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INTEGRATION OF THE LOADED GENERAL KORTEWEG-DE VRIES EQUATION IN TNE CLASS OF RAPIDLY DECREASING COMPLEX-VALUED FUNCTIONS

U.A. Hoitmetov

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Key words: loaded general Korteweg-de Vries equation, Sturm-Liouville operator, Jost solutions, scattering data, Gelfand-Levitan-Marchenko integral equation.

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Abstract. In this paper, the evolution of the scattering data of the Sturm-Liouville operator is derived by the method of the inverse scattering problem, the potential of which is a solution to the loaded general Korteweg-de Vries equation in the class of rapidly decreasing complex-valued functions.

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1 Introduction

The inverse scattering method originated in the works of Gardner, Greene, Kruskal and Miura [6]. They managed to find a global solution to the Cauchy problem for the Korteweg - de Vries (KdV) equation, reducing it to the inverse scattering problem for the self-adjoint Sturm - Liouville operator on the whole line. This inverse scattering problem was first solved by L.D. Faddeev [5], then in the works of V.A. Marchenko [30], B. Levitan [27] and others. Further, P. Lax [25] noticed the universality of the inverse scattering method and generalized the KdV equation by introducing the concept of the higher-order KdV equation.

The classical KdV equation is derived within the framework of the first order of perturbation theory, but such an approximation may not be enough to describe large-amplitude waves. One of the ways to refine such models is to take into account the corrections of the larger orders of smallness of approximation in the evolution equation. The first step in this direction was taken in [26], where it was proposed to use an asymptotic procedure based on the introduction of two small parameters characterizing nonlinearity and variance to derive higher order KdV equations. Such an equation was obtained for internal waves in a two-layer fluid in [16]. A more detailed theoretical analysis of the properties of solitary waves in liquids with arbitrary vertical density and flux distributions is given in [7], in which corrections to the shape and velocity of solitary waves are calculated for various models such as stratification of fluids. General integral expressions for the coefficients of the extended KdV equation for internal waves in a fluid with an arbitrary density stratification were obtained in [24]. In the problem for surface waves in the work of S.A. Kordyukova [17] it was discovered the emergence of the KdV hierarchy, where the KdV approximation becomes unusable. In the works of N.A. Kudryashov, M.B. Sukharev [20], M. Daniel, K. Porsezyan, M. Lakshmanana [4], Yu. Bagderina [1], Li Zhi, R. Sibgatullin [36] nonlinear evolutionary equations of the fifth order were considered for describing water waves. The KdV hierarchy is considered in the works of R. Schimming [34], S.A. Kordyukova [17], R.A. Krenkel, M.A. Manna, J.G. Pereira [19], N.A. Kudryashov [21] and others.

Kundu et al [23] have considered the deformed KdV equation

$$u_t + 6uu_x + u_{xxx} = g_x$$

and shown that it admits a Lax representation, infinitely many generalised symmetries, etc., provided that the deforming function $g(x, t)$ satisfies a differential constraint given by

$$g_{xxx} + 2u_x g + 4ug_x = 0.$$

In the recent work of S.S. Kumar and R. Sakhadevan [22] there was considered the deformed KdV equation of the fifth order, given by the formula

$$u_t + \alpha(u_{xxx} + 6uu_x) + \beta(u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x) = g_x,$$

where α and β are real constants, and they showed that this equation admits a Lax representation provided that the deformed function $g(x,t)$ satisfies the differential constraint

$$g_{xxx} + 2u_x g + 4u g_x = 0.$$

In [33], the $(1+1)$-dimensional geophysical KdV equation is investigated, which is given as

$$u_t - \omega_0 u_x + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} = 0$$

where ω_0 is a parameter, u is function with respect to x, t.

The KdV equation is also found in applied mechanics. For example, in the works of A.A. Lugovtsov [28], [29] the system of equations describing the propagation of one-dimensional nonlinear waves in an inhomogeneous gas-liquid medium is reduced to one equation of the form

$$u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \left[\frac{k}{2\tau} + \delta(\tau)\right]u = 0.$$

In particular, for $\mu = 0,\ k = 1,\ \delta = 0$ it is shown that, under certain conditions, cylindrical waves can exist in the form of solitons.

