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IPHP TRANSFORMATIONS ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD WITH RESPECT TO A CLASS OF LIFT METRICS

M. Zohrehvand

Communicated by J.A. Tussupov

Key words: g-natural metrics, infinitesimal paraholomorphically projective transformations, adapted almost paracomplex structure.

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Abstract. Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle. In this article, we study the infinitesimal paraholomorphically projective (IPHP) transformations on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are real constants with $\alpha(\alpha + \gamma) - \beta^2 \neq 0$ and g^S, g^C and g^V are diagonal lift, complete lift and vertical lift of g, respectively. We determine this type of transformations and then prove that if $(TM_n, \tilde{g})$ has a non-affine infinitesimal paraholomorphically projective transformation, then M_n and TM_n are locally flat.

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1 Introduction

Let M_n be a connected manifold of dimension n and TM_n its tangent bundle. In this paper, we assume that all geometric objects will be discussed in the class C^∞, with the dimension $n > 1$. Moreover, the set of all tensor fields of type (r, s) on M_n and TM_n are denoted by $\Im^r_s(M_n)$ and $\Im^r_s(TM_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called a projective transformation. Also, a transformation on M_n which preserves the connection is called an affine transformation. Therefore, we can say that an affine transformation is a projective transformation which preserves the affine parameter with the geodesics.

Let V be a vector field on M_n and $\{\phi_t\}$ its local one-parameter group. V is called an infinitesimal projective (affine) transformation, if every ϕ_t is a projective (affine) transformation on M_n.

It is well known that, the necessary and sufficient conditions for a vector field V to be an infinitesimal projective transformation are such that, for every $X, Y \in \Im^1_0(M_n)$,

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

where Ω is a 1-form on M_n and L_V is the Lie derivation with respect to V. In this case Ω is called the associated 1-form of V. In the case of $\Omega = 0$, one can see that V is an infinitesimal affine transformation [10].

Almost paracomplex structures on a manifold were introduced by Rasevskii in [8]. An almost paracomplex structure on a manifold M_n is a tensor field $\varphi \in \Im^1_1(M_n)$, where $\varphi^2 = Id$, $\varphi \neq Id$ and the two eigenbundles T^+M_n and T^-M_n corresponding to the eigenvalues ± 1 of φ, have the same rank. In this case, (M_n, φ) is called an almost paracomplex manifold. It would be noted that, in this case, the dimension of M_n is necessarily even. If the both distributions T^+M_n and T^-M_n are integrable, we say that almost paracomplex structure φ is integrable and then (M_n, φ) is called a paracomplex manifold. For more details, one can refer to [3, 4, 9].

Let ∇ be an affine connection on an almost paracomplex manifold (M_n, φ). An infinitesimal paraholomorphically projective (IPHP) transformation on M_n is a vector field V on M_n such that for any $X, Y \in \Im_0^1(M_n)$, we have

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X + \Omega(\varphi X)\varphi Y + \Omega(\varphi Y)\varphi X,$$

where Ω is a 1-form on M_n. It is also called the associated 1-form of V [5, 7]. If $\Omega = 0$, it is obvious that V is an affine transformation.

Let $g = (g_{ji})$ be a Riemannian metric on M_n. It is well known that we can define from g several (pseudo-) Riemannian metrics on TM_n, where they are called the lift metrics of g, as follows: 1) complete lift metric or lift metric II is denoted by g^C, 2) diagonal lift metric or Sasaki metric or lift metric I+III is denoted by g^S, 3) lift metric I+II and 4) lift metric II+III, where I:= $g_{ji}dx^j dx^i$, II:= $2g_{ji}dx^j\delta y^i$ and III:= $g_{ji}\delta y^j \delta y^i$ are bilinear differential forms defined globally on TM_n. It should be noted that in literature I:= $g_{ji}dx^j dx^i$ is called the vertical lift of g and denoted by g^V. For more details on lift metrics, one can refer to [11].

Abbassi and Sarih in [1] defined the "g-natural metrics" on TM_n of a Riemannian metric g and studied a special class of this metrics in [2], that it is denoted by

$$\tilde{g} := \alpha g^S + \beta g^C + \gamma g^V,$$

where α, β and γ are real constants with $\alpha > 0$ and $\lambda := \alpha(\alpha+\gamma) - \beta^2 > 0$. In this case, $\tilde{g}$ is a Riemannian metric on TM_n.

Infinitesimal paraholomorphically projective transformations on the tangent bundle of a Riemannian manifold (M_n, g) with respect to the Levi-Civita connection of Sasaki metric g^S are determined in [6]. Moreover, it is proved that if (TM_n, g^S) admits a non-affine paraholomorphically projective transformation, then M_n and TM_n are locally flat.

The main goal of this paper is studying infinitesimal paraholomorphically projective transformations on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric

$$\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V, \tag{1.1}$$

where α, β and γ are real constants and $\lambda := \alpha(\alpha+\gamma) - \beta^2 \neq 0$. It is obvious that the metric $\tilde{g}$ is a generalization of above lift metrics.

In fact, we prove the following theorems.

