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DISCONTINUOUS MATRIX STURM–LIOUVILLE PROBLEMS

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Key words: matrix Sturm–Liouville problem, transmission conditions, maximal operator, minimal operator, self-adjoint operator, spectral resolution.

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Abstract. In this paper, we investigate discontinuous matrix Sturm–Liouville problems. We establish an existence and uniqueness result. Next, we introduce the corresponding maximal and minimal operators for this problem and some properties of these operators are investigated. Moreover, we give a criterion under which these operators are self-adjoint. Finally, we give an eigenfunction expansion.

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1 Introduction

Recently, discontinuous differential equations have become a very active area of research since these equations describe processes that experience a sudden change of their state at certain moments. Such processes arise in some problems of the theory of the mass and heat transfer, radio science, various physical transfer problems, and geophysics (see [26, 27, 25, 9, 31, 32, 7, 34, 33, 11, 12, 13, 14, 1, 21, 28, 29, 30]).

The study of matrix-valued Sturm–Liouville equations has become an important area of research because such equations arise in a variety of physical problems (for example, see [17, 20, 22, 15, 16]). Although matrix Sturm–Liouville equations are more difficult than the scalar Sturm–Liouville equations the matrix-valued Sturm–Liouville equations have intensively been investigated during the last two decades (see [2, 4, 18, 19, 35, 8, 23, 10] and references therein). In this study, we investigate discontinuous matrix Sturm–Liouville equations. In the analysis that follows, we will largely follow the development of the theory in [3, 5, 6, 36, 24].

This paper is organized as follows. In Section 2, an existence and uniqueness theorem is proved for discontinuous matrix Sturm–Liouville equation. Next, the corresponding maximal and minimal operators for this equation are constructed and some properties of this operators are investigated. In Section 3, a criterion under which discontinuous matrix Sturm–Liouville operators are self-adjoint is given. Finally, an eigenfunction expansion is constructed in Section 4.

2 Discontinuous matrix Sturm–Liouville equation

Consider the following matrix Sturm–Liouville equation

$$-\left(P(x)z'(x)\right)' + Q\left(x\right)z(x) = \lambda R\left(x\right)z\left(x\right),\ x\in[a,c)\cup(c,b], \tag{2.1}$$

where $-\infty < a < c < b < +\infty$, $\lambda\in\mathbb{C}$; $P(x)$, $Q\left(x\right)$ and $R\left(x\right)$ are $n\times n$ complex Hermitian matrix-valued functions, defined on $[a,c)\cup(c,b]$, $\det P\left(x\right)\neq 0$,

$R(x)$ is a positive and the entries of the matrices $P^{-1}(t)$, $Q(t)$ and $R(x)$ are Lebesgue measurable and integrable functions on $[a,c)\cup(c,b]$.

Now, we can convert equation (2.1) into the Hamiltonian system. Let

$$J=\begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix},\ \mathcal{Z}(x)=\begin{pmatrix} z(x) \\ P(x)z'(x) \end{pmatrix},$$

$$V_1(x)=\begin{pmatrix} R(x) & O_n \\ O_n & O_n \end{pmatrix},\ V_2(x)=\begin{pmatrix} -Q(x) & O_n \\ O_n & P^{-1}(x) \end{pmatrix}.$$

From equation (2.1), we get

$$\Gamma(\mathcal{Z}) := J\mathcal{Z}'(x)-V_2(x)\mathcal{Z}(x)=\lambda V_1(x)\mathcal{Z}(x),\ x\in[a,c)\cup(c,b]. \tag{2.2}$$

Let

$$\begin{aligned} &L^2_{V_1}[(a,c)\cup(c,b);E] \\ &:=\left\{\mathcal{Z}:\int_a^c(V_1\mathcal{Z},\mathcal{Z})_E\,dx+\int_c^b(V_1\mathcal{Z},\mathcal{Z})_E\,dx<\infty\right\} \end{aligned}$$

be the Hilbert space of $2n$-dimensional vector-valued functions $\mathcal{X},\mathcal{Y}$ with the inner product

$$\begin{aligned} (\mathcal{X},\mathcal{Y}) &:= \int_a^c(V_1\mathcal{X},\mathcal{Y})_E\,dx+\int_c^b(V_1\mathcal{X},\mathcal{Y})_E\,dx \\ &= \int_a^c\mathcal{Y}^*V_1\mathcal{X}dx+\int_c^b\mathcal{Y}^*V_1\mathcal{X}dx, \end{aligned}$$

where $E:=\mathbb{C}^{2n}$ is the $2n$-dimensional Euclidean space.

Theorem 2.1. *Let $K\in\mathbb{C}^{2n}$ and $\lambda\in\mathbb{C}$. Then equation* (2.2) *has a unique solution such that*

$$\mathcal{Z}(a,\lambda)=K,\ \mathcal{Z}(c+,\lambda)=C\mathcal{Z}(c-,\lambda), \tag{2.3}$$

where C is the $2n\times 2n$ matrix with entries from $\mathbb{R}$ such that $CJC^*=J$.

Proof. An integration yields

$$\begin{aligned} \mathcal{Z}(x,\lambda) &= K-\int_a^c J[\lambda V_1(t,\lambda)+V_2(t,\lambda)]\mathcal{Z}(t,\lambda)\,dt \\ &+\int_c^x J[\lambda V_1(t,\lambda)+V_2(t,\lambda)]\mathcal{Z}(t,\lambda)\,dt, \end{aligned} \tag{2.4}$$

where $x\in[a,c)\cup(c,b]$. Conversely, every solution of equation (2.4) is also a solution of equation (2.2).