For the first time the term "loaded equation" was used in the works of A.M. Nakhushev [32], where the most general definition of a loaded equation is given and various loaded equations are classified in detail, for example, loaded differential, integral, integro-differential, functional equations, etc., and numerous applications are described. In the literature, it is customary to call *loaded differential equations* equations, containing in the coefficients or in the right-hand side any functionals of the solution, in particular, the values of the solution or its derivatives on manifolds of lower dimension. The study of such equations is of interest both from the point of view of constructing a general theory of differential equations and from the point of view of applications. Among the works devoted to loaded equations, one should especially note the works of A.M. Nakhushev [31], [32], A.I. Kozhanov [18] and others.

Note that solutions of the KdV equation and the general KdV equation with a self-consistent source from the class of rapidly decreasing complex-valued functions were considered in [14], [9]. Integration of the loaded KdV equation in the class of periodic functions was studied in [15], [35], and in the class of rapidly decreasing complex-valued functions in [10], [13]. Also, the coefficient inverse problem for a parabolic equation was studied in [11], the nonlinear problem for a parabolic equation with an unknown coefficient at the time derivative was considered in [8], and the Sommerfeld inverse problem for the Helmholtz equation was considered in [12].

Let

$$H = -\frac{1}{2}\frac{d^3}{dx^3} + 2u\frac{d}{dx} + u',$$

where $u = u(x,t)$ and the prime means the partial derivative with respect to x. According to [27], there exist polynomials P_k (in u and derivatives of u with respect to x) such that

$$HP_k = P'_{k+1}.$$

For example

$$P_0 = -\frac{1}{2}, \quad P_1 = -\frac{1}{2}u, \quad P_2 = \frac{1}{4}u_{xx} - \frac{3}{4}u^2, \quad P_3 = -\frac{1}{8}u_{xxxx} + \frac{5}{4}uu_{xx} + \frac{5}{8}(u_x)^2 - \frac{5}{4}u^3$$

etc.

We put

$$L(t) \equiv -\frac{d^2}{dx^2} + u(x,t).$$

Operator (see [27])

$$B_q = \sum_{k=0}^{q} \left(\frac{1}{2} P_k' - P_k \frac{d}{dx} \right) (2L)^{q-k}$$

satisfies the Lax relation

$$[B_q, L] = B_q L - L B_q = -P_{q+1}'.$$

Let c_0, c_1, $c_2, \ldots,$ c_p be arbitrary real numbers. We introduce the following further notations:

$$X_q = -P_{q+1}', \quad Y_p = \sum_{q=0}^{p} c_q B_q, \quad Z_p = \sum_{q=0}^{p} c_q X_q.$$

Then, the following equality holds:

$$[Y_p, L] = Z_p.$$

The equation

$$u_t = Z_p(u),$$

is called the general KdV equation. In particular, for $p = 1$, $c_0 = 0$, $c_1 = 4$ and $p = 2$, $c_0 = 0$, $c_1 = 0$, $c_2 = 8$, respectively, we have

$$u_t - 6uu_x + u_{xxx} = 0, \qquad u_t = u_{xxxxx} - 20u_x u_{xx} - 10uu_{xx} + 30u^2 u_x.$$

We consider the loaded general KdV equation of the form

$$u_t = Z_p(u) + \gamma(t) F(u(0,t)) u_x \,, \tag{1.1}$$

where $F(u(0,t))$ is a polynomial in $u(0,t)$ and $\gamma(t)$ is a given continuous function.

Equation (1.1) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where the initial function $u_0(x)$ is complex-valued and has the properties:

1) for some $\varepsilon > 0$

$$\int_{-\infty}^{\infty} |u_0(x)|\, e^{\varepsilon|x|} dx < \infty, \tag{1.3}$$

2) the operator $L(0) = -\dfrac{d^2}{dx^2} + u_0(x)$ has exactly N complex eigenvalues $\lambda_1(0)$, $\lambda_2(0), \ldots,$ $\lambda_N(0)$ with multiplicities $m_1(0)$, $m_2(0), \ldots, m_N(0)$ and has no spectral singularities.

