B(\ker(\pi))$. Hence, R_{a_α} is invertible, whenever $\alpha \geq \alpha_0$, for some α_0. By surjectivity, there is a $b_\alpha \in \ker(\pi)$ such that $b_\alpha \cdot a_\alpha = a_\alpha$. Then $(b \cdot b_\alpha - b) \cdot a_\alpha = 0$, for $b \in \ker(\pi)$. By injectivity, we get $b{\cdot}b_\alpha = b$. This shows that $e_r = b_\alpha$ is a right identity for $\ker(\pi)$. Let $m = e_A{\otimes}e_A{-}e_r$. Then

$$\varphi(a)m - m\varphi(a) = (\varphi(a) \otimes e_A - e_A \otimes \varphi(a)) \cdot m = 0 \quad (a \in A)$$

and also

$$\pi(m) \cdot \varphi(a) = \varphi(a).$$

Therefore, m is a $\varphi-$diagonal for A, that is, A is $\varphi-$contractible. □

Theorem 2.5. *Let A be $\varphi-$approximate biprojective and unital. Then A is $\varphi-$pseudo contractible.*

Proof. Since A is $\varphi-$approximate biprojective, so there is a net $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)(\alpha \in \Delta)$ such that $\lim_\alpha \pi \circ \theta_\alpha \circ \varphi(a) = \varphi(a),\ (a \in A)$. Let e_A be the unit and define $m_\alpha = \theta_\alpha(e_A)$, for every $a \in A$, then we have

$$\varphi(a) \cdot m_\alpha = \varphi(a) \cdot \theta_\alpha(e_A) = \theta_\alpha(\varphi(a)e_A) = \theta_\alpha(\varphi(a)),$$

and similarly

$$m_\alpha \cdot \varphi(a) = \theta_\alpha(e_A) \cdot \varphi(a) = \theta_\alpha(e_A\varphi(a)) = \theta_\alpha(\varphi(a)).$$

Hence, $\varphi(a) \cdot m_\alpha = m_\alpha \cdot \varphi(a)$. Also, for $a \in A$,

$$\begin{aligned} \pi(m_\alpha) \cdot \varphi(a) &= \pi(\theta_\alpha(e_A)) \cdot \varphi(a) \\ &= \pi(\theta_\alpha\varphi(e_A)) \cdot \varphi(a) \to \varphi(a). \end{aligned}$$

That is, (m_α) is a central $\varphi-$approximate diagonal for A, this means that A is $\varphi-$pseudo contractible. □

Proposition 2.4. *Let A be $\varphi-$pseudo contractible. Then A is $\varphi-$approximate biprojective.*

Proof. Suppose that $(m_\alpha) \subset A \hat{\otimes} A$ is a central $\varphi-$approximate diagonal for A. Define $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)$ by $\theta_\alpha(a) := a \cdot m_\alpha$. Then for every $a \in A$, we have

$$\lim_\alpha \pi \circ \theta_\alpha \circ \varphi(a) = \lim_\alpha \pi(\varphi(a) \cdot m_\alpha) = \varphi(a).$$

□

Proposition 2.5. *Let A be $\varphi-$approximate biprojective with a central approximate identity (e_β). Then A is $\varphi-$pseudo contractible.*

Proof. Since A is $\varphi-$approximate biprojective, so there is a net $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)\ (\alpha \in \Delta)$ such that $\lim_\alpha \pi \circ \theta_\alpha \circ \varphi(a) = \varphi(a),\ (a \in A)$. Take a finite subset $F = \{a_1, \cdots, a_n\} \subset A$ and let $\varepsilon > 0$. Set $M = \max_i \|\varphi(a_i)\| + 1$. Now, choose β (depending on $F,\ \epsilon$) such that e_β satisfies $\|\varphi(e_\beta)\varphi(a_j) - \varphi(a_j)\| < \varepsilon,\ j = 1, \cdots, n$. For this β choose α such that $\|\pi\theta_\alpha(\varphi(e_\beta)) - \varphi(e_\beta)\| < \frac{\varepsilon}{M}$. Then for $j = 1, \cdots, n$, we have

$$\begin{aligned} \|\pi\theta_\alpha(\varphi(e_\beta))\varphi(a_j) - \varphi(a_j)\| &\leq \|\varphi(e_\beta)\varphi(a_j) - \varphi(a_j)\| \\ &+ \|\pi\theta_\alpha(\varphi(e_\beta)) - \varphi(e_\beta)\|\|\varphi(a_j)\| \leq 2\varepsilon. \end{aligned}$$

