ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 3 (2022), 23 – 32

ON AN INDIRECT REPRESENTATION OF EVOLUTIONARY EQUATIONS IN THE FORM OF BIRKHOFF'S EQUATIONS

S.A. Budochkina, H.P. Vu

Communicated by S.N. Kharin

Key words: evolutionary equations, Birkhoff's equations, Pfaffian action.

AMS Mathematics Subject Classification: 49D29, 35A15.

Abstract. In the paper, the problem of an indirect representation of an evolutionary operator equation with the first order time derivative in the form of an operator Birkhoff's equation is solved and the corresponding Pfaffian action is constructed.

DOI: https://doi.org/10.32523/2077-9879-2022-13-3-23-32

1 Introduction

The following system of equations

$$N^i(u) \equiv \sum_{j=1}^{2n} \left(\frac{\partial \mathcal{R}_j}{\partial u^i} - \frac{\partial \mathcal{R}_i}{\partial u^j} \right) \dot{u}^j - \left[\frac{\partial \mathcal{B}}{\partial u^i} + \frac{\partial \mathcal{R}_i}{\partial t} \right] = 0, \quad i = \overline{1, 2n}, \tag{1.1}$$

is called Birkhoff's equations [14] and it is derived from the stationarity condition of the Pfaffian action

$$F_N[u] = \int_{t_0}^{t_1} \left[\sum_{i=1}^{2n} \mathcal{R}_i(t, u) \cdot \dot{u}^i - \mathcal{B}(t, u) \right] dt. \tag{1.2}$$

Equations (1.1) were widely investigated, in particular, in [11, 14]. An operator approach to Birkhoff's equations was developed by V.M. Savchin in [16], where he proved that the problem of a direct representation of an evolutionary operator equation in the form of an operator Birkhoff's equation was closely related to the problem of constructing the corresponding Pfaffian action. There is a large number of works devoted to the construction of direct and indirect variational formulations of different types of equations and their systems (see, e.g., [3, 4, 5, 6, 7, 15, 16, 18, 19, 20, 21, 22, 23]). In classical mechanics there exist methods of construction generalized Lagrangians of systems based on the properties of their motion [8, 9]. It is well known [10, 11, 14], that Birkhoff's systems are generalizations of Hamiltonian ones. It should be noted that the problem of a direct and indirect representation of an operator equation with the first order time derivative in the form of Hamiltonian equation is investigated in [1, 2, 17].

In the paper, we apply Savchin's approach for investigaton of an indirect representation of an evolutionary operator equation in the form of an operator Birkhoff's equation.

Below, we use notations and terminology of [3, 4, 15, 18, 19].

The following theorem is needed for the sequel.

Theorem 1.1. [15] *Consider an operator $N : D(N) \subset U \to V$ and a bilinear form $\Phi(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that for any fixed elements $u \in D(N)$, $g, h \in D(N'_u)$ the function $\psi(\varepsilon) = \Phi(N(u+\varepsilon h), g)$ belongs to the class $C^1[0,1]$. For N to be potential on the convex set $D(N)$ relative to Φ it is necessary and sufficient to have*

$$\Phi\left(N'_u h, g\right) = \Phi\left(N'_u g, h\right) \quad \forall u \in D\left(N\right), \forall h, g \in D\left(N'_u\right). \tag{1.3}$$

Under this condition the potential F_N is given by

$$F_N[u] = \int_0^1 \Phi(N(\tilde{u}(\lambda)), u - u_0)\, d\lambda + F_N[u_0], \tag{1.4}$$

where $\tilde{u}(\lambda) = u_0 + \lambda(u - u_0)$, u_0 is a fixed element of $D(N)$.

2 Conditions of an indirect representation of an evolutionary operator equation with the first order time derivative in the form of an operator Birkhoff's equation

Consider an operator equation

$$N(u) \equiv P_{u,t} u_t + Q(t,u) = 0, \qquad u \in D(N), \tag{2.1}$$

$$t \in [t_0, t_1] \subset \mathbb{R}, \qquad u_t \equiv D_t u \equiv \frac{d}{dt} u.$$

Here $\forall t \in [t_0,t_1]$, $\forall u \in U_1$ $P_{u,t} : U_1 \to V_1$ is a linear operator; $Q : [t_0,t_1] \times U_1 \to V_1$ is an arbitrary operator; $D(N)$ is the domain of definition of the operator N, $U = C^1([t_0,t_1]; U_1)$, $V = C([t_0,t_1]; V_1)$, U_1, V_1 are linear normed spaces, $U_1 \subseteq V_1$.