Let the function $u(x,t) = \mathrm{Re}\, u(x,t) + i\mathrm{Im}\, u(x,t)$ possess sufficient smoothness and tends to its limits rather quickly at $x \to \pm\infty$, i.e.

$$\int_{-\infty}^{\infty} \left| \frac{\partial^j u(x,t)}{\partial x^j} \right| e^{\varepsilon|x|}\, dx < \infty, \quad j = 0, 1, \ldots, 2p+1. \tag{1.4}$$

The main goal of this work is to obtain representations for the solution $u(x,t)$ of problem (1.1) - (1.4) within the framework of the inverse scattering method for the operator $L(t)$.

2 Auxiliary results

Consider the equation

$$L(0)y := -y'' + u_0(x)y = k^2 y, \quad x \in \mathbb{R}, \tag{2.1}$$

where the potential $u_0(x)$ is assumed to be complex-valued and satisfies condition (1.3). In this section we describe some properties of the direct and inverse scattering problems for equation (2.1) that are necessary in what follows. It is easy to verify that the following functions are the solutions to equation (2.1) with conditions at infinity for $\mathrm{Im}k > -\frac{\varepsilon}{2}$:

$$e_+(x,k) = e^{ikx} + o(1), \quad x \to \infty; \qquad e_-(x,k) = e^{-ikx} + o(1), \quad x \to -\infty. \tag{2.2}$$

These solutions are called Jost solutions and the following representations hold for them

$$e_\pm(x,\, k) = e^{\pm ikx} \pm \int\limits_x^{\pm\infty} K_\pm(x,\, y)e^{\pm iky}dy. \tag{2.3}$$

Under condition (1.4) these solutions exist, are unique and holomorphic with respect to k in the half-plane $\mathrm{Im}\ k > -\frac{\varepsilon}{2}$. Moreover, the kernels $K_\pm(x,\, y)$ are connected with the potential $u_0(x)$ as follows:

$$u_0(x) = \mp 2\frac{dK_\pm(x,x)}{dx}. \tag{2.4}$$

Note also that the pairs of functions $\{e_\pm(x,\, k),\, e_\pm(x,\, -k)\}$ form in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ fundamental systems of solutions whose Wronskians are equal to

$$W\{e_\pm(x,\, k),\, e_\pm(x,\, -k)\} = \mp 2ik.$$

We denote by $\omega(k)$ and $v(k)$ the Wronskians

$$\begin{aligned} \omega(k) &:= e_-(x,k)e'_+(x,k) - e'_-(x,k)e_+(x,k), \\ v(k) &:= e_+(x,-k)e'_-(x,k) - e_-(x,k)e'_+(x,-k). \end{aligned} \tag{2.5}$$

The function $\omega(k)$ extends analytically to the half-plane $\mathrm{Im}k > -\frac{\varepsilon}{2}$ and has the asymptotics

$$\omega(k) = 2ik\left[1 + O\left(\frac{1}{k}\right)\right], \qquad |k| \to \infty, \tag{2.6}$$

uniformly in each half-plane $\mathrm{Im}k \geq \eta,\ \ \eta > -\frac{\varepsilon}{2}$. In view of asymptotics (2.6) and the analyticity of $\omega(k)$, in the half-plane $\mathrm{Im}k \geq 0$ the function $\omega(k)$ has a finite number of zeroes (in a general case, zeroes are multiple). The absence of spectral singularities of the operator $L(0)$ means that the function $\omega(k)$ has no real zeroes, i.e., $\omega(k) \neq 0,\ k \in \mathbb{R}$. Let non-real zeroes of $\omega(k)$ be equal to $k_1,\, k_2,\, \ldots,\, k_N$ ($\mathrm{Im}k_j > 0,\ j = \overline{1, N}$), then $\lambda_j = k_j^2,\ j = \overline{1, N}$ are eigenvalues of the operator $L(0)$. Denote the multiplicity of the root k_j of the equation $\omega(k) = 0$ by $m_j,\ j = \overline{1, N}$.