Let us define the sequence $\{\mathcal{Z}_m\}_{m\in\mathbb{N}}$ ($\mathbb{N}:=\{1,2,3,...\}$ of successive approximations by

$$\mathcal{Z}_0(x,\lambda)=K,$$

$$\mathcal{Z}_{m+1}(x,\lambda)=K-\int_a^c J[\lambda V_1(t,\lambda)+V_2(t,\lambda)]\mathcal{Z}_m(t,\lambda)\,dt$$

$$+\int_c^x J[\lambda V_1(t,\lambda)+V_2(t,\lambda)]\mathcal{Z}_m(t,\lambda)\,dt,\ m=0,1,2,..., \tag{2.5}$$

where $x \in [a,c) \cup (c,b]$. Then, we will prove that $\{\mathcal{Z}_m\}_{m\in\mathbb{N}}$ converges to a function $\mathcal{Z}$ uniformly on each compact subset of $[a,c) \cup (c,b]$. There exist positive numbers $\eta(\lambda)$ and $\xi(\lambda)$ such that

$$\|J[\lambda V_1(x,\lambda) + V_2(x,\lambda)]\| \leq \eta(\lambda),$$

$$\|\mathcal{Z}_1(x,\lambda)\| \leq \xi(\lambda), \ x \in [a,c) \cup (c,b].$$

Using mathematical induction, we deduce that

$$\|\mathcal{Z}_{m+1}(x,\lambda) - \mathcal{Z}_m(x,\lambda)\| \leq \eta(\lambda) \frac{(\xi(\lambda)(x-a))^m}{m!} \quad (m \in \mathbb{N}).$$

An application of the Weierstrass M-test implies that the sequence $\{\mathcal{Z}_m\}_{m\in\mathbb{N}}$ converges to a function $\mathcal{Z}$ uniformly on each compact subset of $[a,c) \cup (c,b]$. It is clear that the function $\mathcal{Z}$ satisfies (2.3).

Now, we show that equation (2.2) has a unique solution. Assume $\mathcal{Y}$ is another one. Since $\mathcal{Y}$ is continuous, there exists a positive number $\mathcal{M}$ such that $\|\mathcal{Z} - \mathcal{Y}\| \leq \mathcal{M}$. Proceeding as above we see that

$$\|\mathcal{Z}(x,\lambda) - \mathcal{Y}(x,\lambda)\| \leq \mathcal{M}\eta(\lambda) \frac{(x-a)^m}{m!} \quad (m \in \mathbb{N}).$$

Then we get $\mathcal{Z} = \mathcal{Y}$ on the interval $[a,c) \cup (c,b]$ due to

$$\lim_{m\to\infty} \mathcal{M}\eta(\lambda) \frac{(x-a)^m}{m!} = 0.$$

□

Now, we will give the definition of maximal and minimal operators. Denote

$$\mathcal{D}_{\max} := \left\{ \begin{array}{c} \mathcal{Z} \in L^2_{V_1}[(a,c) \cup (c,b); E] : z \text{ and } Pz' \text{ are} \\ \text{absolutely continuous on } [a,c) \cup (c,b], \\ \text{one-sided limits } z(c\pm),\ Pz'(c\pm) \text{ exist and are} \\ \text{finite, } J\mathcal{Z}'(x) - V_2(x)\mathcal{Z}(x) = V_1 F \text{ exists in} \\ [a,c) \cup (c,b],\ F \in L^2_{V_1}[(a,c) \cup (c,b); E] \text{ and} \\ \mathcal{Z}(c+) = C\mathcal{Z}(c-),\ CJC^* = J \end{array} \right\},$$

$$\mathcal{D}_{\min} := \{\mathcal{Z} \in D_{\max} : \mathcal{Z}(a) = \mathcal{Z}(b) = 0\}. \tag{2.6}$$

The operator $T_{\min}$ defined by

$$\begin{array}{l} T_{\min} : \mathcal{D}_{\min} \to L^2_{V_1}[(a,c) \cup (c,b); E], \\ \quad \mathcal{Z} \to T_{\min}\mathcal{Z} = F \text{ if and only if } \Gamma(\mathcal{Z}) = V_1 F. \end{array}$$

is called the minimal operator generated by equation (2.2). Similarly, the operator $T_{\max}$ defined by

$$\begin{array}{l} T_{\max} : \mathcal{D}_{\max} \to L^2_{V_1}[(a,c) \cup (c,b); E], \\ \quad \mathcal{Z} \to T_{\max}\mathcal{Z} = F \text{ if and only if } \Gamma(\mathcal{Z}) = V_1 F. \end{array}$$

is called the maximal operator for the discontinuous matrix Sturm–Liouville equation.

Now, we give the following Green's formula.

Theorem 2.2 (Green's formula)**.** *Let $\mathcal{Z}, \mathcal{Y} \in \mathcal{D}_{\max}$. Then we have*

$$(T_{\max}\mathcal{Z}, \mathcal{Y}) - (\mathcal{Z}, T_{\max}\mathcal{Y}) = [\mathcal{Z},\mathcal{Y}]_b + [\mathcal{Z},\mathcal{Y}]_{c-} - [\mathcal{Z},\mathcal{Y}]_a - [\mathcal{Z},\mathcal{Y}]_{c+} \tag{2.7}$$

where $[\mathcal{Z},\mathcal{Y}]_x := \mathcal{Y}^*(x) J \mathcal{Z}(x),\ x \in [a,c) \cup (c,b]$.