Theorem 1.1. *Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle with the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are real constants, $\alpha \neq 0$ and $\lambda := \alpha(\alpha+\gamma) - \beta^2 \neq 0$, and the adapted almost paracomplex structure φ. Then $\tilde{V}$ is an IPHP transformation with associated 1-form $\tilde{\Omega}$ on TM_n if and only if there exist $\psi \in \Im_0^0(M_n)$, $B = (B^h), D = (D^h) \in \Im_0^1(M_n)$, $\Phi := (\Phi_i) \in \Im_1^0(M_n)$ and $A = (A_i^h)$, $C = (C_i^h) \in \Im_1^1(M_n)$, satisfying*

I. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A_a^h, D^h + y^a C_a^h + 2y^a \Phi_a y^h)$,

II. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, \Phi_i), \quad \Psi_i = \partial_i \psi$,

III. $\nabla_i \Phi_j = 0$,

IV. $\beta(\Phi_c R_{bji}^h + \Phi_b R_{cji}^h) = 0$,

V. $\nabla_i A_j^h = -\frac{\alpha^2}{2\lambda} D^a R_{aji}^h$,

VI. $A_i^a R_{bja}^h = 0$, $A_a^h R_{bji}^a = 0$,

VII. $B^a \nabla_a R_{bji}^h = R_{bji}^a \nabla_a B^h - R_{bja}^h \nabla_i B^a - R_{aji}^h C_b^a - R_{bai}^h C_j^a$,

VIII. $\nabla_i C_j^h = B^a R_{iaj}^h + \frac{\alpha\beta}{2\lambda} D^a R_{aji}^h$,

IX. $R_{kji}^a(\beta \nabla_a B^h - \beta C_a^h + \alpha \nabla_a D^h) = 0$, *X.* $L_B\Gamma^h_{ji} = \nabla_j\nabla_i B^h + B^a R^h_{aji} = 2\Psi_j\delta^h_i + 2\Psi_i\delta^h_j - \frac{\alpha\beta}{2\lambda}D^a(R^h_{aji} + R^h_{aij})$,

XI. $\nabla_j\nabla_i D^h = \frac{\alpha(\alpha+\gamma)}{2\lambda}R^h_{jia}D^a - \frac{\beta^2}{\lambda}R^h_{jai}D^a$,

XII. $\beta D^a\nabla_j R^h_{bai} = -\beta(R^h_{baj}\nabla_i D^a + R^h_{bai}\nabla_j D^a) - \beta R^a_{jib}\nabla_a D^h$
$- \beta R^h_{bai}(2\frac{\beta^2}{\alpha}\nabla_j B^a - 2\frac{\beta^2}{\alpha}C^a_j - \nabla_j D^a)$,

where $\tilde{V} := (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, *and* $\tilde{\Omega} := (\tilde{\Omega}_h, \tilde{\Omega}_{\bar{h}}) = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}}\delta y^h$.

Theorem 1.2. *Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle with the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{g} = \beta g^C + \gamma g^V$ where β and γ are real constants with $\beta \neq 0$, and the adapted almost paracomplex structure φ. Then $\tilde{V}$ is an IPHP transformation with associated 1-form $\tilde{\Omega}$ on TM_n if and only if there exist $\psi \in \Im^0_0(M_n)$, $B = (B^h), D = (D^h) \in \Im^1_0(M_n)$, $\Phi := (\Phi_i) \in \Im^0_1(M_n)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M_n)$, satisfying*

I. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A^h_a, D^h + y^a C^h_a + 2y^a\Phi_a y^h)$,

II. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, \Phi_i), \quad \Psi_i = \partial_i\psi$,

III. $\nabla_i\Phi_j = 0$,

IV. $\nabla_i A^h_j = 0$,

V. $A^a_i R^h_{bja} = 0,\ A^h_a R^a_{bji} = 0$,

VI. $B^a\nabla_a R^h_{bji} = R^h_{abi}\nabla_j B^a + R^h_{jba}\nabla_i B^a + R^h_{jai}C^a_b - R^a_{jbi}C^h_a$,

VII. $\nabla_i C^h_j = B^a R^h_{iaj}$,

VIII. $L_B\Gamma^h_{ji} = \nabla_j\nabla_i B^h + B^a R^h_{aji} = 2\Psi_j\delta^h_i + 2\Psi_i\delta^h_j$,

IX. $L_D\Gamma^h_{ji} = \nabla_j\nabla_i D^h + D^a R^h_{aji} = 0$,

where $\tilde{V} := (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, *and* $\tilde{\Omega} := (\tilde{\Omega}_h, \tilde{\Omega}_{\bar{h}}) = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}}\delta y^h$.

Theorem 1.3. *Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle with the Levi-Civita connection of pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, where α, β and γ are real constants with $\alpha\beta \neq 0$, $\alpha(\alpha+\gamma) - \beta^2 \neq 0$, and the adapted almost paracomplex structure φ. If $(TM_n, \tilde{g})$ admits a non-affine IPHP transformation, such that $\|\Phi\| \neq 0$, then M_n and TM_n are locally flat.*

2 Preliminaries

Here, we give definitions and theorems on M_n and TM_n, that are needed later. The details of them can be founded in [11, 12]. In this paper, indices $a, b, c, i, j, k, \ldots$ have range in $\{1, \ldots, n\}$.