We will also write

$$N(u) \equiv P_u u_t + Q(u) = 0,$$

bearing in mind that the operators P_u and Q may also depend on t.

Operator equation (2.1) can be an ordinary differential equation, a differential equation with partial derivatives, an integro-differential equation, a differential-difference equation, etc., and a system of such equations (see, e.g., [7, 8, 12, 13, 15]).

We will assume that the bilinear form

$$\Phi(\cdot,\cdot) \equiv \int_{t_0}^{t_1} \langle \cdot,\cdot \rangle\, dt : V \times V \to \mathbb{R}$$

is symmetric and nondegenerate.

Denote by $\tilde{N}(u) = M_u N(u)$, where $M_u : D(M_u) \supset R(N) \to V$ and consider the following equation:

$$\tilde{N}(u) \equiv M_u P_u u_t + M_u Q(u) = 0. \tag{2.2}$$

Theorem 2.1. *Let D_t be skew-symmetric on $D(N'_u)$. Equation (2.1) admits an indirect representation in the form of an operator Birkhoff's equation if and only if $\forall u \in D(N)$, $\forall t \in [t_0,t_1]$ the following conditions hold on $D(N'_u)$:*

$$P_u^* M_u^* + M_u P_u = 0, \tag{2.3}$$

$$-\frac{\partial}{\partial t}\left(P_u^* M_u^*\right)+\left[M_u'\left(Q(u);\cdot\right)\right]^*+Q_u'^* M_u^*-M_u'\left(Q(u);\cdot\right)-M_u Q_u'=0, \tag{2.4}$$

$$\left[M_u'\left(P_u u_t;\cdot\right)\right]^*+\left[P_u'\left(u_t;\cdot\right)\right]^* M_u^*-P_u^{*\prime}\left(M_u^*\left(\cdot\right);u_t\right)-P_u^* M_u^{*\prime}\left(\cdot;u_t\right)$$
$$-M_u'\left(P_u u_t;\cdot\right)-M_u P_u'\left(u_t;\cdot\right)=0. \tag{2.5}$$

Proof. We have

$$\tilde{N}_u' h=M_u'\left(P_u u_t;h\right)+M_u P_u'\left(u_t;h\right)+M_u P_u h_t+M_u'\left(Q\left(u\right);h\right)+M_u Q_u' h.$$

Further,

$$\Phi\left(\tilde{N}_u' h,g\right)=\int_{t_0}^{t_1}\left\{\left\langle M_u'\left(P_u u_t;h\right),g\right\rangle+\left\langle M_u\left(P_u'\left(u_t;h\right)+P_u h_t\right),g\right\rangle\right.$$
$$\left.+\left\langle M_u'\left(Q\left(u\right);h\right),g\right\rangle+\left\langle M_u Q_u' h,g\right\rangle\right\}dt$$
$$=\int_{t_0}^{t_1}\left\{\left\langle h,\left[M_u'\left(P_u u_t;\cdot\right)\right]^* g\right\rangle+\left\langle h,\left[P_u'\left(u_t;\cdot\right)\right]^* M_u^* g\right\rangle-\left\langle h,\frac{\partial}{\partial t}\left(P_u^* M_u^*\right)g\right\rangle\right.$$
$$-\left\langle h,P_u^{*\prime}\left(M_u^* g;u_t\right)\right\rangle-\left\langle h,P_u^* M_u^{*\prime}\left(g;u_t\right)\right\rangle-\left\langle h,P_u^* M_u^* g_t\right\rangle$$
$$\left.+\left\langle h,\left[M_u'\left(Q\left(u\right);\cdot\right)\right]^* g\right\rangle+\left\langle h,Q_u'^* M_u^* g\right\rangle\right\}dt$$
$$=\int_{t_0}^{t_1}\left\{\left\langle h,\left\{\left[M_u'\left(P_u u_t;\cdot\right)\right]^*+\left[P_u'\left(u_t;\cdot\right)\right]^* M_u^*-P_u^{*\prime}\left(M_u^*\left(\cdot\right);u_t\right)-P_u^* M_u^{*\prime}\left(\cdot;u_t\right)\right.\right.\right.$$
$$\left.\left.\left.-\frac{\partial}{\partial t}\left(P_u^* M_u^*\right)+\left[M_u'\left(Q\left(u\right);\cdot\right)\right]^*+Q_u'^* M_u^*\right\}g\right\rangle-\left\langle h,P_u^* M_u^* g_t\right\rangle\right\}dt.$$