Lemma 2.1. *The operator $T_{\min}$ is Hermitian.*

Proof. Let $\mathcal{Z},\mathcal{Y}\in\mathcal{D}_{\min}$. Then there exist $F,G\in L^2_{V_1}\left[(a,c)\cup(c,b)\,;E\right]$ such that $\Gamma(\mathcal{Z})=V_1F$ and $\Gamma(\mathcal{Y})=V_1G$. From (2.6) and (2.7), we see that

$$(T_{\min}\mathcal{Z},\mathcal{Y})-(\mathcal{Z},T_{\min}\mathcal{Y})=(F,\mathcal{Y})-(\mathcal{Z},G)$$

$$=\int_a^c\left[\mathcal{Y}^*(t)\,V_1F-G^*(t)\,V_1\mathcal{Z}(t)\right]dt+\int_c^b\left[\mathcal{Y}^*(t)\,V_1F-G^*(t)\,V_1\mathcal{Z}(t)\right]dt$$

$$=\int_a^c\left[\mathcal{Y}^*(t)\,\Gamma(\mathcal{Z})-\Gamma^*(\mathcal{Y})\,\mathcal{Z}(t)\right]dt+\int_c^b\left[\mathcal{Y}^*(t)\,\Gamma(\mathcal{Z})-\Gamma^*(\mathcal{Y})\,\mathcal{Z}(t)\right]dt$$

$$=[\mathcal{Z},\mathcal{Y}]_b+[\mathcal{Z},\mathcal{Y}]_{c-}-[\mathcal{Z},\mathcal{Y}]_a-[\mathcal{Z},\mathcal{Y}]_{c+}=0.$$

The following lemma has a proof similar to that of Lemma 2.1. □

Lemma 2.2. *Let $\mathcal{Z}\in\mathcal{D}_{\min}$ and $\mathcal{Y}\in\mathcal{D}_{\max}$. Then we have the following relation*

$$(T_{\min}\mathcal{Z},\mathcal{Y})=(\mathcal{Z},T_{\max}\mathcal{Y})\,.$$

Lemma 2.3. *Let us denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the null space and the range of an operator T, respectively. Then we have*

$$\mathcal{R}(T_{\min})=\mathcal{N}(T_{\max})^{\perp}\,.$$

Proof. Let $\xi\in\mathcal{R}(T_{\min})$. There exists $\mathcal{Z}\in\mathcal{D}_{\min}$ such that $T_{\min}\mathcal{Z}=\xi$. It follows from Lemma 2.2 that for each $\mathcal{Y}\in\mathcal{N}(T_{\max})$,

$$(\xi,\mathcal{Y})=(T_{\min}\mathcal{Z},\mathcal{Y})=(\mathcal{Z},T_{\max}\mathcal{Y})=0,$$

i.e., $\mathcal{R}(T_{\min})\subset\mathcal{N}(T_{\max})^{\perp}$.

For any given $\xi\in\mathcal{N}(T_{\max})^{\perp}$ and for all $\mathcal{Y}\in\mathcal{N}(T_{\max})$, we have $(\xi,\mathcal{Y})=0$. Let us consider the following problem:

$$\begin{gathered}J\mathcal{Z}'(x)-V_2(x)\,\mathcal{Z}(x)=V_1(x)\,\xi(x)\,,\ \ x\in[a,c)\cup(c,b]\\ \mathcal{Z}(a,\lambda)=0,\ \ \mathcal{Z}(c+,\lambda)=C\mathcal{Z}(c-,\lambda)\end{gathered}\tag{2.8}$$

It follows from Theorem 2.1 that problem (2.8) has a unique solution on $[a,c)\cup(c,b]$. Let $\Psi(x)=(\psi_1,\psi_2,...,\psi_{2n})$ be the fundamental solution of the system

$$\begin{gathered}J\mathcal{Z}'(x)-V_2(x)\,\mathcal{Z}(x)=0,\ \ x\in[a,c)\cup(c,b],\\ \Psi(a)=J,\ \ \mathcal{Z}(c+)=C\mathcal{Z}(c-).\end{gathered}$$

It is clear that $\psi_i \in \mathcal{N}(T_{\max})$ for $1 \leq i \leq 2n$. By Theorem 2.2, for $1 \leq i \leq 2n$, we have

$$0 = (\xi, \psi_i) = \int_a^c \psi_i^*(t) V_1(x) \xi(t)\, dt + \int_c^b \psi_i^*(t) V_1(x) \xi(t)\, dt$$

$$= \int_a^c \psi_i^*(t) \Gamma(\mathcal{Z})(t)\, dt + \int_c^b \psi_i^*(t) \Gamma(\mathcal{Z})(t)\, dt$$

$$= \int_a^c \psi_i^*(t) \Gamma(\mathcal{Z})(t)\, dt + \int_c^b \psi_i^*(t) \Gamma(\mathcal{Z})(t)\, dt$$

$$- \int_a^c \Gamma(\psi_i)^*(t) \mathcal{Z}(t)\, dt - \int_c^b \Gamma(\psi_i)^*(t) \mathcal{Z}(t)\, dt$$

$$= [\mathcal{Z}, \psi_i]_a + [\mathcal{Z}, \psi_i]_{c-} - [\mathcal{Z}, \psi_i]_{c+} - [\mathcal{Z}, \psi_i]_0 = [\mathcal{Z}, \psi_i]_a.$$

This implies that

$$[\mathcal{Z}, \psi_i]_a = \Psi^*(a) J Z(a) = Z(a) = 0,$$

i.e., $\xi \in \mathcal{R}(T_{\min})$. □

Theorem 2.3. *The operator $T_{\min}$ is a densely defined operator, so the operator $T_{\min}$ is symmetric. Furthermore $T_{\min}^* = T_{\max}$.*

Proof. Let $\xi \in \mathcal{D}_{\min}^{\perp}$. Then, for all $\mathcal{Y} \in \mathcal{D}_{\min}$, we have $(\xi, \mathcal{Y}) = 0$. Set $T_{\min}\mathcal{Y}(x) = \phi(x)$.