Setting $m_{F,\varepsilon} = \theta_\alpha(\varphi(e_\beta))$ with the order $(F_1, \varepsilon_1) \prec (F_2, \varepsilon_2)$ if $F_1 \subset F_2,\ \varepsilon_1 > \varepsilon_2$ yields a net $(m_{(F,\varepsilon)})$ satisfying $\pi m_{(F,\varepsilon)}\varphi(a) \longrightarrow \varphi(a)$ for $a \in A$. Also, for each $a \in A$, we get

$$\begin{aligned} \varphi(a) \cdot m_{(F,\varepsilon)} &= \varphi(a) \cdot \theta_\alpha(\varphi(e_\beta)) \\ &= \theta_\alpha(\varphi(a)\varphi(e_\beta)) \\ &= \theta_\alpha(\varphi(e_\beta)\varphi(a)) \\ &= \theta_\alpha(\varphi(e_\beta)) \cdot \varphi(a) = m_{(F,\varepsilon)} \cdot \varphi(a), \end{aligned}$$

by the cenrality of (e_β), therefore A $\varphi-$pseudo contractible.

□

Definition 3. *Let A be a Banach algebra and $\varphi \in Hom(A)$. We say that A is $\varphi-$approximate amenable if every $\varphi-$derivation $D : A \longrightarrow X^*$ is a $\varphi-$approximate inner derivation, which means that there is a net $x_\alpha \subset X^*$ such that $D(a) = \lim_\alpha \varphi(a)x_\alpha - x_\alpha\varphi(a)$ for all $a \in A$.*

Recall that an A-bimodule X is neo-unital if

$$X = A \cdot X \cdot A = \{a \cdot x \cdot b : a,\ b \in A,\ x \in X\}.$$

Theorem 2.6. *Suppose that A is a Banach algebra and $\varphi \in Hom(A)$ such that $\varphi^2 = 1$. If A is $\varphi-$pseudo amenable and has a bounded approximate identity, then A is $\varphi-$approximate amenable.*

Proof. Let (m_α) be a $\varphi-$approximate virtual diagonal for A. Define $\theta : A \hat{\otimes} A \longrightarrow A$ by $a \otimes b \mapsto a \cdot D(\varphi(b))$ and let $D : A \longrightarrow X^*$ be a $\varphi-$derivation . Then for every $a \in A$, we have

$$\theta(\varphi(a)m_\alpha - m_\alpha\varphi(a)) + \theta(m_\alpha)\varphi(a) - \varphi(a)\theta(m_\alpha) + \pi(m_\alpha)D(a) = 0.$$

Hence

$$\pi(m_\alpha)D(a) = -\theta(\varphi(a)m_\alpha - m_\alpha\varphi(a)) - \theta(m_\alpha)\varphi(a) + \varphi(a)\theta(m_\alpha),\ (a \in A).$$

Since A has a bounded approximate identity, we may assume that X is neo-unital [8]. Therefore, $W^* - \lim_\alpha \pi(m_\alpha)D(a) = D(a)$ and $\lim_\alpha \theta(\varphi(a)m_\alpha - m_\alpha\varphi(a)) = 0$. Hence,

$$W^* - \lim \varphi(a)\theta(m_\alpha)\_\theta(m_\alpha)\varphi(a) = D(a), \quad (a \in A).$$

By Goldestin's Theorem we can replace the weak* convergence in the equations by the weak convergence. Applying Mazur's Theorem, we then obtain a net (x_α) of convex combinations $(\theta(m_\alpha))$ such that

$$D(a) = \lim_\alpha \varphi(a)x_\alpha - x_\alpha\varphi(a) \quad (a \in A).$$

Hence, A is $\varphi-$approximate amenable. □

Proposition 2.6. *Let A be a $\varphi-$approximate biflat, has a bounded approximate identity (e_β) and $\varphi^2 = 1$. Then A is $\varphi-$approximate amenable.*