As distinct from $\omega(k)$, the function $v(k)$ is defined only in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$. Functions $\omega(k)$ and $v(k)$ in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ satisfy the equality

$$\omega(k)\omega(-k) - v(k)v(-k) = 4k^2. \tag{2.7}$$

In addition, in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ the following equality is valid:

$$e_-(x,\, k) = \frac{v(k)}{2ik}e_+(x,\, k) + \frac{\omega(k)}{2ik}e_+(x,\, -k). \tag{2.8}$$

There exist the so-called chains of normalizing numbers $\{\chi_0^j,\, \chi_1^j, \dots,\, \chi_{m_j-1}^j\}$ $j = \overline{1, N}$ such that

$$\frac{1}{s!}\left(\left(\frac{d}{dk}\right)^s f_-(x,\,k)\right)_{k=k_j} = \sum_{\nu=0}^{s} \chi_{s-\nu}^j \frac{1}{\nu!}\left(\left(\frac{d}{dk}\right)^\nu f_+(x,\,k)\right)_{k=k_j}, \tag{2.9}$$

$$s = \overline{0,\, m_j - 1},\ \ j = \overline{1,\, N}$$

while $\chi_0^j \neq 0$.

As is known [2, 3], the kernel $K_+(x,\, y)$ of transformation operator (2.3) satisfies the Gelfand-Levitan-Marchenko integral equation

$$K_+(x,\, y) + F_+(x+y) + \int_x^\infty K_+(x,\, s)F_+(s+y)ds = 0,\ \ x \le y, \tag{2.10}$$

where

$$F_+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(k)e^{ikx}\, dk + \sum_{j=1}^{N}\sum_{\nu=0}^{m_j-1} \chi_{m_j-\nu-1}^j \frac{1}{\nu!}\frac{d^\nu}{dk^\nu}\left(\frac{2k(k-k_j)^{m_j}}{\omega(k)} e^{ikx}\right), \tag{2.11}$$

$$S(k) := \frac{\upsilon(k)}{\omega(k)}, \tag{2.12}$$

herewith the potential $u_0(x)$ is given by formula (2.4).

Definition 1. The collection $\left\{S(k),\ \lambda_j,\, \chi_0^j,\, \dots,\, \chi_{m_j-1}^j,\ j = \overline{1,\, N}\right\}$ is said to be the scattering data for the operator $L(0)$.

The problem that implies the determination of the complex-valued potential $u_0(x)$ from the scattering data is called the inverse problem.

One can prove the following lemma by immediate verification.

Lemma 2.1. *If ϕ_j is an eigenfunction of the operator $L(0)$ with the potential $u_0(x)$ that corresponds to the eigenvalue k_j^2, then*

$$\int_{-\infty}^{\infty} u_0(x)\phi_j'\phi_j dx = 0, \quad \int_{-\infty}^{\infty} u_0'(x)\phi_j^2 dx = 0.$$

3 Evolution of scattering data

Let us introduce the notation

$$G(x,t) = \gamma(t)F(u(0,t))u_x \tag{3.1}$$

and we will consider a more general problem, namely, consider

$$u_t - Z_p(u) = G(x,t). \tag{3.2}$$

For equation (3.2) we seek for the Lax pair in the form

$$-\frac{\partial^2 e_-(x,k,t)}{\partial x^2} + (u-k^2)e_-(x,k,t) = 0, \tag{3.3}$$

$$\frac{\partial e_-(x,k,t)}{\partial t} = Y_p e_-(x,k,t) + \frac{1}{2}ik\sum_{l=0}^{p} c_l(2k^2)^l e_-(x,k,t) + \Phi(x,t), \tag{3.4}$$

where $e_-(x,k,t)$ is the Jost solution of the equation

$$L(t)y = k^2 y$$

with asymptotes (2.2). Using the equality

$$\frac{\partial^3 e_-(x,k,t)}{\partial x^2 \partial t} = \frac{\partial^3 e_-(x,k,t)}{\partial t \partial x^2}$$

on the basis of equalities (3.1), (3.2), we obtain

$$-\Phi_{xx} + (u-\lambda)\Phi = -G(x,t)e_-(x,k,t). \tag{3.5}$$

We will seek a solution to this equation in the form

$$\Phi(x,t) = C(x)e_-(x,k,t) + B(x)e_-(x,-k,t).$$

Then for finding $B(x)$ and $C(x)$ we get the system

$$\begin{cases} C'(x)e_-(x,k\,,\,t) + B'(x)e_-(x,-k\,,\,t) = 0 \\ C'(x)e'_-(x,k\,,\,t) + B'(x)e'_-(x,-k\,,\,t) = G(x,t)e_-(x,k\,,\,t)\,, \end{cases}$$

