**ISSN (Print): 2077-9879**
**ISSN (Online): 2617-2658**

# Eurasian Mathematical Journal

**2022, Volume 13, Number 2**

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

**EURASIAN MATHEMATICAL JOURNAL**
ISSN 2077-9879
Volume 13, Number 2 (2022), 62 – 69

# A CRITERION FOR EFFECTIVE COMPLETE DECOMPOSABILITY OF ABELIAN GROUPS

**N.G. Khisamiev, V.A. Roman'kov, S.D. Tynybekova**

Communicated by J.A. Tussupov

**Key words:** completely decomposable abelian group, effectively $\langle p, \omega\rangle$-decomposable abelian group, computable group.

**AMS Mathematics Subject Classification:** 20K25, 03C57, 03D78.

**Abstract.** The notion of effective $\langle p, \omega\rangle$-decomposability of an abelian group is introduced and a criterion for such decomposability is obtained.

**DOI: https://doi.org/10.32523/2077-9879-2022-13-2-62-69**

## 1 Introduction

Effective structure theory and, more particularly, effective algebra are concerned with studying mathematical objects such as groups, rings and fields, but where the objects are given with computable domains and such that the operations are computable functions. See [1], [3], [4], [8], [15], [16]. The study of constructive (i.e., computable) abelian groups was initiated by A.I. Mal'tsev in [13], where he posed a general problem: "Determine what constructive numberings are allowed by abstractly given group". Mal'tsev interested in the algebraic theory of infinite abelian groups and so he used the new effective approach to algebra on the class of abelian groups [13]. He defined an abelian group to be recursive (computable) if there is an effective listing of its elements under which the operation of the group becomes a recursive (computable) function. This numbering of the universe of the group is called a computable presentation or constructivisation of the group. Computable groups are often called constructive.

Computable abelian groups have been intensively studied. For survey of results in the field see, e.g., Khisamiev [10] and Downey [5]. Modern computable abelian group theory combines methods of computable model theory (Ershov and Goncharov [4], Ash and Knight [1]) and pure abelian group theory (Fuchs [7]).

In 1937, Baer [2] introduced the class of completely decomposable groups. A countable torsion-free abelian group $A$ is called *completely decomposable* if the equality

$$A = \oplus\{A_i | i \in \omega\}, \tag{1.1}$$

is true for some subgroups $A_i$ of the rationals $\langle \mathbb{Q}, +\rangle$ under addition.

Khisamiev and Krykpaeva [12] looked at completely decomposable groups from the computability-theoretic point of view. It turned out that even a basic question of the theory of completely decomposable groups, when considered from the effective point of view, may lead to a difficult problem with an unexpected solution; see, e.g., the result of Khisamiev [11]. More results on computable completely decomposable groups can be found in [11], [6], [14].

In [12, 9], the first author introduced the concepts of effective and strong decomposability of an abelian group. Moreover, he also obtained criteria for such decomposability for the class of groups of the form $A = \oplus\{\mathbb{Q}_{p_i} | i \in \omega\}$, where $\mathbb{Q}_{p_i}$ is the additive group of rational numbers whose denominators are powers of some prime number $p_i$. In [11], a criterion for strong decomposability is obtained for the class of groups of the form $A = \oplus\{\mathbb{Q}^{p_i} | i \in \omega\}$, where $\mathbb{Q}^{p_i} = \{m/n | (n, p_i) = 1, m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

## 2 Preliminary results

Let $P = \{p_0, p_1, \ldots\}$ be the set of all prime numbers in ascending order and $S \subseteq \omega$, $S \neq \emptyset$. We set $A_S = \oplus\{A_{p_n} | n \in S\}$. In [5], it was proved that $A_S$ has a decidable (computable) copy if and only if $S \in \Sigma^0_2$ ($\Sigma^0_3$).

In this paper, we introduce the notion of effective $\langle p, \omega\rangle$-decomposability of an abelian group and obtain a criterion for such a decomposition.

In [12], the following definition was introduced.

**Definition 1.** Let there exist a computable numbering $\nu$ of the group $A$ of form (1.1) such that the pair $(A, \nu)$ contains a computably enumerable maximal linearly independent system of elements $\langle a_i \mid a_i \in A_i\rangle$. In this case, the pair $(A, \nu)$ is called a *computably completely decomposed* group, and $A$ itself is called an *effectively completely decomposed* group.