On the other hand,

$$\Phi\left(\tilde{N}_u' g,h\right)=\int_{t_0}^{t_1}\left\{\left\langle h,M_u'\left(P_u u_t;g\right)+M_u P_u'\left(u_t;g\right)+M_u'\left(Q(u);g\right)+M_u Q_u' g+M_u P_u g_t\right\rangle\right\}dt.$$

Thus, from condition (1.3) it follows that

$$\left[M_u'\left(P_u u_t;\cdot\right)\right]^*+\left[P_u'\left(u_t;\cdot\right)\right]^* M_u^*-P_u^{*\prime}\left(M_u^*\left(\cdot\right);u_t\right)-P_u^* M_u^{*\prime}\left(\cdot;u_t\right)-M_u'\left(P_u u_t;\cdot\right)$$
$$-M_u P_u'\left(u_t;\cdot\right)-\frac{\partial}{\partial t}\left(P_u^* M_u^*\right)+\left[M_u'\left(Q\left(u\right);\cdot\right)\right]^*+Q_u'^* M_u^*-M_u'\left(Q\left(u\right);\cdot\right)-M_u Q_u'=0,$$
$$P_u^* M_u^*+M_u P_u=0.$$

Hence, conditions (2.3)-(2.5) are satisfied. □

Remark 1. Denote by

$$\overline{P}_u=M_u P_u,\qquad \overline{Q}(u)=M_u Q(u).$$

Then conditions (2.3)-(2.5) can be written in the form

$$\overline{P}_u^*+\overline{P}_u=0,$$
$$-\frac{\partial\overline{P}_u^*}{\partial t}+\overline{Q}_u'^*-\overline{Q}_u'=0,$$
$$\left[\overline{P}_u'\left(u_t;\cdot\right)\right]^*-\overline{P}_u^{*\prime}\left(\cdot;u_t\right)-\overline{P}_u'\left(u_t;\cdot\right)=0.$$

Remark 2. If $M_u \equiv I$ is the identity operator then from (2.3)-(2.5) we obtain

$$P_u^* + P_u = 0, \tag{2.6}$$

$$-\frac{\partial P_u^*}{\partial t} + Q_u'^* - Q_u' = 0, \tag{2.7}$$

$$\left[P_u'(u_t;\cdot)\right]^* - P_u^{*\prime}(\cdot;u_t) - P_u'(u_t;\cdot) = 0. \tag{2.8}$$

Note that these are conditions of a direct representation of an operator equation with the first order time derivative in the form of operator Birkhoff's equation [16].

Remark 3. Consider a system of ordinary differential equations

$$N^i(u) \equiv \sum_{j=1}^{2n} \mathcal{C}_{ij}(t,u)\dot{u}^j + \mathcal{D}_i(t,u) = 0, \qquad i = \overline{1,2n}. \tag{2.9}$$

In our case

$$P_u = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \dots & \mathcal{C}_{1,2n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \dots & \mathcal{C}_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{2n,1} & \mathcal{C}_{2n,2} & \dots & \mathcal{C}_{2n,2n} \end{pmatrix}, \qquad Q(u) = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{2n} \end{pmatrix}.$$

Let us assume that

$$M_u = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1,2n} \\ M_{21} & M_{22} & \dots & M_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{2n,1} & M_{2n,2} & \dots & M_{2n,2n} \end{pmatrix}, \tag{2.10}$$

where $M_{ij} = M_{ij}(t,u),\ i,j = \overline{1,2n}$.