Let M_n be a manifold that covered by coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The tangent bundle of M_n is defined by $TM_n := \bigcup_{x\in M} T_x(M_n)$, where $T_x(M_n)$ is the tangent space of M_n at any point $x \in M_n$. The elements of TM_n are denoted by (x, y) where $y \in T_x(M_n)$ and the natural projection $\pi : TM_n \to M_n$, is given by $\pi(x, y) := x$.

Let (M_n, g) be a Riemannian manifold and ∇ the Levi-Civita connection associated with g. The coefficients of ∇ with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji}, i.e. $\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h$.

Using the Levi-Civita Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM_n, as follows

$$E_i := \partial_i - y^b\Gamma^h_{bi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame field is called the adapted frame on TM_n. By define $\delta y^h := dy^h + y^b\Gamma^h_{ab}dx^a$, one can see that $\{dx^h, \delta y^h\}$, is the dual frame of $\{E_i, E_{\bar{i}}\}$. By the straightforward calculations, the following lemmas are proved.

Lemma 2.1. *The Lie brackets of the adapted frame $\{E_i, E_{\bar{i}}\}$ satisfy the following identities:*

1. $[E_j, E_i] = y^b R^a_{ijb} E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,

where R^a_{ijb} are the coefficients of the Riemannian curvature tensor of ∇.

Lemma 2.2. *Let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a vector field on TM_n. Then*

1. $[\tilde{V}, E_i] = -(E_i \tilde{V}^a) E_a + (\tilde{V}^c y^b R^a_{icb} - \tilde{V}^{\bar{b}} \Gamma^a_{bi} - E_i \tilde{V}^{\bar{a}}) E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = -(E_{\bar{i}} \tilde{V}^a) E_a + (\tilde{V}^b \Gamma^a_{bi} - E_{\bar{i}} \tilde{V}^{\bar{a}}) E_{\bar{a}}$.

Using the adapted frame $\{E_h, E_{\bar{h}}\}$, we can define a tensor field $\varphi \in \Im^1_1(TM_n)$, as follows

$$\varphi(E_h) = E_h, \quad \varphi(E_{\bar{h}}) = -E_{\bar{h}}.$$

We see that $\varphi \neq Id$ and $\varphi^2 = Id$. Thus φ is a paracomplex structure on TM_n which is called an adapted paracomplex structure. It is well known that φ is integrable if and only if M_n is locally flat.

Let $g = (g_{ji})$ be a Riemannian metrics on a manifold M_n. We can define several Riemannian or pseudo-Riemannian metrics on TM_n, from g, as follows

II: $2g_{ji} dx^j \delta y^i$,

I+II: $g_{ji} dx^j dx^i + 2g_{ji} dx^j \delta y^i$,

I+III: $g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i$,

II+III: $2g_{ji} dx^j \delta y^i + g_{ji} \delta y^j \delta y^i$

where

I: $g_{ji} dx^j dx^i$,

II: $2g_{ji} dx^j \delta y^i$,

III: $g_{ji} \delta y^j \delta y^i$,

are all quadratic differential forms which are globally defined on TM_n. It should be mentioned that the metric II is called the complete lift metric and denoted by g^C, the metric I+III is called the Sasakian metric and denoted by g^S, and the quadratic form I is called the vertical lift and denoted by g^V. For more details, one can refer to [10].

Abbassi and Sarih in [2] studied a special class of g-natural metrics on TM_n that denoted by

$$\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V,$$

where α, β and γ are constants with $\alpha > 0$, and $\alpha(\alpha + \gamma) - \beta^2 > 0$.

Now, let $\tilde{g} := \alpha g^S + \beta g^C + \gamma g^V$, where α, β, and γ are real constants with $\lambda := \alpha(\alpha + \gamma) - \beta^2 \neq 0$. In this case, one can see that $\tilde{g}$ is the generalization of the above lifted metrics, for example, putting $\alpha = \beta = 1$ and $\gamma = -1$, then $\tilde{g} = g^S + g^C - g^V$ which is the metric II+III.

The coefficients of Levi-Civita connection $\tilde{\nabla}$ of the pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$, with respect to the frame field $\{E_i, E_{\bar{i}}\}$ are computed in [2]. In fact, we have the following lemma.