In what follows, we assume that the base set of the group $A$ is the set $\omega = \{0, 1, 2, \ldots\}$. The element $k \in A_i$, $i \in \omega$, will be denoted by $a_{ik}$, and by $x, y, z, x_0, y_0, z_0, \ldots$ – arbitrary elements of the group $A$; $p, q, p_0, q_0, \ldots$ are prime numbers.

We introduce the following predicates:

$$R(x, p, n, y) \rightleftharpoons (x = p^n y), \tag{2.1}$$

$$D(x, p) \rightleftharpoons \forall n \exists x R(x, p, n, x). \tag{2.2}$$

**Definition 2.** If on the group $A$ of form (1.1), the formula

$$H(p, n, a) \leftrightharpoons \exists x R(a, p, n, x) \wedge \forall y \urcorner R(a, p, n+1, y),$$

is true, where the predicate $R$ is defined by formula (2.1), then we say that the *p-height* of the element $a \in A$ is equal to $n$ and denote this fact by $h_p(a) = n$; if $A \models D(a, p)$, where the predicate $D$ is defined in (2.2), then we say that the *p-height* of $a$ is equal to $\omega$ and denote $h_p(a) = \omega$.

For any nonzero element $a$ of the group $A$ of form (1.1), we introduce the following sets:

$$H_{<\omega}(a) \leftrightharpoons \{p \mid 0 < h_p(a) < \omega,\ p\ -\ \text{prime number}\}, \tag{2.3}$$

$$H_{\omega}(a) \leftrightharpoons \{p \mid h_p(a) = \omega,\ p\ -\ \text{prime number}\}. \tag{2.4}$$

**Lemma 2.1.** *For any subgroup $A \leq \mathbb{Q}$, and any nonzero elements $a, b \in A$, the following statements hold.*

(a) *The set*

$$H_{<\omega}(a) \vartriangle H_{<\omega}(b) \rightleftharpoons (H_{<\omega}(a) \setminus H_{<\omega}(b)) \cup (H_{<\omega}(b) \setminus H_{<\omega}(a))$$

*is finite.*

(b) *The following equality is true:*

$$H_{\omega}(a) = H_{\omega}(b),$$

*where the sets $H_{<\omega}(a)$, $H_{\omega}(a)$ are defined by* (2.3) *and* (2.4) *respectively.*

*Proof.* (a) Suppose the set

$$H_{<\omega}(a) \setminus H_{<\omega}(b) \tag{2.5}$$

is infinite. Since $a, b \in A \setminus \{0\}$ and $A \leq \langle \mathbb{Q}, +\rangle$, then there are two coprime numbers $m, n \neq 0$ such that

$$ma = nb. \tag{2.6}$$

From the condition (2.5) it follows that for some prime number $p$ the following equality holds:

$$p \in (H_{<\omega}(a) \setminus H_{<\omega}(b)). \tag{2.7}$$

Also $p$ is coprime to $m, n$ from (2.6). Then for some integers $k_\varepsilon$, $l_\varepsilon$, $\varepsilon < 2$ we have

$$\begin{cases} pk_0 + ml_0 = 1, \\ pk_1 + nl_1 = 1. \end{cases}$$

Then

$$\begin{cases} pk_0a + ml_0a = a, \\ pk_1b + nl_1b = b. \end{cases}$$

From the second equality of this system and (2.6) we have $pk_1b + ml_1a = b$. Then by (2.7) we obtain the contradiction $p \in H_{<\omega}(b)$. Thus, the set $H_{<\omega}(a) \backslash H_{<\omega}(b)$ is finite.

The finiteness of $H_{<\omega}(b) \backslash H_{<\omega}(a)$ can be proved in the similar way. Then $H_{<\omega}(a) \vartriangle H_{<\omega}(b)$ is also finite. Statement (a) of the lemma is proved.

(b) Let the nonzero elements $a, b \in A$ and a prime number $p$ be given for which $p \in H_\omega(a)$, i.e.,

$$A \models \forall n \exists a_{p,n}(p^n a_{p,n} = a). \tag{2.8}$$

We will prove that $p \in H_\omega(b)$. Since $A \leqslant Q$ and $a, b \in A \setminus \{0\}$ then there exist coprime nonzero numbers $m, r$, such that $ma = rb$. Then by (2.8) we obtain that

$$A \models \forall n \exists b_{p,n}(p^n b_{p,n} = rb). \tag{2.9}$$

Without loss of generality of reasoning, we can assume that the numbers $r$ and $p$ are coprime. Then for any number $k \in \omega \setminus \{0\}$ there are integers $s_k$ and $t_k$ such that the equality $rs_k + p^k t_k = 1$ holds. Then $rs_kb + p^k t_k b = b$ and (2.9) implies that for each $k > 0$ the element $b$ divides $p^k$, and so $p \in H_\omega(b)$.