Denote by

$$\overline{\mathcal{C}}_{ij} = \sum_{k=1}^{2n} M_{ik}\mathcal{C}_{kj}, \qquad \overline{\mathcal{D}}_i = \sum_{k=1}^{2n} M_{ik}\mathcal{D}_k, \qquad i,j = \overline{1,2n}.$$

From (2.3)-(2.5), we get

$$\overline{\mathcal{C}}_{ij} + \overline{\mathcal{C}}_{ji} = 0, \tag{2.11}$$

$$\frac{\partial \overline{\mathcal{C}}_{ij}}{\partial t} = \frac{\partial \overline{\mathcal{D}}_i}{\partial u^j} - \frac{\partial \overline{\mathcal{D}}_j}{\partial u^i}, \tag{2.12}$$

$$\frac{\partial \overline{\mathcal{C}}_{ij}}{\partial u^k} + \frac{\partial \overline{\mathcal{C}}_{ki}}{\partial u^j} + \frac{\partial \overline{\mathcal{C}}_{jk}}{\partial u^i} = 0, \quad i,j,k = \overline{1,2n}. \tag{2.13}$$

Note that (2.11)-(2.13) are conditions of an indirect representation of system (2.9) in the form of classical Birkhoff's equations [11].

If (2.10) is the identity matrix, then conditions (2.11)-(2.13) take the form

$$\mathcal{C}_{ij} + \mathcal{C}_{ji} = 0,$$

$$\frac{\partial \mathcal{C}_{ij}}{\partial t} = \frac{\partial \mathcal{D}_i}{\partial u^j} - \frac{\partial \mathcal{D}_j}{\partial u^i},$$

$$\frac{\partial \mathcal{C}_{ij}}{\partial u^k} + \frac{\partial \mathcal{C}_{ki}}{\partial u^j} + \frac{\partial \mathcal{C}_{jk}}{\partial u^i} = 0, \quad i,j,k = \overline{1,2n}.$$

These are conditions of a direct representation of system (2.9) in the form of classical Birkhoff's equations [11].

3 Construction of a Pfaffian action

Theorem 3.1. *If conditions* (2.3) – (2.5) *are fulfilled then the corresponding Pfaffian action is given by*

$$F_{\widetilde{N}}\left[u\right]=\int\limits_{t_0}^{t_1}\left\{\langle M_u\mathcal{R}(u),u_t\rangle+\mathcal{B}_M[u]\right\}dt+F_{\widetilde{N}}\left[u_0\right],\tag{3.1}$$

where

$$\Phi\left\langle M_u\mathcal{R}(u),u_t\right\rangle=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle -M_{\tilde{u}(\lambda)}P_{\tilde{u}(\lambda)}(u-u_0),\frac{\partial\tilde{u}(\lambda)}{\partial t}\right\rangle d\lambda dt,\tag{3.2}$$

$$\mathcal{B}_M[u]=\int\limits_{0}^{1}\left\langle M_{\tilde{u}(\lambda)}Q(\tilde{u}(\lambda)),u-u_0\right\rangle d\lambda,\tag{3.3}$$

$\tilde{u}(\lambda)=u_0+\lambda(u-u_0)$; u_0 *is a fixed element of* $D(N)$.

Proof. Taking into consideration formula (1.4) and condition (2.3) we get

$$F_{\widetilde{N}}\left[u\right]-F_{\widetilde{N}}\left[u_0\right]=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle\widetilde{N}\left(\tilde{u}\left(\lambda\right)\right),u-u_0\right\rangle d\lambda dt$$

$$=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle M_{\tilde{u}(\lambda)}P_{\tilde{u}(\lambda)}\frac{\partial\tilde{u}(\lambda)}{\partial t},u-u_0\right\rangle d\lambda dt+\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle M_{\tilde{u}(\lambda)}Q\left(\tilde{u}(\lambda)\right),u-u_0\right\rangle d\lambda dt$$

$$=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle P^*_{\tilde{u}(\lambda)}M^*_{\tilde{u}(\lambda)}(u-u_0),\frac{\partial\tilde{u}(\lambda)}{\partial t}\right\rangle d\lambda dt+\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle M_{\tilde{u}(\lambda)}Q(\tilde{u}(\lambda)),u-u_0\right\rangle d\lambda dt$$

$$=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle -M_{\tilde{u}(\lambda)}P_{\tilde{u}(\lambda)}(u-u_0),\frac{\partial\tilde{u}(\lambda)}{\partial t}\right\rangle d\lambda dt+\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle M_{\tilde{u}(\lambda)}Q(\tilde{u}(\lambda)),u-u_0\right\rangle d\lambda dt.$$