Lemma 2.3. *Let $\tilde{\nabla}$ be the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{g} = \alpha g^S + \beta g^C + \gamma g^V$ on TM_n, where α, β, and γ are real constants with $\lambda := \alpha(\alpha + \gamma) - \beta^2 \neq 0$. Then we have*

$$\tilde{\nabla}_{E_j} E_i = \{\Gamma^h_{ji} + \tfrac{\alpha\beta}{2\lambda} y^k (R^h_{kji} + R^h_{kij})\} E_h + y^k \{\tfrac{\beta^2}{\lambda} R^h_{jki} - \tfrac{\alpha(\alpha+\gamma)}{2\lambda} R^h_{jik}\} E_{\bar{h}},$$

$$\tilde{\nabla}_{E_j} E_{\bar{i}} = \tfrac{\alpha^2}{2\lambda} y^k R^h_{kij} E_h + (\Gamma^h_{ji} - \tfrac{\alpha\beta}{2\lambda} y^k R^h_{kij}) E_{\bar{h}},$$

$$\tilde{\nabla}_{E_{\bar{j}}} E_i = \tfrac{\alpha^2}{2\lambda} y^k R^h_{kji} E_h - \tfrac{\alpha\beta}{2\lambda} y^k R^h_{kji} E_{\bar{h}},$$

$$\tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.$$

where Γ^h_{ji} denotes the coefficients of Riemannian connection ∇ with respect to g.

3 Proof of theorems

In this section, we only prove Theorems 1.1 and 1.3, because the proof of Theorem 1.2 is similar to that of Theorem 1.1.

Proof of Theorem 1.1

First, we prove the necessary conditions. Let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal paraholomorphically projective transformation and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$ its the associated 1-form on TM_n. Thus for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TM_n)$, we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X} + \tilde{\Omega}(\varphi\tilde{X})\varphi\tilde{Y} + \tilde{\Omega}(\varphi\tilde{Y})\varphi\tilde{X}. \tag{3.1}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = 2\tilde{\Omega}_{\bar{j}} E_{\bar{i}} + 2\tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we obtain

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^h - \frac{\alpha^2}{2\lambda} y^b (R^h_{iba} \partial_{\bar{j}} \tilde{V}^a + R^h_{jba} \partial_{\bar{i}} \tilde{V}^a) = 0, \tag{3.2}$$

and

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} + \frac{\alpha\beta}{2\lambda} y^b (R^h_{iba} \partial_{\bar{j}} \tilde{V}^a + R^h_{jba} \partial_{\bar{i}} \tilde{V}^a) = \tilde{\Omega}_{\bar{j}} \delta^h_i + \tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.3}$$

One can see that (3.2) can be rewritten as follows:

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^h = \frac{\alpha^2}{2\lambda} \{\partial_{\bar{j}}(y^b R^h_{iba} \tilde{V}^a) + \partial_{\bar{i}}(y^b R^h_{jba} \tilde{V}^a)\}. \tag{3.4}$$

By differentiating with respect to y^k from (3.4) we have

$$\begin{aligned} \partial_{\bar{k}} \partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^h &= \frac{\alpha^2}{2\lambda} \{\partial_{\bar{k}} \partial_{\bar{j}}(y^b R^h_{iba} \tilde{V}^a) + \partial_{\bar{k}} \partial_{\bar{i}}(y^b R^h_{jba} \tilde{V}^a)\} \\ &= \frac{\alpha^2}{2\lambda} \{\partial_{\bar{j}} \partial_{\bar{i}}(y^b R^h_{iba} \tilde{V}^a) + \partial_{\bar{j}} \partial_{\bar{k}}(y^b R^h_{jba} \tilde{V}^a)\} \\ &= \frac{\alpha^2}{2\lambda} \{\partial_{\bar{i}} \partial_{\bar{k}}(y^b R^h_{iba} \tilde{V}^a) + \partial_{\bar{i}} \partial_{\bar{j}}(y^b R^h_{jba} \tilde{V}^a)\}, \end{aligned} \tag{3.5}$$

because the left-hand side is symmetric with respect to i, j, k. From (3.5), we obtain that

$$\partial_{\bar{k}} \partial_{\bar{j}} (\partial_{\bar{i}} \tilde{V}^h - \frac{\alpha^2}{\lambda} y^b R^h_{iba} \tilde{V}^a) = 0. \tag{3.6}$$

Thus we can put

$$P^h_{ji} := \partial_{\bar{j}} (\partial_{\bar{i}} \tilde{V}^h - \frac{\alpha^2}{\lambda} y^b R^h_{iba} \tilde{V}^a), \tag{3.7}$$

and

$$A^h_i + y^a P^h_{ai} = \partial_{\bar{i}} \tilde{V}^h - \frac{\alpha^2}{\lambda} y^b R^h_{iba} \tilde{V}^a, \tag{3.8}$$

where P^h_{ji} and A^h_i are certain functions on M_n. By straightforward calculations, one can see that $A = (A^h_i) \in \Im^1_1(M_n)$ and $P = (P^h_{ji}) \in \Im^1_2(M_n)$.