Similarly we can prove that if $p \in H_\omega(b)$ then $p \in H_\omega(a)$. Therefore, $H_\omega(a) = H_\omega(b)$.

□

**Definition 3.** [7, p. 129]. A $p$-heigh sequence

$$\chi(a) = \langle h_{p_0}(a), \ldots, h_{p_n}(a), \ldots \rangle$$

is called *characteristic* of the element $a \in A \setminus \{0\}$, where $p_i$ is the $i$-th prime and $h_p(a)$ is introduced by definition 2. Characteristics $\langle k_0, \ldots, k_n, \ldots \rangle$ and $\langle l_0, \ldots, l_n, \ldots \rangle$ are *equivalent* if $k_n \neq l_n$ holds only for a finite set of numbers $n$ and only if $k_n$ and $l_n$ are finite.

**Theorem**. Baer [2] (see also [7, p. 132]). *Two groups $A$ and $B$ of rank 1 are isomorphic if and only if there exist two nonzero elements $a \in A$ and $b \in B$ with the equivalent characteristics $\chi(a)$ and $\chi(b)$.*

**Lemma 2.2.** *Let $A$ be a subgroup of the additive group of rationals $\langle \mathbb{Q}, + \rangle$. Suppose that the element $a \in A \backslash \{0\}$ satisfies the following conditions:*

(a1) $H_\omega(a) \neq \emptyset$;

(a2) $H_{<\omega}(a)$ *is finite.*

*Then there exists $b \in A$ for which*

(c1) $H_{<\omega}(b) = \emptyset$;

(c2) *$A$ is isomorphic to $B \leq \langle \mathbb{Q}, + \rangle$ generated by the following set of elements:*

$$S(b) = \{b_{p,n} \mid p^n b_{p,n} = b, p \in H_\omega(b), n \in \omega\}, \tag{2.10}$$

*where the sets $H_{<\omega}(a)$ and $H_\omega(a)$ are defined by (2.3) and (2.4), respectively.*

*Proof.* (c1) Condition (a2) implies that

$$H_{<\omega}(a) = \{q_0, \dots, q_{s-1}\} \tag{2.11}$$

for some $s > 0$ and prime numbers $q_i$, $i < s$. Then for every $i < s$ there existds a number $m_i \in \omega \setminus \{0\}$ such that

$$A \models \exists a_{i,m_i}(q_i^{m_i} a_{i,m_i} = a \wedge \forall x\ q_i^{m_i+1} x \neq a).$$

Therefore

$$q_0^{m_0} q_1^{m_1} \dots q_{s-1}^{m_{s-1}} b = a \tag{2.12}$$

for some $b \in A$. By (2.11) we obtain the equality

$$H_{<\omega}(b) = \emptyset. \tag{2.13}$$

Statement (c1) is proved.

(c2) Let the elements $a$ and $b$ are defined as in lemma 2.2 and formula (2.12), respectively. By (2.11), (2.13) and (b) of lemma 2.1 we have that the characteristics $\chi(a)$ and $\chi(b)$ are equivalent. Then the Baer theorem implies

$$A = gr\{c \mid \exists m \exists n(mc = na)\} \text{ and } B = gr\{d \mid \exists m \exists n(md = nb)\}.$$

□

**Definition 4.** Let $A = \oplus\{A_i \mid i \in \omega\}$, where each $A_i$ is a subgroup of $\langle \mathbb{Q}, + \rangle$. Then $A$ is called $\langle p, \omega \rangle$-*decomposable* if each $A_i$ contains an element $a_i \in A_i$ for which $H_{<\omega}(a_i)$ defined by (2.3) is finite.

By lemma 2.1 we obtain the following statement.

**Corollary 2.1.** *Let $A$ be the group of form* (1.1). *Suppose that $A$ is $\langle p, \omega \rangle$-decomposable. Then for any nonzero elements $a_i, b_i \in A_i$, $i \in \omega$ the following statements hold:*

- *The sets $H_{<\omega}(a_i)$ and $H_{<\omega}(b_i)$ are finite.*
- $H_\omega(a_i) = H_\omega(b_i)$.