The use of (3.2), (3.3) yields Pfaffian action (3.1). □

Remark 4. If $M_u\equiv I$ is the identity operator and conditions (2.6)-(2.8) hold then the corresponding Pfaffian action is given by

$$F_N\left[u\right]=\int\limits_{t_0}^{t_1}\left\{\langle\mathcal{R}(u),u_t\rangle+\mathcal{B}[u]\right\}dt+F_N\left[u_0\right],\tag{3.4}$$

where

$$\Phi\left\langle\mathcal{R}(u),u_t\right\rangle=\int\limits_{t_0}^{t_1}\int\limits_{0}^{1}\left\langle -P_{\tilde{u}(\lambda)}(u-u_0),\frac{\partial\tilde{u}(\lambda)}{\partial t}\right\rangle d\lambda dt,$$

$$\mathcal{B}[u]=\int\limits_{0}^{1}\langle Q(\tilde{u}(\lambda)),u-u_0\rangle d\lambda,$$

$\tilde{u}(\lambda)=u_0+\lambda(u-u_0)$; u_0 is a fixed element of $D(N)$.

Note that Pfaffian action (3.4) was constructed in [16].

4 Birkhoffian structure of an evolutionary operator equation with the first order time derivative

Theorem 4.1. *Conditions* (2.3) – (2.5) *are fulfilled if and only if equation* (2.2) *has the Birkhoffian structure*

$$\tilde{N}(u) \equiv \overline{P}_u u_t + \overline{Q}(u) \equiv \left(\overline{\mathcal{R}}'^*_u - \overline{\mathcal{R}}'_u\right) u_t - \frac{\partial \overline{\mathcal{R}}}{\partial t}(u) + grad_{\Phi_1} \overline{\mathcal{B}}[u] = 0, \tag{4.1}$$

where

$$\overline{P}_u = M_u P_u, \quad \overline{Q}(u) = M_u Q(u), \quad \overline{\mathcal{R}}(u) = M_u \mathcal{R}(u), \quad \overline{\mathcal{B}}[u] = \mathcal{B}_M[u]. \tag{4.2}$$

Proof. Let conditions (2.3)-(2.5) be satisfied. Then the corresponding Pfaffian action is given by (3.1) and

$$\delta F_{\tilde{N}}[u,h] = \int\limits_{t_0}^{t_1} \{\langle M'_u(\mathcal{R}(u);h), u_t\rangle + \langle M_u \mathcal{R}'_u h, u_t\rangle + \langle M_u \mathcal{R}(u), h_t\rangle$$

$$+ \langle grad_{\Phi_1} \mathcal{B}_M[u], h\rangle\} \, dt = \int\limits_{t_0}^{t_1} \left\{\left\langle [M'_u(\mathcal{R}(u); \cdot)]^* u_t, h\right\rangle + \langle h, \mathcal{R}'^*_u M^*_u u_t\rangle\right.$$

$$\left. - \left\langle \frac{\partial}{\partial t} (M_u \mathcal{R}(u)) + M'_u (\mathcal{R}(u); u_t) + M_u \mathcal{R}'_u u_t, h \right\rangle + \langle grad_{\Phi_1} \mathcal{B}_M[u], h\rangle \right\} dt$$

$$= \int\limits_{t_0}^{t_1} \left\langle \tilde{N}(u), h \right\rangle dt.$$

Hence

$$M_u P_u = [M'_u(\mathcal{R}(u); \cdot)]^* + \mathcal{R}'^*_u M^*_u - M'_u (\mathcal{R}(u); \cdot) - M_u \mathcal{R}'_u,$$

$$M_u Q(u) = -\frac{\partial}{\partial t} (M_u \mathcal{R}(u)) + grad_{\Phi_1} \mathcal{B}_M[u].$$

Bearing in mind notations (4.2), we obtain

$$\overline{P}_u = \overline{\mathcal{R}}'^*_u - \overline{\mathcal{R}}'_u,$$

$$\overline{Q}(u) = -\frac{\partial \overline{\mathcal{R}}}{\partial t}(u) + grad_{\Phi_1} \overline{\mathcal{B}}[u].$$

This means that equation (2.2) is of the Birkhoffian type and, therefore, equation (2.1) is indirectly represented in the form of an operator Birkhoff's equation.