Using (3.2) and (3.7), we have

$$P^h_{ji} + P^h_{ij} = 2\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h - \frac{\alpha^2}{\lambda} y^b (R^h_{iba}\partial_{\bar{j}}\tilde{V}^a + R^h_{jba}\partial_{\bar{i}}\tilde{V}^a) = 0. \tag{3.9}$$

Thus P^h_{ji} is antisymmetric with respect to i, j, and therefore

$$2P^h_{ji} = P^h_{ji} - P^h_{ij} = \frac{\alpha^2}{\lambda}\{\partial_{\bar{i}}(y^a R^h_{jab}\tilde{V}^b) - \partial_{\bar{j}}(y^a R^h_{ias}\tilde{V}^b)\}, \tag{3.10}$$

and then

$$\begin{aligned} 2y^j P^h_{ji} &= \frac{\alpha^2}{\lambda}\{y^j\partial_{\bar{i}}(y^b R^h_{jba}\tilde{V}^a) - y^j\partial_{\bar{j}}(y^b R^h_{iba}\tilde{V}^a)\} \\ &= -\frac{2\alpha^2}{\lambda} y^j R^h_{ija}\tilde{V}^a - \frac{\alpha^2}{\lambda} y^j y^b R^h_{iba}\partial_{\bar{j}}\tilde{V}^a. \end{aligned} \tag{3.11}$$

By substituting (3.11) into (3.8) we obtain

$$\partial_{\bar{i}}\tilde{V}^h = A^h_i - \frac{\alpha^2}{2\lambda} y^j y^b R^h_{iba}\partial_{\bar{j}}\tilde{V}^a, \tag{3.12}$$

from which we have

$$y^i\partial_{\bar{i}}\tilde{V}^h = y^i A^h_i. \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$\partial_{\bar{i}}\tilde{V}^h = A^h_i - \frac{\alpha^2}{2\lambda} y^a y^b R^h_{iac} A^c_b, \tag{3.14}$$

and then

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h = -\frac{\alpha^2}{2\lambda} y^b (R^h_{iba}A^a_j + R^h_{ija}A^a_b). \tag{3.15}$$

On the other hand, substituting (3.14) into (3.2), we obtain

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h = \frac{\alpha^2}{2\lambda} y^b (R^h_{iba}A^a_j + R^h_{jba}A^a_i) - \frac{\alpha^4}{4\lambda} y^b y^c y^d (R^h_{iba}R^a_{jce}A^e_d + R^h_{jba}R^a_{ice}A^e_d). \tag{3.16}$$

Comparing (3.15) and (3.16), we obtain

$$\alpha(2R^h_{jba}A^a_i + R^h_{jia}A^a_b + R^h_{iba}A^a_j) = 0, \tag{3.17}$$

from which

$$\alpha(R^h_{jba}A^a_i + R^h_{iba}A^a_j) = 0. \tag{3.18}$$

By using (3.18) and the first Bianchi identity, one can see that

$$\alpha(R^h_{bja}A^a_i) = 0, \tag{3.19}$$

thus

$$R^h_{bja}A^a_i = 0, \tag{3.20}$$

by virtue of $\alpha \neq 0$. From (3.14) and (3.19), we get

$$\partial_{\bar{i}}\tilde{V}^h = A^h_i. \tag{3.21}$$

Thus we can put

$$\tilde{V}^h = B^h + A^h_a y^a. \tag{3.22}$$

where B^h are certain functions on M_n. It is easy to see that $B := (B^h) \in \Im_0^1(M_n)$.

Substituting (3.21) in (3.3) and by using (3.19), one can see that

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}} = 2\tilde{\Omega}_{\bar{j}}\delta_i^h + 2\tilde{\Omega}_{\bar{i}}\delta_j^h. \tag{3.23}$$

From (3.23), we have

$$\tilde{\Omega}_{\bar{j}} = \partial_{\bar{j}}\tilde{\varphi}, \tag{3.24}$$

where

$$\tilde{\varphi} := \frac{1}{2(n+1)}\partial_{\bar{a}}\tilde{V}^{\bar{a}}. \tag{3.25}$$

Substituting (3.24) into (3.23), we get

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}} = 2\partial_{\bar{j}}\tilde{\varphi}\delta_i^h + 2\partial_{\bar{i}}\tilde{\varphi}\delta_j^h. \tag{3.26}$$

By a similar way, one can see that, there exist $\Phi = (\Phi_i) \in \Im_1^0(M_n)$, $D = (D^h) \in \Im_0^1(M_n)$ and $C = (C_i^h) \in \Im_1^1(M_n)$, satisfying

$$\tilde{\Omega}_{\bar{i}} = \Phi_i, \tag{3.27}$$

and

$$\tilde{V}^{\bar{h}} = D^h + C_a^h y^a + 2y^a\Phi_a y^h. \tag{3.28}$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_i) = 0,$$