whose solution has the form

$$C(x) = -\frac{1}{2ik}\int_{-\infty}^{x} e_-(x,k,t)\, e_-(x,-k,t)G(x,t)\, dx,$$

$$B(x) = \frac{1}{2ik}\int_{-\infty}^{x} e_-^2(x,k,t)G(x,t)\, dx.$$

Therefore, equation (3.4) can be rewritten as follows:

$$\begin{aligned} \frac{\partial e_-(x,k,t)}{\partial t} &= Y_p e_-(x,k,t) + \frac{1}{2}ik\sum_{l=0}^{p} c_l (2k^2)^l e_-(x,k,t) \\ &\quad -\frac{1}{2ik}e_-(x,k,t)\int_{-\infty}^{x} e_-(x,k,t)e_-(x,-k,t)G(x,t)\, dx \\ &\quad +\frac{1}{2ik}e_-(x,-k,t)\int_{-\infty}^{x} e_-^2(x,k,t)G(x,t)dx\,. \end{aligned} \tag{3.6}$$

Taking the limit in equality (3.6) as $x \to \infty$, in view of (1.3), (2.2), (2.7) and (2.8) we get

$$\begin{aligned} \frac{\partial \omega(k,t)}{\partial t} &= -\frac{\omega(k,t)}{2ik}\int_{-\infty}^{\infty} e_-(x,k,t)\, e_-(x,-k,t)G(x,t)\, dx \\ &\quad -\frac{v(-k,t)}{2ik}\int_{-\infty}^{\infty} e_-^2(x,k,t)G(x,t)\, dx\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial v(k,t)}{\partial t} &= ik\sum_{l=0}^{p} c_l \big(2k^2\big)^l v(k,t) - \frac{v(k,t)}{2ik}\int_{-\infty}^{\infty} e_-(x,k,t)e_-(x,-k,t)G(x,t)\, dx \\ &\quad -\frac{\omega(-k,t)}{2ik}\int_{-\infty}^{\infty} e_-^2(x,k,t)G(x,t)\, dx\,. \end{aligned} \tag{3.8}$$

Multiplying (3.8) by $\omega(k,t)$ and subtracting (3.7) multiplied by $v(k,t)$ from it, in accordance with (2.7) and (2.12) we get

$$\frac{\partial S(k,t)}{\partial t} = ik\sum_{l=0}^{p} c_l \left(2k^2\right)^l S(k,t) + \frac{2ik}{\omega^2(k,t)}\int_{-\infty}^{\infty} e_-^2(x,k,t)G(x,t)\, dx. \tag{3.9}$$

Lemma 3.1. *The following identities hold:*

$$\int_{-\infty}^{\infty} G(x,t)e_-^2(x,\,k,t)\,dx = \gamma(t)F(u(0,t))v(k,t)\omega(k,t),$$
$$\int_{-\infty}^{\infty} G(x,\,t)e_-(x,\,k,\,t)e_-(x,\,-k,\,t)\,dx = \gamma(t)F(u(0,t))v(k,t)v(-k,t). \tag{3.10}$$

Proof. Indeed, by using expression (2.8), we get

$$\int_{-\infty}^{\infty} G(x,t)e_-^2(x,\,k,\,t)\,dx = \gamma(t)F(u(0,t))\int_{-\infty}^{\infty} e_-^2(x,\,k;\,t)u_x(x,t)\,dx$$
$$= -2\gamma(t)F(u(0,t))\int_{-\infty}^{\infty}\big(e_-''(x,k,\,t)+k^2e_-(x,k,t)\big)e_-'(x,k,\,t)\,dx$$
$$= -\gamma(t)F(u(0,t))\int_{-\infty}^{\infty}\Big[\Big(\big(e_-'(x,k;t)\big)^2\Big)'+k^2\big(e_-^2(x,k,t)\big)'\Big]\,dx$$
$$= -\gamma(t)F(u(0,t))\lim_{R\to\infty}\Big[k^2e_-^2(x,k,t)+\big(e_-'(x,k,t)\big)^2\Big]\Big|_{-R}^{R}$$
$$= \gamma(t)F(u(0,t))v(k,t)\omega(k,t).$$