On the other hand, let equation (2.2) be of the Birkhoffian structure. As it is shown above, equation (4.1) is derived from the stationarity condition of Pfaffian action (3.1). It signifies that operators P_u and Q must satisfy conditions (2.3)-(2.5). □

Remark 5. If $M_u \equiv I$ is the identity operator then

$$P_u = \mathcal{R}'^*_u - \mathcal{R}'_u,$$

$$Q(u) = -\frac{\partial \mathcal{R}}{\partial t}(u) + grad_{\Phi_1} \mathcal{B}[u],$$

i.e. equation (2.1) is directly represented in the form of an operator Birkhoff's equation

$$N(u) \equiv P_u u_t + Q(u) \equiv (\mathcal{R}'^*_u - \mathcal{R}'_u)\, u_t - \frac{\partial \mathcal{R}}{\partial t}(u) + grad_{\Phi_1} \mathcal{B}[u] = 0 \tag{4.3}$$

(see [16]).

Remark 6. Suppose that

$$\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_{2n})^T, \qquad \mathcal{B} = -\mathcal{B},$$

where $\mathcal{R}_i$ $(i = \overline{1,2n})$, $\mathcal{B}$ are functions of the variable t and unknown vector function $u(t) = (u^1(t), u^2(t), ..., u^{2n}(t))^T$.

Then

$$\mathcal{R}'_u = \begin{pmatrix} \frac{\partial \mathcal{R}_1}{\partial u^1} & \frac{\partial \mathcal{R}_1}{\partial u^2} & \cdots & \frac{\partial \mathcal{R}_1}{\partial u^{2n}} \\ \frac{\partial \mathcal{R}_2}{\partial u^1} & \frac{\partial \mathcal{R}_2}{\partial u^2} & \cdots & \frac{\partial \mathcal{R}_2}{\partial u^{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{R}_{2n}}{\partial u^1} & \frac{\partial \mathcal{R}_{2n}}{\partial u^2} & \cdots & \frac{\partial \mathcal{R}_{2n}}{\partial u^{2n}} \end{pmatrix}, \quad \mathcal{R}'^*_u = \begin{pmatrix} \frac{\partial \mathcal{R}_1}{\partial u^1} & \frac{\partial \mathcal{R}_2}{\partial u^1} & \cdots & \frac{\partial \mathcal{R}_{2n}}{\partial u^1} \\ \frac{\partial \mathcal{R}_1}{\partial u^2} & \frac{\partial \mathcal{R}_2}{\partial u^2} & \cdots & \frac{\partial \mathcal{R}_{2n}}{\partial u^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{R}_1}{\partial u^{2n}} & \frac{\partial \mathcal{R}_2}{\partial u^{2n}} & \cdots & \frac{\partial \mathcal{R}_{2n}}{\partial u^{2n}} \end{pmatrix}.$$

In this case, (3.4) is classical Pfaffian action (1.2) and from (4.3) we get the structure of classical Birkhoff's equations (1.1).

5 Example

Consider the following partial differential equation:

$$N(u) \equiv \alpha u_t + \beta u_{txx} + f(x,t,u,u_x) = 0, \quad (x,t) \in \mathcal{Q} = (a,b) \times (t_0,t_1), \tag{5.1}$$

where α, β are constants, $f \in C^2(\overline{\mathcal{Q}} \times \mathbb{R}^2)$.

Define $D(N)$ by

$$D(N) = \{u \in C^3(\overline{\mathcal{Q}}) : u|_{t=t_0} = \varphi_1(x),\ u|_{t=t_1} = \varphi_2(x)\ (x \in (a,b)),\ \ u|_{x=a} = \psi_1(t),$$

$$u|_{x=b} = \psi_2(t),\ u_x|_{x=a} = \psi_3(t),\ u_{tx}|_{x=a} = 0,\ \int\limits_a^b u(x,t)dx = \psi_4(t)\ (t \in (t_0,t_1))\}, \tag{5.2}$$

where φ_i $(i = 1,2)$, ψ_j $(j = \overline{1,4})$ are continuous functions.

We introduce the classical bilinear form

$$\Phi(v,g) = \int\limits_{t_0}^{t_1} \int\limits_a^b v(x,t)g(x,t)dxdt. \tag{5.3}$$

In this case

$$P_u \equiv P = \alpha I + \beta D_x^2, \qquad Q(u) = f(x,t,u,u_x),$$

where I is the identity operator.

Note that equation (5.1) does not admit a direct representation in the form of operator Birkhoff's equation, because $P = P^*$ and condition (2.6) is not fulfilled.