or

$$(L_{\tilde{V}}\tilde{\nabla})(E_j, E_{\bar{i}}) = 0,$$

and using (3.22) and (3.28), we get

$$\begin{aligned}
0 =& \Big\{ \big(\nabla_i A_j^h + \frac{\alpha^2}{2\lambda}D^a R_{aji}^h\big) + \frac{y^b}{2\lambda}\big(\alpha^2(B^a\nabla_a R_{bji}^h - R_{bji}^a\nabla_a B^h \\
&+ R_{bja}^h\nabla_i B^a + C_b^a R_{aji}^h + C_j^a R_{bai}^h) + \alpha\beta R_{bji}^a A_a^h\big) \\
&+ \frac{\alpha^2}{2\lambda}y^b y^c\big(A_c^a\nabla_a R_{bji}^h + 4\Phi_c R_{bji}^h - R_{bji}^a\nabla_a A_c^h + R_{bja}^h\nabla_i A_c^a\big)\Big\}E_h \\
&+ \Big\{\big(\nabla_i C_j^h - B^a R_{iaj}^h - \frac{\alpha\beta}{2\lambda}D^a R_{aji}^h\big) - \frac{y^b}{2\lambda}\big(\alpha^2 R_{bji}^a\nabla_a D^h \\
&+ \alpha\beta(B^a\nabla_a R_{bji}^h + R_{bja}^h\nabla_i B^a + R_{aji}^h C_b^a - R_{bji}^a C_a^h + R_{bai}^h C_j^a) \\
&- 4\lambda(\nabla_i\Phi_j\delta_b^h + \nabla_i\Phi_b\delta_j^h)\big) + \frac{y^b y^c}{2\lambda}\big(\alpha^2(R_{bji}^a B^d R_{adc}^h - R_{bji}^a\nabla_a C_c^h) \\
&- \alpha\beta(A_c^a\nabla_a R_{bji}^h + R_{bja}^h\nabla_i A_c^a - 2R_{bji}^a\Phi_a\delta_c^h + 2R_{bji}^h\Phi_c)\big) \\
&- \frac{\alpha^2}{2\lambda}y^b y^c y^d R_{bji}^a\nabla_a 2\Phi_d\delta_c^h\Big\}E_{\bar{h}}.
\end{aligned} \tag{3.29}$$

Comparing both sides of (3.29), we obtain

$$\nabla_i A_j^h = -\frac{\alpha^2}{2\lambda}D^a R_{aji}^h, \tag{3.30}$$

$$\alpha B^a\nabla_a R_{bji}^h = \alpha R_{bji}^a\nabla_a B^h - \alpha R_{bja}^h\nabla_i B^a - \alpha R_{aji}^h C_b^a - \alpha R_{bai}^h C_j^a + \beta R_{bji}^a A_a^h, \tag{3.31}$$

$$\nabla_i C_j^h = B^a R_{iaj}^h + \frac{\alpha\beta}{2\lambda}D^a R_{aji}^h, \tag{3.32}$$

$$\begin{aligned}A_c^a\nabla_a R_{bji}^h+A_b^a\nabla_a R_{cji}^h=&R_{bji}^a\nabla_a A_c^h+R_{cji}^a\nabla_a A_b^h-R_{bja}^h\nabla_i A_c^a\\&-R_{cja}^h\nabla_i A_b^a-4\Phi_c R_{bji}^h-4\Phi_b R_{cji}^h,\end{aligned}\tag{3.33}$$

$$\begin{aligned}4\lambda(\nabla_i\Phi_j\delta_b^h+\nabla_i\Phi_b\delta_j^h)=&\alpha^2 R_{bji}^a\nabla_a D^h+\alpha\beta(B^a\nabla_a R_{bji}^h+R_{bja}^h\nabla_i B^a\\&+R_{aji}^h C_b^a-R_{bji}^a C_a^h+R_{bai}^h C_j^a),\end{aligned}\tag{3.34}$$

$$\begin{aligned}\beta(A_c^a\nabla_a R_{bji}^h+A_b^a\nabla_a R_{cji}^h)=&-\beta(R_{bja}^h\nabla_i A_c^a+R_{cja}^h\nabla_i A_b^a+2R_{bji}^a\Phi_a\delta_c^h\\&+2R_{cji}^a\Phi_a\delta_b^h-2R_{bji}^h\Phi_c-2R_{cji}^h\Phi_b)+\alpha(R_{bji}^a B^d R_{adc}^h\\&-R_{bji}^a\nabla_a C_c^h+R_{cji}^a B^d R_{adb}^h-R_{cji}^a\nabla_a C_b^h).\end{aligned}\tag{3.35}$$

By changing indices j and b in (3.34), we get

$$\alpha R_{bji}^a\nabla_a D^h+\beta(B^a\nabla_a R_{bji}^h+R_{bja}^h\nabla_i B^a+R_{aji}^h C_b^a-R_{bji}^a C_a^h+R_{bai}^h C_j^a)=0.\tag{3.36}$$

By contracting h and j in (3.34) and using (3.36), we get

$$\nabla_i\Phi_k=0.\tag{3.37}$$

Substituting (3.31) into (3.36), we obtain

$$R_{bji}^a(\beta^2 A_a^h+\alpha\beta\nabla_a B^h-\alpha\beta C_a^h+\alpha^2\nabla_a D^h)=0.\tag{3.38}$$