Let $\mathcal{Z}(.)$ be any solution of the system

$$J\mathcal{Z}'(x) - V_2(x) \mathcal{Z}(x) = V_1(x) \xi(x), \;\; x \in [a, c) \cup (c, b].$$

It follows from Theorem 2.2 that

$$(\mathcal{Z}, \phi) - (\xi, \mathcal{Y})$$

$$= \int_a^c \phi^*(t) V_1(t) \mathcal{Z}(t)\, dt + \int_c^b \phi^*(t) V_1(t) \mathcal{Z}(t)\, dt$$

$$- \int_a^c \mathcal{Y}^*(t) V_1(t) \xi(t)\, dt - \int_c^b \mathcal{Y}^*(t) V_1(t) \xi(t)\, dt$$

$$= \int_a^c \Gamma(\mathcal{Y})^*(t) \mathcal{Z}(t)\, dt + \int_c^b \Gamma(\mathcal{Y})^*(t) \mathcal{Z}(t)\, dt$$

$$- \int_a^c \mathcal{Y}^*(t) \Gamma(\mathcal{Z})(t)\, dt - \int_c^b \mathcal{Y}^*(t) \Gamma(\mathcal{Z})(t)\, dt$$

$$= -[\mathcal{Y}, \mathcal{Z}]_a - [\mathcal{Z}, \psi_i]_{c-} + [\mathcal{Z}, \psi_i]_{c+} + [\mathcal{Y}, \mathcal{Z}]_0 = 0.$$

It follows from Lemma 2.3 that $\mathcal{Z} \in \mathcal{R}(T_{\min})^{\perp} = \mathcal{N}(T_{\max})$. Thus $\xi = 0$, i.e., $\mathcal{D}_{\min}^{\perp} = \{0\}$.

Let us denote by $\mathcal{D}_{\min}^{*}$ the domain of the operator $T_{\min}^{*}$. Now, we will prove that $\mathcal{D}_{\min}^{*}=\mathcal{D}_{\max}$, and $T_{\min}^{*}\mathcal{Z}=T_{\max}\mathcal{Z}$ for all $\mathcal{Z}\in\mathcal{D}_{\min}^{*}$. It follows from Lemma 2.2 that $(\mathcal{Z},T_{\min}\mathcal{Y})=(T_{\max}\mathcal{Z},\mathcal{Y})$, where $\mathcal{Z}\in\mathcal{D}_{\min}$ and $\mathcal{Y}\in\mathcal{D}_{\max}$. Hence, the functional $(\mathcal{Z},T_{\min}(.))$ is continuous on $\mathcal{D}_{\min}$ and $\mathcal{Z}\in\mathcal{D}_{\min}^{*}$, i.e., $\mathcal{D}_{\max}\subset\mathcal{D}_{\min}^{*}$.

Now, we will prove that $\mathcal{D}_{\min}^{*}\subset\mathcal{D}_{\max}$. If $\mathcal{Z}\in\mathcal{D}_{\min}^{*}$, then $\mathcal{Z},\phi\in L_{V_1}^{2}\left[(a,c)\cup(c,b);E\right]$, where $\phi:=T_{\min}^{*}\mathcal{Z}$. Assume that $\mathcal{U}$ is a solution of the equation

$$J\mathcal{U}'(x)-V_2(x)\mathcal{U}(x)=V_1(x)\phi(x). \tag{2.9}$$

It follows from Lemma 2.2 that$(\phi,\mathcal{Y})=(T_{\max}\mathcal{U},\mathcal{Y})=(\mathcal{U},T_{\min}\mathcal{Y})$. This implies that

$$\begin{aligned}(\mathcal{Z}-\mathcal{U},T_{\min}\mathcal{Y})&=(\mathcal{Z},T_{\min}\mathcal{Y})-(\mathcal{U},T_{\min}\mathcal{Y})\\&=(T_{\min}^{*}\mathcal{Z},\mathcal{Y})-(\phi,\mathcal{Y})=0,\end{aligned}$$

i.e., $\mathcal{Y}-\mathcal{U}\in\mathcal{R}(T_{\min})^{\perp}$. By Lemma 2.3, we conclude that $\mathcal{Y}-\mathcal{U}\in\mathcal{N}(T_{\max})$.

Using (2.9), we deduce that

$$\begin{aligned}&J\mathcal{Z}'(x)-V_2(x)\mathcal{Z}(x)\\&=J\mathcal{U}'(x)-V_2(x)\mathcal{U}(x)=V_1(x)\phi(x),\end{aligned}$$

where $x\in[a,c)\cup(c,b]$. Since $\mathcal{Z},\phi\in L_{V_1}^{2}\left[(a,c)\cup(c,b);E\right]$, we see that $\mathcal{Z}\in\mathcal{D}_{\max}$ and $T_{\max}\mathcal{Z}=\phi=T_{\min}^{*}\mathcal{Z}$. □

3 Self-adjoint discontinuous matrix Sturm–Liouville operators

Now, we will give a criterion under which discontinuous matrix Sturm–Liouville operators are self-adjoint.