Let $M = D_x^{-1}$, where

$$D_x^{-1} v(x,t) = \int\limits_a^x v(y,t)dy.$$

Equation (5.1) admits an indirect representation in the form of operator Birkhoff's equation if and only if $\forall u \in D(N)$, $\forall t \in [t_0,t_1]$ the following condition holds on $D(N'_u)$:

$$-\frac{\partial f}{\partial u} D_x^{-1} + D_x\left(\frac{\partial f}{\partial u_x}\right) D_x^{-1} - D_x^{-1}\left(\frac{\partial f}{\partial u}(\cdot)\right) + D_x^{-1}\left(D_x\left(\frac{\partial f}{\partial u_x}\right)(\cdot)\right) = 0. \tag{5.4}$$

Indeed, we have

$$M^* = -D_x^{-1}, \qquad Q'_u = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial u_x} D_x, \qquad Q'^*_u = \frac{\partial f}{\partial u} - D_x\left(\frac{\partial f}{\partial u_x}\right) - \frac{\partial f}{\partial u_x} D_x$$

and

(2.3) $\Longrightarrow -\alpha D_x^{-1}h - \beta h_x + \alpha D_x^{-1}h + \beta h_x = 0$,

(2.4) $\Longrightarrow -\frac{\partial f}{\partial u} D_x^{-1}h + D_x\left(\frac{\partial f}{\partial u_x}\right) D_x^{-1}h + \frac{\partial f}{\partial u_x} h - D_x^{-1}\left(\frac{\partial f}{\partial u} h\right) - D_x^{-1}\left(\frac{\partial f}{\partial u_x} h_x\right)$

$= -\frac{\partial f}{\partial u} D_x^{-1}h + D_x\left(\frac{\partial f}{\partial u_x}\right) D_x^{-1}h + \frac{\partial f}{\partial u_x} h - D_x^{-1}\left(\frac{\partial f}{\partial u} h\right) - \frac{\partial f}{\partial u_x} h + D_x^{-1}\left(D_x\left(\frac{\partial f}{\partial u_x}\right) h\right)$

$= -\frac{\partial f}{\partial u} D_x^{-1}h + D_x\left(\frac{\partial f}{\partial u_x}\right) D_x^{-1}h - D_x^{-1}\left(\frac{\partial f}{\partial u} h\right) + D_x^{-1}\left(D_x\left(\frac{\partial f}{\partial u_x}\right) h\right) = 0$,

(2.5) $\Longrightarrow 0 = 0$.

Hence, from condition (2.4) we obtain condition (5.4).

From (3.2) we find

$$\mathcal{R} = -\frac{1}{2}\alpha I - \frac{1}{2}\beta D_x^2.$$

Note that under condition (5.4) the following operator

$$N_1(u) \equiv D_x^{-1} f(x, t, u, u_x)$$

is potential on $D(N)$ (5.2) relative to bilinear form (5.3) and in this case

$$\mathcal{B}_M[u] \equiv B_M[u] - B_M[u_0] = \int_0^1 \int_a^b D_x^{-1} f(x, t, \tilde{u}(\lambda), \tilde{u}_x(\lambda)) \cdot (u - u_0) dx d\lambda,$$

where $\tilde{u}(\lambda) = u_0 + \lambda(u - u_0)$; u_0 is a fixed element of $D(N)$.

So functional (3.1) takes the form

$$F_{\widetilde{N}}[u] = \int_{t_0}^{t_1} \left\{ -\frac{1}{2} \int_a^b \left(\alpha D_x^{-1} u \cdot u_t + \beta u_x u_t\right) dx + B_M[u] \right\} dt.$$

References

[1] S.A. Budochkina, *On a representation of an operator equation with first time derivative in the form of a B_u-Hamiltonian equation.* Differential Equations, 49 (2013), no. 2, 176–186.

[2] S.A. Budochkina, V.M. Savchin, *On B_u-Hamiltonian equations in mechanics of infinite-dimensional systems.* Doklady Mathematics, 84 (2011), no. 1, 525–526.

[3] S.A. Budochkina, V.M. Savchin, *On direct variational formulations for second order evolutionary equations.* Eurasian Mathematical Journal, 3 (2012), no. 4, 23–34.