**Definition 5.** Let $A$ be a $\langle p, \omega \rangle$-decomposable abelian group of form (1.1), and $\langle a_i \mid a_i \in A_i \rangle$ be a maximal linearly independent system of elements in $A$. Then the sequence of sets

$$\chi(A) = \langle H_\omega(a_i) \mid i \in \omega \rangle, \tag{2.14}$$

with $H_\omega(a_i)$ is defined by (2.4), is called *characteristic* of $A$.

**Definition 6.** Let $A$ be a $\langle p, \omega \rangle$-decomposable abelian group of form (1.1). Suppose that $A$ is effectively completely decomposable. Then $A$ is said to be *effectively $\langle p, \omega \rangle$-decomposable* group.

The following statement follows directly from the definitions.

**Corollary 2.2.** *Abelian group $A$ of form* (1.1) *is effectively $\langle p, \omega \rangle$-decomposable if and only if it is $\langle p, \omega \rangle$-decomposable and there exists a countable numbering $\nu$ and a countable function $f(i)$ such that*

$$\langle a_i \mid a_i = \nu(f(i)),\ a_i \in A_i \rangle$$

*is a maximal linearly independent system of nonzero elements of $A$.*

**Definition 7.** If predicate $R(i, p, n, x)$ for $p \in P$, $i, n, x \in \omega$ satisfies the conditions

- $R(i, p, 0, i)$;
- $R(i, p, n, x) \wedge R(i, p, n, y) \to x = y$;

- $(R(i,p,n,x) \wedge 0 < m < n) \to (\exists y(R(i,p,m,y) \wedge R(y,p,n-m,x)))$,

then it is called *F-predicate.*

For a $F$-predicate $R(i,p,n,x)$ we introduce the following sets:

$$F(i) \leftrightharpoons \{p \mid \exists n_p > 0\ (\exists x R(i,p,n_p,x) \wedge \forall y \neg R(i,p,n_p+1,y))\}, \tag{2.15}$$

$$I(i) \leftrightharpoons \{p \mid \forall n \exists x\ R(i,p,n,x)\}. \tag{2.16}$$

## 3 Main results

If $p$ is a prime number, $p \in F(i)$ ($p \in I(i)$) where $F(i)$ ($I(i)$) is defined by (2.15) ((2.16)), then $p$ is called *$\langle i,p\rangle$-finite* (*$\langle i,p\rangle$-infinite*).

$F$-predicate $R(i,p,n,x)$ is called *$F$-finite* in the case when $F(i)$ is finite for each $i \in \omega$.

**Theorem 3.1.** *Let the group*

$$A = \oplus\{A_i \mid A_i \le \langle \mathbb{Q},+\rangle,\ i \in \omega\}, \tag{3.1}$$

*be $\langle p,\omega\rangle$-decomposable. Then it is effectively $\langle p,\omega\rangle$-decomposable if and only if there exist a computable function $f(i)$ and a computable $F$-finite predicate $R(i,p,n,x)$ such that the following equality holds:*

$$\chi(A) = \langle I(f(i)) \mid i \in \omega\rangle, \tag{3.2}$$

*where the characteristic $\chi(A)$ and the set $I(i)$ are defined by* (2.14) *and* (2.16) *respectively.*

*Proof.* Let $A$ be a $\langle p,\omega\rangle$-decomposable group. Then there are a computable numbering $\nu$ of $A$ and a computable fuction $f(i)$ such that the sequence $\langle a_i \mid a_i = \nu(f(i)),\ a_i \in A_i\rangle$ is maximal linearly independent system of nonzero elements. □

Let a predicate $R(i,p,n,x)$ be defined as

$$R(i,p,n,x) \Leftrightarrow (A,\nu) \models p^n\nu(x) = \nu(i). \tag{3.3}$$

Since $A$ is effectively $\langle p,\omega\rangle$-decomposable then by corollary 2.1 predicate $R(i,p,n,x)$ is computable $F$-finite for which equality (3.2) is valid. Therefore, we have proved the necessity of the conditions of the theorem.

Now let $R(i,p,n,x)$ be a $F$-finite predicate and $f(i)$ be a computable function satisfying the equality (3.2), where $i,n,x \in \omega$, $p \in P$. For any $i \in \omega$ and $p \in P$ we will build a group $B_{i,p} \le \langle \mathbb{Q},+,0\rangle$.

Step 0. We introduce the element $b_{i,p,0} = b_i$ and the number $k_{i,p,0} = f(i)$.