Let

$$\mathcal{D}:=\left\{\mathcal{Z}\in D_{\max}:\Sigma\mathcal{Z}(a)+\Lambda\mathcal{Z}(b)=0\right\}, \tag{3.1}$$

where Σ, Λ are $m\times 2n$ matrices such that $rank\,(\Sigma:\Lambda)=m$. We define the operator T by

$$T:\mathcal{D}\to L_{V_1}^{2}\left[(a,c)\cup(c,b);E\right], \tag{3.2}$$
$$\mathcal{Z}\to T\mathcal{Z}=F \text{ if and only if } \Gamma(\mathcal{Z})=V_1F \tag{3.3}$$

Let Ω and Υ be $(4n-m)\times 2n$ matrices, chosen so that $rank\,(\Omega:\Upsilon)=4n-m$ and

$$\begin{pmatrix}\Sigma&\Lambda\\\Omega&\Upsilon\end{pmatrix}$$

is nonsingular. Let

$$\begin{pmatrix}\widetilde{\Sigma}&\widetilde{\Lambda}\\\widetilde{\Omega}&\widetilde{\Upsilon}\end{pmatrix}$$

be chosen so that

$$\begin{pmatrix}-J&0\\0&J\end{pmatrix}=\begin{pmatrix}\widetilde{\Sigma}&\widetilde{\Lambda}\\\widetilde{\Omega}&\widetilde{\Upsilon}\end{pmatrix}^{*}\begin{pmatrix}\Sigma&\Lambda\\\Omega&\Upsilon\end{pmatrix}. \tag{3.4}$$

Then we have the following theorem.

Theorem 3.1. *The following relation holds*

$$(T_{\max}\mathcal{Z},\mathcal{Y})-(\mathcal{Z},T_{\max}\mathcal{Y})=\left[\widetilde{\Sigma}\mathcal{Y}(a)+\widetilde{\Lambda}\mathcal{Y}(b)\right]^{*}[\Sigma\mathcal{Z}(a)+\Lambda\mathcal{Z}(b)]$$
$$+\left[\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)\right]^{*}[\Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)],$$

where $\mathcal{Z},\mathcal{Y}\in\mathcal{D}_{\max}$.

Proof. By virtue of (2.7) and (3.4), we conclude that

$$(T_{\max}\mathcal{Z},\mathcal{Y})-(\mathcal{Z},T_{\max}\mathcal{Y})$$

$$=[\mathcal{Z},\mathcal{Y}]_{b}+[\mathcal{Z},\mathcal{Y}]_{c-}-[\mathcal{Z},\mathcal{Y}]_{a}-[\mathcal{Z},\mathcal{Y}]_{c+}$$

$$=\begin{pmatrix}\mathcal{Y}^{*}(a) & \mathcal{Y}^{*}(b)\end{pmatrix}\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}\mathcal{Z}(a)\\ \mathcal{Z}(b)\end{pmatrix}$$

$$=\begin{pmatrix}\mathcal{Y}^{*}(a) & \mathcal{Y}^{*}(b)\end{pmatrix}\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Upsilon}\end{pmatrix}^{*}\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Upsilon\end{pmatrix}\begin{pmatrix}\mathcal{Z}(a)\\ \mathcal{Z}(b)\end{pmatrix}$$

$$=\left[\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Upsilon}\end{pmatrix}\begin{pmatrix}\mathcal{Y}(a)\\ \mathcal{Y}(b)\end{pmatrix}\right]^{*}\left[\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Upsilon\end{pmatrix}\begin{pmatrix}\mathcal{Z}(a)\\ \mathcal{Z}(b)\end{pmatrix}\right]$$

$$=\begin{pmatrix}\widetilde{\Sigma}\mathcal{Y}(a)+\widetilde{\Lambda}\mathcal{Y}(b)\\ \widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)\end{pmatrix}^{*}\begin{pmatrix}\Sigma\mathcal{Z}(a)+\Lambda\mathcal{Z}(b)\\ \Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)\end{pmatrix}.$$

□

Now, we will describe the adjoint of the operator T.

Theorem 3.2. *Let* $\mathcal{Y}\in\mathcal{D}^{*}$, *where*

$$\mathcal{D}^{*}:=\left\{\mathcal{Y}\in D_{\max}:\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)=0\right\}.$$

Then $T^{*}\mathcal{Y}=F_{1}$ *if and only if*

$$J\mathcal{Y}'-V_{2}(x)\mathcal{Y}(x)=V_{1}(x)F_{1}(x).$$

Proof. It is clear that $T_{\min}\subset T^{*}\subset T_{\max}$ since $T_{\min}\subset T\subset T_{\max}$. Let $\mathcal{Z}\in\mathcal{D}$ and $\mathcal{Y}\in\mathcal{D}^{*}$. By Theorem 3.1, we conclude that

$$(T\mathcal{Z},\mathcal{Y})-(\mathcal{Z},T^{*}\mathcal{Y})=\left[\widetilde{\Sigma}\mathcal{Y}(a)+\widetilde{\Lambda}\mathcal{Y}(b)\right]^{*}[\Sigma\mathcal{Z}(a)+\Lambda\mathcal{Z}(b)]$$
$$+\left[\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)\right]^{*}[\Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)].$$

Then

$$0=\left[\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)\right]^{*}[\Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)].$$

Thus we get $\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)=0$, since $\Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)$ is arbitrary.

Conversely, if $\mathcal{Y}$ satisfies the criteria listed above then $\mathcal{Y}\in\mathcal{D}^*$.