From (3.33) and (3.35), we obtain that

$$\beta(\Phi_c R_{bji}^h+\Phi_b R_{cji}^h)=0.\tag{3.39}$$

From

$$(L_{\tilde V}\tilde\nabla)(E_j,E_i)=2\tilde\Omega_j\delta_i^h+2\tilde\Omega_i\delta_j^h,$$

using (3.19), (3.22), (3.28) and (3.37), we obtain

$$\begin{aligned}2\tilde\Omega_j\delta_i^h+2\tilde\Omega_i\delta_j^h=&\nabla_j\nabla_i B^h+B^a R_{aji}^h+\frac{\alpha\beta}{2\lambda}D^a(R_{aji}^h+R_{aij}^h)+\frac{y^b}{2\lambda}\{2\lambda\nabla_j\nabla_i A_b^h\\&+\alpha\beta\big(B^a(\nabla_a R_{bji}^h+\nabla_a R_{bij}^h)-(R_{bji}^a+R_{bij}^a)\nabla_a B^h\\&+(R_{bai}^h+R_{bia}^h)\nabla_j B^a+(R_{baj}^h+R_{bja}^h)\nabla_i B^a+(R_{aji}^h+R_{aij}^h)C_b^a\big)\\&-2\beta^2 R_{jbi}^a A_a^h+\alpha(\alpha+\gamma)R_{jib}^a A_a^h+\alpha^2(R_{bai}^h\nabla_j D^a+R_{baj}^h\nabla_i D^a)\}\\&+\frac{y^b y^c}{2\lambda}\{\alpha\beta\big(A_c^a(\nabla_a R_{bji}^h+\nabla_a R_{bij}^h)-(R_{bji}^a+R_{bij}^a)\nabla_a A_c^h\\&+(R_{bai}^h+R_{bia}^h)\nabla_j A_c^a+(R_{baj}^h+R_{bja}^h)\nabla_i A_c^a+2\Phi_b(R_{cji}^h+R_{cij}^h)\big)\\&-\alpha^2(R_{bai}^h B^d R_{jdc}^a+R_{baj}^h B^d R_{idc}^a-R_{bai}^h\nabla_j C_c^a+R_{baj}^h\nabla_i C_c^a)\},\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}0=&\nabla_j\nabla_i D^h+\frac{\beta^2}{\lambda}R_{jai}^h D^a-\frac{\alpha(\alpha+\gamma)}{2\lambda}R_{jia}^h D^a\\&+\frac{y^b}{2\lambda}\{2\lambda\big(\nabla_j\nabla_i C_b^h-\nabla_j(B^a R_{iab}^h)\big)+2\beta^2(B^a\nabla_a R_{jbi}^h+R_{abi}^h\nabla_j B^a\\&+R_{jba}^h\nabla_i B^a+R_{jai}^h C_b^a-R_{jbi}^a C_a^h)-\alpha(\alpha+\gamma)(B^a\nabla_a R_{jib}^h+R_{aib}^h\nabla_j B^a\\&+R_{jab}^h\nabla_i B^a+R_{jia}^h C_b^a-R_{jib}^a C_a^h)-\alpha\beta\big(R_{bai}^h\nabla_i D^a+R_{baj}^h\nabla_i D^a\\&+(R_{bji}^a+R_{bij}^a)\nabla_a D^h\big)\}+\frac{y^b y^c}{2\lambda}\{\big(\alpha(\alpha+\gamma)R_{jib}^a\Phi_a-2\beta^2 R_{jbi}^a\Phi_a\big)\delta_c^h\\&+2\beta^2(A_c^a\nabla_a R_{jbi}^h+R_{abi}^h\nabla_j A_c^a+R_{jba}^h\nabla_i A_c^a)\\&-\alpha(\alpha+\gamma)(A_c^a\nabla_a R_{jib}^h+R_{aib}^h\nabla_j A_c^a+R_{jab}^h\nabla_i A_c^a)\\&+\alpha\beta\big((R_{bji}^a+R_{bij}^a)B^d R_{adc}^h+R_{bai}^h B^d R_{jdc}^a+R_{baj}^h B^d R_{idc}^a\\&-(R_{bji}^a+R_{bij}^a)\nabla_a C_c^h-R_{bai}^h\nabla_j C_c^a-R_{baj}^h\nabla_i C_c^a\big)\}.\end{aligned}\tag{3.41}$$

By changing the indices i and j in (3.40), we get

$$\nabla_j \nabla_i A_b^h - \nabla_i \nabla_j A_b^h = R_{ijk}^a A_a^h. \tag{3.42}$$

By contracting h and i in (3.42) and using (3.30), we obtain

$$\nabla_j \nabla_a A_b^a = -\frac{\alpha^2}{2\lambda} \nabla_a (R_{cbj}^a D^c) = 0. \tag{3.43}$$

By contracting h and i in (3.40) and using (3.36) and (3.43), we have

$$\begin{aligned} 2(n+1)\tilde{\Omega}_j =& \nabla_j \nabla_a B^a - \frac{\alpha\beta}{2\lambda} D^a R_{aj} - \frac{y^b y^c}{2\lambda} \{\alpha\beta(A_c^a \nabla_a R_{bj} + R_{ba} \nabla_j A_c^a \\ &+ 2\Phi_b R_{cj}) + \alpha^2 (R_{baj}^d B^e R_{dec}^a - R_{baj}^d \nabla_d C_c^a)\}. \end{aligned} \tag{3.44}$$