Step $t+1$. We define on the $t$-th first steps the element $b_{i,p,t}$ and number $k_{t,p,t} \in \omega$.

The following cases are possible:

a) If the predicate $\exists x \le tR(k_{i,p,t},p,1,x)$ is true, we introduce an element $b_{i,p,t+1}$, a number $k_{i,p,t+1} = x$ and a relation $pb_{i,p,t+1} = b_{i,p,t}$. If on the step $t$ the element $b_{i,p,t}$ was marked by $*$ then it is deleted.

The step $t+1$ is over. We go to the next step.

b) If the predicate $\exists x \le tR(k_{i,p,t},p,1,x)$ is not true, then we set $b_{i,p,t+1} = b_{i,p,t}$, $k_{i,p,t+1} = k_{i,p,t}$ and mark $b_{i,p,t}$ by $*$.

The step $t+1$ is over. We go to the next step.

We define the groups $B_{i,p}$, $B_i$, $B(R)$ as:

$$B_{i,p} \leftrightharpoons gr(\{b_{i,p,t} \mid t \in \omega\}), \tag{3.4}$$

$$B_i \leftrightharpoons gr(\{b_{i,p,t} \mid p \in P,\ t \in \omega\}), \tag{3.5}$$

$$B(R) \leftrightharpoons \oplus\{B_i \mid i \in \omega\}. \tag{3.6}$$

The construction is complete.

To complete the proof of the theorem, we need the following two lemmas.

**Lemma 3.1.** *For any pair $\langle i, p\rangle$, $i \in \omega$, $p \in P$ the group $B_{i,p}$ defined by* (3.4) *is either infinite cyclic or group of the form $\mathbb{Q}_{p_i}$ for some prime number $p$.*

*Proof.* For the predicate of the form $R(i, p, n, x)$ the following cases are possible:

- $\exists n > 0\ (\exists x R(f(i), p, n, x) \wedge \forall y \neg R(f(i), p, n+1, y))$.
  The construction of $B_{i,p}$ implies that there exists a sequence $s_0, \ldots, s_n$, $s_0 = 0$, for which $\{b_{i,p,k} \mid k \in \omega\} = \{b_{i,p,s_l} \mid l \leq n\}$ and $pb_{i,p,s_{l+1}} = b_{i,p,s_l}$, $l < n$. Therefore, $B_{i,p}$ is isomorphic to the infinite cyclic group generated by $b_{i,p,s_n}$.

- $\forall n \exists x R(f(i), p, n, x)$.
  Thus, the construction $B_{i,p}$ implies that there exists a sequence $\langle s_l \mid l \in \omega\rangle$, $s_0 = 0$, for which $\{b_{i,p,k} \mid k \in \omega\} = \{b_{i,p,s_l} \mid l \in \omega\}$ and $pb_{i,p,s_{l+1}} = b_{i,p,s_l}$, $l \in \omega$. Therefore, $B_{i,p}$ is isomorphic to $A_p = \langle \{\frac{m}{p^n} \mid m \in Z, n \in \omega\}, +, 0\rangle$.

□

**Lemma 3.2.** *For any $i \in \omega$,*

$$H_\omega(a_i) = \emptyset \Leftrightarrow B_i \ - \ \text{is an infinite cyclic group,}$$

*where $\langle a_i \mid a_i \in A_i\rangle$ is a maximal linearly independent system of nonzero elements in $A$, and the set $H_\omega(a)$ and group $B_i$ are defined by* (2.4) *and* (3.5) *respectively.*

*Proof.* Let $H_\omega(a_i) = \emptyset$. Then (3.2) implies

$$I(f(i)) = \emptyset. \tag{3.7}$$

It follows that for any prime $p$ the following formula is true:

$$\exists x R(f(i), p, 1, x) \rightarrow (\exists m_p > 0(\exists z R(f(i), p, m_p, z) \wedge \forall y \neg R(i, p, m_p + 1, y))). \tag{3.8}$$

Therefore, for any $p$

$$p \in F(f(i)) \Leftrightarrow \exists x R(f(i), p, 1, x),$$

where the set $F(f(i))$ is defined by (2.15). Since $R$ is $F$-finite, then $F(f(i))$ is finite. Then (3.7) implies that $B_i$ is isomorphic to the group generated by the set $\{b_{i,p,t_p} \mid m_p b_{i,p,t_p} = b_i, p \in F(f(i))\}$, where $m_p$ is defined by (3.8) and the elements $b_{i,p,t_p}$ were marked by $*$. Therefore, $B_i$ is infinite cyclic group. We proved that the conditions of the lemma are sufficient.