[4] S.A. Budotchkina, V.M. Savchin, *On indirect variational formulations for operator equations.* Journal of Function Spaces and Applications, 5 (2007), no. 3, 231–242.

[5] V.M. Filippov, *Variational principles for nonpotential operators.* Peoples' Friendship University of Russia, Moscow, 1985 (in Russian).

[6] V.M. Filippov, S.R. Mikhailova, Gondo Yake, *Construction of variational factors for quasilinear second order partial differential equations.* Computer Physics Communications, 126 (2000), no. 1-2, 67–71.

[7] V.M. Filippov, V.M. Savchin, S.G. Shorokhov, *Variational principles for nonpotential operators.* Journal of Mathematical Sciences, 68 (1994), no. 3, 275–398.

[8] A.S. Galiullin, *Inverse problems of dynamics.* Nauka, Moscow, 1981 (in Russian).

[9] A.S. Galiullin, *Invariance of action and inverse problems of dynamics.* Differential Equations, 20 (1984), no. 8, 1318–1325 (in Russian).

[10] A.S. Galiullin, *Generalizations of Hamiltonian systems.* Differential Equations, 24 (1988), no. 5, 483–490.

[11] A.S. Galiullin, G.G. Gafarov, R.P. Malayshka, A.M. Khvan, *Analytical dynamics of Helmholtz, Birkhoff, Nambu systems.* Advances in Physical Sciences, Moscow, 1997 (in Russian).

[12] M.T. Jenaliyev, M.I. Ramazanov, M.T. Kosmakova, Zh.M. Tuleutaeva, *On the solution to a two-dimensional heat conduction problem in a degenerate domain.* Eurasian Mathematical Journal, 11 (2020), no. 3, 89–94.

[13] M.T. Jenaliyev, M.I. Ramazanov, M.G. Yergaliyev, *On an inverse problem for a parabolic equation in a degenerate angular domain.* Eurasian Mathematical Journal, 12 (2021), no. 2, 25–38.

[14] R.M. Santilli, *Foundations of theoretical mechanics, II: Birkhoffian generalization of Hamiltonian mechanics.* Springer-Verlag, New-York-Berlin, 1983.

[15] V.M. Savchin, *Mathematical methods of mechanics of infinite dimensional nonpotential systems.* Peoples' Friendship University of Russia, Moscow, 1991 (in Russian).

[16] V.M. Savchin, *An operator approach to Birkhoff's equations.* Bulletin of Peoples' Friendship University of Russia. Series Mathematics, no. 2 (2) (1995), 111–123.

[17] V.M. Savchin, *Potential operators with the first time derivative and Hamiltonian systems.* Proceedings of the International Conference dedicated to Corresponding Member of RAS, Professor L.D. Kudryavtsev on the occasion of his 75th anniversary. Moscow: Peoples' Friendship University of Russia. 2 (1998), 147–151 (in Russian).

[18] V.M. Savchin, S.A. Budochkina, *On the structure of a variational equation of evolution type with the second t-derivative.* Differential Equations, 39 (2003), no. 1, 127–134.

[19] V.M. Savchin, S.A. Budochkina, *On the existence of a variational principle for an operator equation with the second derivative with respect to "time".* Mathematical Notes, 80 (2006), no. 1, 83–90.

[20] M.I. Tleubergenov, D.T. Azhymbaev, *Stochastical problem of Helmholtz for Birkhoff systems.* Bulletin of the Karaganda University. Mathematics Series, 93 (2019), no. 1, 78–87.

[21] M.I. Tleubergenov, G.T. Ibraeva, *On the solvability of the main inverse problem for stochastic differential systems.* Ukrainian Mathematical Journal, 71 (2019), no. 1, 157–165.

[22] M.I. Tleubergenov, G.T. Ibraeva, *On inverse problem of closure of differential systems with degenerate diffusion.* Eurasian Mathematical Journal, 10 (2019), no. 2, 93–102.

[23] M.I. Tleubergenov, G.T. Ibraeva, *On the closure of stochastic differential equations of motion.* Eurasian Mathematical Journal, 12 (2021), no. 2, 82–89.

Svetlana Aleksandrovna Budochkina, Hue Phuong Vu
S.M. Nikol'skii Mathematical Institute
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklay St,
117198 Moscow, Russian Federation
E-mails: budochkina-sa@rudn.ru, vuhuephuong0812@gmail.com

Received: 04.05.2020