We will find parametric boundary conditions for $\mathcal{D}$ and $\mathcal{D}^*$. Recall that

$$\Omega\mathcal{Z}(a)+\Upsilon\mathcal{Z}(b)=F_2,\ \ \Sigma\mathcal{Z}(a)+\Lambda\mathcal{Z}(b)=0, \tag{3.5}$$

where F_2 is arbitrary. Hence, we obtain

$$\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Upsilon\end{pmatrix}\begin{pmatrix}\mathcal{Z}(a)\\ \mathcal{Z}(b)\end{pmatrix}=\begin{pmatrix}0\\ F_2\end{pmatrix}. \tag{3.6}$$

If we multiply both sides of (3.6) by

$$\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Upsilon}\end{pmatrix}^*,$$

then we deduce that

$$\begin{pmatrix}\mathcal{Z}(a)\\ \mathcal{Z}(b)\end{pmatrix}=\begin{pmatrix}J\widetilde{\Omega}^*F_2\\ -J\widetilde{\Upsilon}^*F_2\end{pmatrix}. \tag{3.7}$$

Similarly, one can find parametric boundary conditions for $\mathcal{D}^*$. Since

$$\widetilde{\Omega}\mathcal{Y}(a)+\widetilde{\Upsilon}\mathcal{Y}(b)=0,\ \ \widetilde{\Sigma}\mathcal{Y}(a)+\widetilde{\Lambda}\mathcal{Y}(b)=F_3,$$

where F_3 is arbitrary, we have

$$\begin{pmatrix}\mathcal{Y}^*(a) & \mathcal{Y}^*(b)\end{pmatrix}\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Upsilon}\end{pmatrix}^*=\begin{pmatrix}F_3^* & 0\end{pmatrix}. \tag{3.8}$$

Multiplying both sides of (3.8) by

$$\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Upsilon\end{pmatrix}\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}$$

it follows that

$$\mathcal{Y}(a)=-J\Sigma^*F_3,\ \ \mathcal{Y}(b)=J\Lambda^*F_3. \tag{3.9}$$

□

Now, we have the following theorem.

Theorem 3.3. *$\Sigma J\Sigma^*=\Lambda J\Lambda^*$ and $m=2n$ if and only if T is a self-adjoint operator.*

Proof. Let $\Sigma J\Sigma^*=\Lambda J\Lambda^*$. Then we get

$$\begin{pmatrix}-\Sigma J & \Lambda J\end{pmatrix}\begin{pmatrix}\Sigma^*\\ \Lambda^*\end{pmatrix}=0.$$

That is, the columns $\begin{pmatrix}\Sigma^*\\ \Lambda^*\end{pmatrix}$ satisfy the equation

$$\begin{pmatrix}-\Sigma J & \Lambda J\end{pmatrix}X=0.$$

By virtue of (3.5) and (3.7), we conclude that

$$\begin{pmatrix} -\Sigma J & \Lambda J \end{pmatrix} \begin{pmatrix} \widetilde{\Omega}^* \\ \widetilde{\Upsilon}^* \end{pmatrix} = 0.$$

Thus, there must be a constant, nonsingular matrix K such that

$$\begin{pmatrix} \widetilde{\Omega}^* \\ \widetilde{\Upsilon}^* \end{pmatrix} K^* = \begin{pmatrix} \Sigma^* \\ \Lambda^* \end{pmatrix}.$$

or

$$\begin{pmatrix} \Sigma & \Lambda \end{pmatrix} = K \begin{pmatrix} \widetilde{\Omega} & \widetilde{\Upsilon} \end{pmatrix}$$

The conditions $\Sigma \mathcal{Z}(a) + \Lambda \mathcal{Z}(b) = 0$ and $\Omega \mathcal{Z}(a) + \Upsilon \mathcal{Z}(b) = 0$ are equivalent. Since the forms of T and T^* are the same, we see that $T = T^*$.

Conversely, let T be a self-adjoint operator. Then $\mathcal{Z}$ satisfies the boundary conditions for $\mathcal{D}$, i.e., $\Sigma \mathcal{Z}(a) + \Lambda \mathcal{Z}(b) = 0$. By (3.9), we get

$$\begin{aligned} \Sigma(-J\Sigma^* F_3) + \Lambda(J\Lambda^* F_3) = 0 \\ [\Sigma J \Sigma^* - \Lambda J \Lambda^*] F_3 = 0. \end{aligned}$$

Then we have $\Sigma J \Sigma^* = \Lambda J \Lambda^*$, since F_3 is arbitrary. □

4 Eigenfunction expansions

Let

$$\mathcal{D}_1 := \{\mathcal{Z} \in \mathcal{D} : \Sigma J \Sigma^* = \Lambda J \Lambda^*\}, \tag{4.1}$$

where $\mathcal{D}$ is defined in (3.1). We define the self-adjoint operator T_1 by

$$T_1 : \mathcal{D}_1 \to L^2_{V_1}[(a,c) \cup (c,b); E], \tag{4.2}$$

$$\mathcal{Z} \to T_1 \mathcal{Z} \Leftrightarrow J\mathcal{Z}' - V_2 \mathcal{Z} = V_1 F. \tag{4.3}$$

Let $Z(x,\lambda)$ be a fundamental matrix solution of the equation $\Gamma(\mathcal{Z}) = 0$ satisfying $Z(a,\lambda) = I$. It is clear that

$$Z^*(x,\lambda) J Z(x,\lambda) = J \tag{4.4}$$

for all $x \in [a,c) \cup (c,b]$ ([24]).

Theorem 4.1. *The resolvent operator of T_1 is given by the formula*

$$R_1(\lambda) F(x) = (T_1 - \lambda I)^{-1} F(x)$$

$$= \int_a^c G(x,t,\lambda) V_1(t) F(t)\, dt + \int_c^b G(x,t,\lambda) V_1(t) F(t)\, dt,$$

where $G(x,t,\lambda)$ is the matrix Green function defined as

$$G(x,t,\lambda) =$$

$$\begin{cases} Z(x,\lambda)[\Sigma + \Lambda Z(b,\lambda)]^{-1} \Sigma J Z^*(t,\overline{\lambda}), & a \le t \le x \le b,\ t \ne c,\ x \ne c \\ -Z(x,\lambda)[\Sigma + \Lambda Z(b,\lambda)]^{-1} \Lambda J Z^*(t,\overline{\lambda}), & a \le x \le t \le b,\ t \ne c,\ x \ne c. \end{cases}$$