Let $B_i$ be infinite cyclic group. Then $I(f(i)) = \emptyset$. From (3.2) we obtain the necessity of the conditions of the lemma. □

From the computability of the predicate $R(i, p, n, x)$ and the construction of the groups $B_{i,p}$, $B_i$ and $B(R)$, the construction by formulas (3.4) – (3.6), it follows that there exists a computable numbering $\nu$ of the group $B(R)$ such that in the pair $(B(R), \nu)$ the sequence elements $\langle b_i \mid i \in \omega\rangle$ is computably enumerable. Therefore, the group $B(R)$ is effectively $\langle p, \omega\rangle$-decomposable. From (3.2) by Lemmas 2.2, 3.1, 3.2 it follows that the groups $A$ and $B(R)$ are isomorphic. Thus, the group $A$ is also effectively $\langle p, \omega\rangle$-decomposable. Theorem is proved. □

Let $A$ be a $\langle p, \omega\rangle$-decomposable abelian group and $\chi(A) = \langle S_i | i \in \omega\rangle$. Define

$$A(\omega) = \oplus\{A_i | S_i \neq \emptyset\} \text{ and } A(\emptyset) = \oplus\{A_i | S_i = \emptyset\}.$$

The subgroup $A(\omega)$ $(A(\emptyset))$ of $A$, defined in this way, is said to be *noncyclic (cyclic) summand* of $A$.

**Corollary 3.1.** *If an abelian group $A$ is effectively $\langle p, \omega\rangle$-decomposable, then there exists a computable enumeration $\nu$ of $A$ for which the cyclic summand $A(\emptyset)$ is computably enumerable in $(A, \nu)$.*

*Proof.* Indeed, in the proof of Theorem 1 we associate to $(A, \nu)$ the computable predicate $R(i, p, n, x)$ and determine the computably enumerated pair $(A(R), \nu(R))$ such that $A$ is isomorphic to $A(R)$. By this construction it is directly follows that the cyclic summand is generated by the $*$-marked elements. Therefore, the cyclic summand of $A(R)$ is computably enumerated in $(A(R), \nu(R))$. $\square$

**Corollary 3.2.** *If the abelian group $A(R)$ is effectively $\langle p, \omega\rangle$-decomposable, then the noncyclic summand $A(\omega)$ is computable.*

*Proof.* Let $R$ and $(A(R), \nu(R))$ be as in the proof of corollary 2.1. The noncyclic summand $A(R)(\omega)$ of $A(R)$ is isomorphic to $A(R)/A(R)(\emptyset)$, where $A(R)(\emptyset)$ is the cyclic summand of $A(R)$. By corollary 2.1 the subgroup $A(R)(\emptyset)$ is computably enumerable in $(A(R), \nu(R))$. Thus, the factor-group $A(R)/A(R)(\emptyset)$ is a computably enumerably defined abelian torsion-free. In [10], it was proved that such group is computable. It follows that $A(R)/A(R)(\emptyset)$ is also computable. Therefore, the noncyclic summand $A(R)(\omega)$ of $A(R)$ is computable. $\square$

Corollaries 3.1 and 3.2 imply

**Corollary 3.3.** *If an abelian group $A$ is effectively $\langle p, \omega\rangle$-decomposable, then there exists a computable enumeration $\nu$ of $A$ such that the subgroups $A(\omega)$ and $A(\emptyset)$ are recursive in $(A, \nu)$.*

## Acknowledgments

The research of the first author was partially supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, grant no. AP08855497 and the research of the second author was funded in accordance with the state task of the IM SB RAS, project no. FWNF-2022-0003.

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Nazif Garifullinovich Khisamiev, Saule Djunusovna Tynybekova
Departments of Information Systems and Higher Mathematics
L.N. Gumilyov Eurasian National University
2 Satpaev St.
Nur-Sultan, 010008
Republic of Kazakhstan
E-mails: hisamiev@mail.ru, saule.tynybekovna@bk.ru

Vitalii Anatolievich Roman'kov
Laboratory of Combinatorial and Computational Methods of Algebra and Logic
Sobolev Institute of Mathematics SB RAS, Omsk Branch
13 Pevtsov St.
Omsk, 644099
Russia
E-mail: romankov48@mail.ru


Received: 10.05.2021