One can prove the second equality in (3.10) analogously. □

In view of Lemma 3.1 and formula (3.7) we get $\omega_t(k,\,t)=0$. Therefore, we deduce that

$$\frac{d\lambda_j(t)}{dt}=0,$$
$$\frac{\partial S(k,t)}{\partial t}=\left[ik\sum_{l=0}^{p}c_l\big(2k^2\big)^l+2ik\gamma(t)F(u(0,t))\right]S(k,t),\ \left(|\mathrm{Im}k|<\frac{\varepsilon}{2}\right).$$

Let us now describe the evolution of the normalizing chain $\{\chi_0^n,\,\chi_1^n,\dots,\,\chi_{m_n-1}^n\}$ that corresponds to the complex eigenvalue $\lambda_n,\ \ n=\overline{1,N}$. For this, we rewrite equality (3.6) in the following form:

$$\frac{\partial e_-(x,k,t)}{\partial t}=Y_pe_-(x,k,t)+\frac{1}{2}ik\sum_{l=0}^{p}c_l(2k^2)^le_-(x,k,t)$$
$$-\frac{1}{2ik}\left[e_-(x,k,t)\int_{-\infty}^{x}e_-(x,k,t)e_-(x,-k,t)G(x,t)\,dx\right.$$
$$\left.-e_-(x,-k,t)\int_{-\infty}^{x}e_-^2(x,k,t)G(x,t)\,dx\right]=Y_pe_-(x,k,t)+\frac{1}{2}ik\sum_{l=0}^{p}c_l(2k^2)^le_-(x,k,t)$$
$$-\frac{\gamma(t)F(u(0,t))e_-(x,k,\,t)}{2ik}\Big[e_-(x,\,k,\,t)e_-(x,\,-k,t)u(x,t)$$
$$-\int_{-\infty}^{x}u(x,t)\big(e_-'(x,\,k,\,t)e_-(x,\,-k,t)+e_-(x,\,k,t)e_-'(x,\,-k,\,t)\big)\,dx\Big]$$
$$+\frac{\gamma(t)F(u(0,t))e_-(x,\,-k,\,t)}{2ik}\left[e_-^2(x,\,k,\,t)u(x,t)-\int_{-\infty}^{x}2e_-'(x,\,k,t)e_-(x,\,k,\,t)u(x,t)\,dx\right]$$
$$=Y_pe_-(x,\sqrt{\lambda},t)+\frac{1}{2}ik\sum_{l=0}^{p}c_l(2k^2)^le_-(x,\sqrt{\lambda},t)$$
$$+\gamma(t)F(u(0,t))e_-'(x,k,t)+ik\gamma(t)F(u(0,t))e_-(x,k,t).$$

Therefore, we have

$$\frac{\partial e_-(x,k,t)}{\partial t} = Y_p e_-(x,k,t) + \frac{1}{2} ik \sum_{l=0}^{p} c_l (2k^2)^l e_-(x,k,t)$$

$$+\gamma(t)F(u(0,t))e'_-(x,k,t) + ik\gamma(t)F(u(0,t))e_-(x,k,t). \tag{3.11}$$

Differentiating equality (3.11) $m_n - 1$ times in k, setting $k = k_n$, $x \to \infty$, using equality (2.8) and equating the coefficients of $(ix)^l \cdot e^{ik_n x}$, $l = m_n - 1$, $m_n - 2, \dots, 0$, we find an analogue of the Gardner-Greene-Kruskal-Miura equations

$$\frac{d\chi_r^n}{dt} = i\left(\sum_{q=0}^{p} c_q 2^q k_n^{2q+1} + 2k_n\gamma(t)F(u(0,t))\right)\chi_r^n$$

$$+i\left(\sum_{q=0}^{p} c_q 2^q (2q+1)k_n^{2q} + 2\gamma(t)F(u(0,t))\right)\chi_{r-1}^n$$

$$+i\sum_{l=2}^{r}\left[\sum_{q=0}^{p} c_q 2^q \frac{1}{l\,!}\frac{(2q+1)\,!}{(2q+1-l)\,!}k_n^{2q+1-l}\right]\chi_{r-l}^n.$$

Thus, we have proved the following theorem.