On the other hand, by using (3.20), (3.30), (3.32) and the second Bianchi identity, the last part of right-hand side in (3.44) vanishes. Thus (3.44) is rewritten in the form

$$\tilde{\Omega}_i = \Psi_i \tag{3.45}$$

where

$$\Psi_i := \frac{1}{2(n+1)} (\nabla_i \nabla_a B^a - \frac{\alpha\beta}{2\lambda} D^a R_{ai}). \tag{3.46}$$

From (3.32) and (3.46), we have

$$\Psi_i := \frac{1}{2(n+1)} (\nabla_i \nabla_a B^a + \nabla_i C_a^a). \tag{3.47}$$

Putting $\psi := \frac{1}{2(n+1)}(\nabla_a B^a + C_a^a)$, one can see that

$$\Psi_i = \partial_i \psi. \tag{3.48}$$

Substituting (3.45) into (3.40), and comparing both sides, we have

$$L_B \Gamma_{ji}^h = \nabla_j \nabla_i B^h + B^a R_{aji}^h = 2\Psi_j \delta_i^h + 2\Psi_i \delta_j^h - \frac{\alpha\beta}{2\lambda} D^a (R_{aji}^h + R_{aij}^h), \tag{3.49}$$

and

$$\begin{aligned} 2\lambda \nabla_j \nabla_i A_b^h =& -\alpha\beta(B^a(\nabla_a R_{bji}^h + \nabla_a R_{bij}^h) - (R_{bji}^a + R_{bij}^a)\nabla_a B^h \\ &+ (R_{bai}^h + R_{bia}^h)\nabla_j B^a + (R_{baj}^h + R_{bja}^h)\nabla_i B^a \\ &+ (R_{aji}^h + R_{aij}^h) C_b^a) + 2\beta^2 R_{jbi}^a A_a^h - \alpha(\alpha+\gamma) R_{jib}^a A_a^h \\ &- \alpha^2 (R_{bai}^h \nabla_j D^a + R_{baj}^h \nabla_i D^a) \end{aligned} \tag{3.50}$$

Substituting (3.30) and (3.31) into (3.50), we have

$$\begin{aligned} \alpha^2 \nabla_j (R_{abi}^h D^a) =& \alpha R_{bai}^h (\alpha \nabla_j D^a - \beta C_j^a + \beta \nabla_j B^a) \\ &+ \alpha R_{baj}^h (\alpha \nabla_i D^a - \beta C_i^a + \beta \nabla_i B^a) + \lambda R_{jib}^a A_a^h. \end{aligned} \tag{3.51}$$

From (3.51), we get

$$R_{jib}^a A_a^h = 0. \tag{3.52}$$

From (3.41) we obtain

$$\nabla_j \nabla_i D^h = \frac{\alpha(\alpha+\gamma)}{2\lambda} R_{jia}^h D^a - \frac{\beta^2}{\lambda} R_{jai}^h D^a, \tag{3.53}$$

and

$$\begin{aligned}2\lambda\big(\nabla_j\nabla_i C_b^h - \nabla_j(B^c R_{icb}^h)\big) = &-2\beta^2(B^a\nabla_a R_{jbi}^h + R_{abi}^h\nabla_j B^a + R_{jba}^h\nabla_i B^a + R_{jai}^h C_b^a \\ &- R_{jbi}^a C_a^h) + \alpha(\alpha+\gamma)(B^a\nabla_a R_{jib}^h + R_{aib}^h\nabla_j B^a \\ &+ R_{jab}^h\nabla_i B^a + R_{jia}^h C_b^a - R_{jib}^a C_a^h) + \alpha\beta\big(R_{bai}^h\nabla_i D^a \\ &+ R_{baj}^h\nabla_i D^a + (R_{bji}^a + R_{bij}^a)\nabla_a D^h\big).\end{aligned} \tag{3.54}$$

Substituting (3.31), (3.32) and (3.38) into (3.54), we have

$$\begin{aligned}\beta D^a\nabla_j R_{bai}^h = &-\beta(R_{baj}^h\nabla_i D^a + R_{bai}^h\nabla_j D^a) - \beta R_{jib}^a\nabla_a D^h \\ &-\beta R_{bai}^h(2\frac{\beta^2}{\alpha}\nabla_j B^a - 2\frac{\beta^2}{\alpha}C_j^a - \nabla_j D^a).\end{aligned} \tag{3.55}$$

Proof of Theorem 1.3

Let $\tilde{V}$ be a non-affine infinitesimal paraholomorphically projective transformation on TM_n. By using (1.1) in Theorem 1.1, one can see that $\nabla_i\|\Phi\|^2 = 0$. Thus $\|\Phi\|$ is constant on M_n. Let $\|\Phi\| \neq 0$, then from (1.1) in Theorem 1.1, $\|\Phi\|(\Phi^a R_{aji}^h) = 0$. Thus $\Phi^a R_{aji}^h = 0$ and one can see that $\|\Phi\| R_{aji}^h = 0$. Therefore M_n is locally flat, by virtue of $\|\Phi\| \neq 0$. It is easy to see that TM_n also is locally flat.

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