Proof. Let $(e_\beta)_{\beta\in I}$ be an approximate identity for A and let $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)^{**}(\alpha \in\Delta)$ satisfy $\pi^{**} \circ \theta_\alpha \circ \varphi(a) \to \varphi(a),\ (a \in A)$. Then for every $a \in A$ and $f \in (A \hat{\otimes} A)^*$, we obtain

$$\begin{aligned}\lim_\beta \lim_\alpha \langle f, \theta_\alpha(\varphi(e_\beta)) \cdot \varphi(a) - \varphi(a) \cdot \theta_\alpha(\varphi(e_\beta))\rangle &= \lim_\beta \lim_\alpha \langle f, \theta_\alpha(\varphi(e_\beta)\varphi(a)\rangle \\ &- \varphi(a)\varphi(e_\beta))\rangle \\ &= \lim_\beta \lim_\alpha \langle f,\ \theta_\alpha\,(\varphi(e_\beta a - ae_\beta)\,)\rangle = 0.\end{aligned}$$

Also, for $a \in A$ and $\psi \in A^*$, we have

$$\lim_\beta \lim_\alpha \langle \psi, \varphi(a) \cdot \pi^{**} \circ \theta_\alpha(\varphi(e_\beta))\rangle \quad = \quad \lim_\beta \langle \psi, \varphi(a)e_\beta\rangle = \varphi(a).$$

Let $E = I \times \Delta^I$, where Δ^I is the set of all functions from I to Δ. Consider the product ordering on E defined by

$$(\beta, \alpha) \leq_E (\acute{\beta}, \acute{\alpha}) \Leftrightarrow \beta \leq_I \acute{\beta},\ \alpha \leq_{\Delta^I} \acute{\alpha} \quad (\beta, \acute{\beta} \in I,\ \alpha, \acute{\alpha} \in\Delta^I).$$

For each $\lambda = (\beta, \alpha) \in E$, we define $m_\lambda = \theta_\alpha(\varphi(e_\beta))$. Using the iterated limit theorem [9, Theorem 2.4], the above calculation gives

$$w^* - \lim_\lambda (m_\lambda \cdot \varphi(a) - \varphi(a) \cdot m_\lambda) = 0 \ \ (a \in A),$$

and

$$w^* - \lim_\lambda \varphi(a) \cdot \pi^{**}(m_\lambda) = \varphi(a) \ \ (a \in A).$$

By Goldestin's Theorem we can assume that $(m_\lambda) \subset (A \hat{\otimes} A)$ and replace the weak* convergence in equations by the weak convergence. Applying Mazur's Theorem, we then obtain a net $(\acute{m}_\lambda) \subset (A \hat{\otimes} A)$ of convex combinations (m_λ) such that

$$\acute{m}_\lambda \cdot \varphi(a) - \varphi(a) \cdot \acute{m}_\lambda \to 0,$$

and

$$\varphi(a) \cdot \pi^{**}(\acute{m}_\lambda) \to \varphi(a) \ \ (a \in A).$$

Hence, A is $\varphi-$pseudo amenable, and by Theorem 2.6, A is $\varphi-$approximate amenable. □

Corollary 2.2. *Let A be a $\varphi-$approximate biprojective with a bounded central approximate identity (e_β) such that $\varphi^2 = 1$. Then A is $\varphi-$approximate amenable.*

Proof. A is $\varphi-$pseudo amenable by Proposition 2.14. Since (e_β) is bounded, by Theorem 2.3 in [4] A is $\varphi-$approximate biflat. So by Proposition 2.6, A is $\varphi-$approximate amenable.

□

Proposition 2.7. *Suppose that A is $\varphi-$approximate amenable. Then $\varphi(A)$ has a bounded approximate identity.*

Proof. Let $\mathfrak{A}$ be the Banach $A-$bimodule whose underlying space is A and

$$a.x := ax, \quad \text{and} \quad x.a := 0, \quad (a \in A, \ x \in \mathfrak{A}).$$

Define $D : A \longrightarrow \mathfrak{A}^{**}$ by $a \mapsto \hat{\varphi(a)}$. So, D is a $\varphi-$derivation. Thus, there is a net $(m_\alpha) \subset \mathfrak{A}^{**}$ with

$$D(a) = \lim_\alpha \varphi(a) \cdot m_\alpha - m_\alpha \cdot \varphi(a) = \lim_\alpha \varphi(a) \cdot m_\alpha = \hat{\varphi(a)}, \ \ (a \in A).$$