Proof. Let $\mathcal{Z}$ satisfy the equation $\Gamma(\mathcal{Z}) = V_1 F$. By using the method of variation of constants, we seek a solution of the form

$$\mathcal{Z}(x,\lambda) = Z(x,\lambda) K(x,\lambda),$$

where $K(x,\lambda)$ is a $2n+1$ vector function. Then we have

$$\begin{aligned} J\mathcal{Z}' &= JZ'K + JZK', \\ (\lambda V_1 + V_2)\mathcal{Z} &= (\lambda V_1 + V_2) ZK. \end{aligned}$$

Hence

$$\begin{aligned} V_1 F &= J\mathcal{Z}' - (\lambda V_1 + V_2)\mathcal{Z} \\ &= JZ'K + JZK' - (\lambda V_1 + V_2) ZK \\ &= [JZ' - (\lambda V_1 + V_2) Z] K + JZK' = JZK' \end{aligned}$$

i.e., $K' = [JZ]^{-1} V_1 F$. It follows from (4.4) that $K' = -JZ^*\left(x,\overline{\lambda}\right) V_1 F$. Then, we conclude that

$$\begin{aligned} \mathcal{Z}(x,\lambda) = &-Z(x,\lambda)\int_a^c JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt \\ &- Z(x,\lambda)\int_c^x JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt + Z(x,\lambda) K_1. \end{aligned}$$

By the condition $\Sigma Z(a) + \Lambda Z(b) = 0$, we get

$$\begin{aligned} \mathcal{Z}(a) &= K_1, \\ \mathcal{Z}(b) &= -Z(b,\lambda)\int_a^c JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt \\ &\quad - Z(b,\lambda)\int_c^b JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt + Z(b,\lambda) K_1. \end{aligned}$$

Thus, we get

$$\begin{aligned} \mathcal{Z}(x,\lambda) = &-Z(x,\lambda)[\Sigma + \Lambda Z(b)]^{-1}\Sigma\int_a^c JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt \\ &- Z(x,\lambda)[\Sigma + \Lambda Z(b)]^{-1}\Sigma\int_c^x JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt \\ &+ Y(x,\lambda)[\Sigma + \Lambda Z(b)]^{-1}\Lambda\int_x^b JZ^*\left(x,\overline{\lambda}\right) V_1(t) F(t)\,dt. \end{aligned}$$

□

Theorem 4.2. *The operator $R(\lambda)$ exists for all nonreal λ, and is a bounded operator. It exists also for all real λ for which* $\det[\Sigma + \Lambda Z(b)] \neq 0$ *as a bounded operator. The spectrum of T_1 consists entirely of isolated eigenvalues, zeros of the equation* $\det[\Sigma + \Lambda Z(b)] = 0$. *Furthermore, eigenfunctions associated with different eigenvalues are mutually orthogonal.*

Proof. It is clear that the operator $R(\lambda)$ exists for all real λ except the zeros of the equation $\det[\Sigma + \Lambda Z(b)] = 0$. Since T_1 is a self-adjoint operator, it follows that the operator $R(\lambda)$ exists for all nonreal λ. The spectrum of T_1 consists entirely of isolated eigenvalues, zeros of $\det[\Sigma + \Lambda Z(b)] = 0$ because $\det[\Sigma + \Lambda Z(b)]$ is analytic in λ and is not identically zero. These zeros can accumulate only at $\pm\infty$.

Now, we will prove that the operator $R(\lambda)$ is a bounded operator. Let

$$f(\eta) = V_1^{1/2}(\eta) F(\eta)$$

and

$$W(x,\eta,\lambda) = V_1^{1/2}(\eta) G(x,t,\lambda) V_1^{1/2}(x),$$

where $V^{1/2}$ is a square root of the matrix V_1. Then, we have

$$\|R(\lambda) F\|^2 = \|\mathcal{Z}\|^2 = \int_a^c \mathcal{Z}^* V_1 \mathcal{Z} dx + \int_c^b \mathcal{Z}^* V_1 \mathcal{Z} dx$$

$$= \int_a^c \left[\int_a^c G(x,\eta,\lambda) V_1(\eta) F(\eta) d\eta\right]^* V_1(x) \left[\int_a^c G(x,\eta,\lambda) V_1(\eta) F(\eta) d\eta\right] dx$$

$$+ \int_c^b \left[\int_c^b G(x,\eta,\lambda) V_1(\eta) F(\eta) d\eta\right]^* V_1(x) \left[\int_c^b G(x,\eta,\lambda) V_1(\eta) F(\eta) d\eta\right] dx$$

$$= \int_a^c \left[\int_a^c f^*(\eta) W^*(x,\eta,\lambda) d\eta\right] \left[\int_a^c W(x,t,\lambda) f(t) dt\right] dx$$

$$+ \int_c^b \left[\int_c^b f^*(\eta) W^*(x,\eta,\lambda) d\eta\right] \left[\int_c^b W(x,t,\lambda) f(t) dt\right] dx.$$

By using Cauchy–Schwarz's inequality, we get $\|\mathcal{Z}\|^2 \leq \|W\|^2 \|f\|^2$, where

$$\|W\|^2 = \int_a^c \int_a^c \sum_{i=1}^{2n} \sum_{j=1}^{2n} |W_{ij}(x,\eta,\lambda)|^2 d\eta dx$$

$$+ \int_c^b \int_c^b \sum_{i=1}^{2n} \sum_{j=1}^{2n} |W_{ij}(x,\eta,\lambda)|^2 d\eta dx.$$

Finally, it is easily seen that eigenfunctions associated with different eigenvalues are mutually orthogonal since T_1 is a self-adjoint operator. □

There is no loss of generality in assuming that zero is not an eigenvalue. Then, the solution of the following problem

$$J\mathcal{Z}' - V_2\mathcal{Z} = V_1 F,$$
$$\mathcal{Z}(c+) = C\mathcal{Z}(c-),$$
$$\Sigma\mathcal{Z}(a) + \Lambda\mathcal{Z}(b) = 0,$$

is given by

$$\mathcal{Z}(x) = \int_a^c G(x,t)\, V_1(t)\, F(t)\, dt + \int_c^b G(x,t)\, V_1(t)\, F(t)\, dt,$$

where $G(x,t) = G(x,t,0)$.