Take finite sets $F \subset A$, $\psi \subset A^*$ and let $\varepsilon > 0$. By Goldestin's Theorem there is a net $(e_\alpha)_{\alpha=\alpha(F,\psi,\varepsilon)}$ such that

$$\begin{aligned} |<\phi, \varphi(a)e_\alpha> - <\phi, \varphi(a)>| &\leq |<\phi, \varphi(a)e_\alpha> - <\varphi(a)m_\alpha, \phi>| \\ &+ |<\varphi(a)m_\alpha - \hat{\varphi(a)}, \phi>| \\ &= |<\phi \cdot \varphi(a), e_\alpha> - <m_\alpha, \phi \cdot \varphi(a)>| \\ &+ |<\varphi(a)m_\alpha - \hat{\varphi(a)}, \phi>| \\ &\leq \varepsilon \quad (a \in F, \ \phi \in \psi). \end{aligned}$$

Then for $a \in A$, $w - \lim_\alpha \varphi(a)e_\alpha = \varphi(a)$. Applying Mazur's Theorem, we obtain a net (e_α) of convex combinations $(e_\alpha)_{\alpha=\alpha(F,\psi,\varepsilon)}$ such that $\lim_\alpha \varphi(a)e_\alpha = \varphi(a)$ $(a \in A)$ i.e. (e_α) is a bounded right approximate identity for $\varphi(A)$.

In a similar way, we obtain a left approximate identity $(f_\beta)_\beta$ for $\varphi(A)$, Define $e_{\alpha,\beta} := e_\alpha + f_\beta - e_\alpha f_\beta$. Then, for any $a \in A$, we have

$$\begin{aligned} \|\varphi(a)e_{\alpha,\beta} - \varphi(a)\| &\leq \|\varphi(a)e_\alpha - \varphi(a)\| + \|\varphi(a)f_\beta - \|\varphi(a)e_\alpha f_\beta\| \\ &\leq \|\varphi(a)e_\alpha - \varphi(a)\| + \|\varphi(a) - \|\varphi(a)e_\alpha\|\|f_\beta\| \longrightarrow 0. \end{aligned}$$

Similarly, $\varphi(a) = \lim_{\alpha,\beta} e_{\alpha,\beta}\varphi(a)$ for $a \in A$. □

In the next examples, we construct a $\varphi-$approximate biprojective Banach algebra which is not a biprojective Banach algebra.

Example 1. The Banach algebra l^1 with respect to pointwise product is a non-amenable and biprojective Banach algebra [3, Example 4.1.42]. Hence, $(l^1)^\sharp$ (unitization of l^1) is not biprojective. If we define $\varphi : (l^1)^\sharp \longrightarrow (l^1)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in l^1 and\ \lambda \in \mathbb{C}$, then Example 3.2 [11] shows that $(l^1)^\sharp$ is a $\varphi-$pseudo contactible Banach algebra, hence by Proposition 2.4, $(l^1)^\sharp$ is $\varphi-$approximate biprojective.

On the other hand, since φ is an idempotent homomorphism then by Example 2.1 [5], $(l^1)^\sharp$ is $\varphi-$approximate biprojective.

Example 2. Let $\mathcal{V}$ be a Banach space, and let $f \in \mathcal{V}^*$ be a non-zero element such that $\|f\| \leq 1$. Then $\mathcal{V}$ equipped with the product defined by $ab := f(a)b$ for $a, b \in \nu$ is a Banach algebra which is denoted by $\mathcal{V}_f$. In general, $\mathcal{V}_f$ is a non- commutative and non-unital Banach algebra without right approximate identity, but it is not amenable. Hence, $(\mathcal{V}_f)^\sharp$ (unitization of $\mathcal{V}_f$) is not biprojective. If we define $\varphi : (\mathcal{V}_f)^\sharp \longrightarrow (\mathcal{V}_f)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in \mathcal{V}_f$ and $\lambda \in \mathbb{C}$, then Example 3.2 [11] shows that $(\mathcal{V}_f)^\sharp$ is a $\varphi-$pseudo contactible Banach algebra. Thus, by Proposition 2.4, $(\mathcal{V}_f)^\sharp$ is $\varphi-$approximate biprojective.

Here, we now give an example of a $\varphi-$approximate biprojective Banach algebra which is not a $\varphi-$approximate amenable Banach algebra.

Example 3. Let $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$ under the standard operator norm. We see that A has no identity and right approximate identity. Therefore, A is not a $\varphi-$approximate amenable Banach algebra. For

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we define

$$\theta\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = a(f \otimes f) + b(f \otimes g).$$

Then for $a \in A$ and $\varphi \in Hom(A)$, we have $\pi \circ \theta \circ \varphi(a) = \varphi(a)$. So, A is a $\varphi-$approximate biprojective Banach algebra.

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