Let $\mathcal{Z} = T_2 F = T_1^{-1} F$. Then we have the following theorems.

Theorem 4.3. *T_2 is a bounded operator and*

$$\|T_2\| = \sup\left\{\left|\lambda_m^{-1}\right| : \lambda_m \in \sigma(T_1)\right\}.$$

Proof. If $T_1 \chi_m = \lambda_m \chi_m$ $(m \in \mathbb{N})$, then $T_2 \chi_m = \tau_m \chi_m$, where $\tau_m = \frac{1}{\lambda_m}$. Then, we have

$$\|T_2\| = \sup_{\substack{\chi \in L^2_{V_1}[(a,c)\cup(c,b);E] \\ \|\chi\|=1}} |(T_2 \chi, \chi)|$$

$$= \sup\{|\tau_m| : \tau_m \in \sigma(T_2)\} = \sup\left\{\left|\lambda_m^{-1}\right| : \lambda_m \in \sigma(T_1)\right\}.$$

□

Now, we shall order the eigenvalues of T_2 such that

$$|\tau_1| \geq |\tau_2| \geq ... \geq |\tau_m| \geq ...,$$

where

$$\lim_{m\to\infty} |\tau_m| = 0. \tag{4.5}$$

Let us define $\{T_{2,m}\}_{m=1}^{\infty}$ by

$$T_{2,m} F = T_2 F - \sum_{i=1}^{m-1} \tau_i \chi_i (F, \chi_i).$$

Theorem 4.4. *$\|T_{2,m}\| = |\tau_m|$ $(m \in \mathbb{N})$, and*

$$\lim_{m\to\infty} T_{2,m} = 0. \tag{4.6}$$

Proof. It is clear that

$$T_{2,m} \chi_j = \begin{cases} 0, & \text{if } 1 \leq j \leq m-1 \\ \tau_j \chi_j, & \text{if } m \leq j < \infty. \end{cases}$$

Further $T_{2,m}$ is bounded and self-adjoint. Then, we have

$$\|T_{2,m}\| = \sup_{\substack{\chi \in L^2_{V_1}[(a,c)\cup(c,b);E] \\ \|\chi\|=1}} |(T_{2,m} \chi, \chi)|$$

$$= \sup_{\substack{\chi \in L^2_{V_1}[(a,c)\cup(c,b);E] \\ \|\chi\|=1 \\ \chi \neq \chi_1, \dots, \chi_{m-1}}} |(T_{2,m} \chi, \chi)| = |\tau_m|.$$

It follows from (4.5) that

$$\lim_{m\to\infty} T_{2,m} = 0.$$

□

Theorem 4.5. *Let $F \in L^2_{V_1}[(a,c)\cup(c,b);E]$ and $\mathcal{Z} \in \mathcal{D}_1$. Then we have*

$$F = \sum_{i=1}^{\infty} \chi_i (F, \chi_i), \ T_2F = \sum_{i=1}^{\infty} \tau_i \chi_i (F, \chi_i),$$

$$T_1\mathcal{Z} = \sum_{i=1}^{\infty} \lambda_i \chi_i (\mathcal{Z}, \chi_i).$$

Proof. It follows from (4.6) that

$$T_2F = \sum_{i=1}^{\infty} \tau_i \chi_i (F, \chi_i). \tag{4.7}$$

Applying T_1 to equality (4.7), we conclude that

$$F = \sum_{i=1}^{\infty} \chi_i (F, \chi_i).$$

Further,

$$(F, \chi_i) = (T_1\mathcal{Z}, \chi_i) = (\mathcal{Z}, T_1\chi_i) = \lambda_i (\mathcal{Z}, \chi_i).$$

Thus, we get

$$T_1\mathcal{Z} = \sum_{i=1}^{\infty} \lambda_i \chi_i (\mathcal{Z}, \chi_i).$$

□

Theorem 4.6. *There exists a collection of projection operators $\{E(\lambda)\}$ satisfying*
(a) $\lim_{\lambda\to\infty} E(\lambda) = I$, $\lim_{\lambda\to-\infty} E(\lambda) = 0$,
(b) $E(\lambda_1) \leq E(\lambda_2)$ *when* $\lambda_1 \leq \lambda_2$,
(c) $E(\lambda)$ *is continuous from above,*
(d) for all $F \in L^2_{V_1}[(a,c)\cup(c,b);E]$ *and* $\mathcal{Z} \in \mathcal{D}_1$,

$$F = \int_{-\infty}^{\infty} dE(\lambda) F, \ T_2F = \int_{-\infty}^{\infty} \frac{1}{\lambda} dE(\lambda) F,$$

$$T_1\mathcal{Z} = \int_{-\infty}^{\infty} \lambda dE(\lambda) \mathcal{Z}.$$

Proof. Let us define

$$P_iF = \chi_i (F, \chi_i),$$

where P_i is a projection operator. If we define

$$E(\lambda) F = \sum_{\lambda_i \leq \lambda} P_i F,$$

then $E(\lambda)$ generates a Stieltjes measure. The integrals in (d) are obtained from this series. □

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