Theorem 3.1. *If a complex-valued function $u(x,t)$ is a solution to the Cauchy problem (1.1)-(1.4), then the scattering data $\left\{S(k,t),\ \lambda_j(t),\ \chi_0^j(t), \chi_1^j(t), \dots, \chi_{m_j-1}^j(t),\ j = \overline{1, N}\right\}$ of the non-self-adjoint operator $L(t)$, $t > 0$, with the potential $u(x,t)$ depend on t in the following way:*

$$\frac{\partial S(k,t)}{\partial t} = \left[ik\sum_{l=0}^{p} c_l \left(2k^2\right)^l + 2ik\gamma(t)F(u(0,t))\right] S(k,t), \quad \left(|\mathrm{Im}k| < \frac{\varepsilon}{2}\right)$$

$$\lambda_n(t) = \lambda_n(0),$$

$$\frac{d\chi_r^n}{dt} = i\left(\sum_{q=0}^{p} c_q 2^q k_n^{2q+1} + 2k_n\gamma(t)F(u(0,t))\right)\chi_r^n$$

$$+i\left(\sum_{q=0}^{p} c_q 2^q (2q+1)k_n^{2q} + 2\gamma(t)F(u(0,t))\right)\chi_{r-1}^n$$

$$+i\sum_{l=2}^{r}\left[\sum_{q=0}^{p} c_q 2^q \frac{1}{l\,!}\frac{(2q+1)\,!}{(2q+1-l)\,!}k_n^{2q+1-l}\right]\chi_{r-l}^n,$$

$$n = \overline{1, N},\ r = 0, 1, \dots, m_n - 1.$$

The obtained equalities completely determine the evolution of the scattering data, which makes it possible to apply the method of the inverse scattering problem to solve problem (1.1)-(1.4).

4 Examples

In conclusion, we present examples illustrating the application of Theorem 3.1.
Example 4.1. Consider the problem

$$u_t = u_{xxxxx} - 20u_x u_{xx} - 10uu_{xx} + 30u^2 u_x + \gamma(t)u(0,t)u_x, \tag{4.1}$$

$$u(x,0) = \frac{8a^2 e^{2iax}}{(1+e^{2iax})^2},\ \mathrm{Im}a > 0,\ \ x \in \mathbb{R}, \tag{4.2}$$

where

$$\gamma(t) = -8a^2(t^2+1) + \frac{\sqrt{t^2+1}}{2ia^3}.$$

One can easily find the scattering data of the operator $L(0)$, namely:

$$\lambda(0) = k^2 = a^2; \quad v(k,0) = 0, \quad S(k,0) = 0, \quad \chi_0(0) = 1.$$

In view of Theorem 3.1 we get

$$\lambda(t) = \lambda(0) = a^2, \quad S(k,\, t) = 0, \quad \chi_0(t) = e^{\beta(t)},$$

where

$$\beta(t) = 32ia^5 t + 2ia \int_0^t \gamma(\tau)u(0,\tau)\, d\tau.$$

Substituting these data into formula (2.12), we find the kernel

$$F_+(x,t) = -2iae^{iax+\beta(t)}$$

of the Gelfand-Levitan-Marchenko integral equation. Furthermore, by solving the integral equation

$$K_+(x,\, y;\, t) - 2iae^{\beta(t)} \cdot e^{ia(x+y)} - 2iae^{\beta(t)} \cdot e^{iay} \int_x^\infty K_+(x,\, s;\, t)e^{ias}ds = 0,$$

we get

$$K_+(x,\, y;\, t) = \frac{2iae^{\beta(t)} \cdot e^{ia(x+y)}}{1 + e^{\beta(t)} \cdot e^{2iax}}.$$

Hence, we find the solution to the Cauchy problem (4.1)-(4.2)

$$u(x,\, t) = \frac{8a^2 e^{2iax+2\mathrm{arcsh}\, t}}{(1+e^{2iax+2\mathrm{arcsh}\, t})^2}.$$

Example 4.2. In equation (1.1), if we assume that $\gamma(t) \equiv 0$ and $p = 2,\ c_0 = 0,\ c_1 = 0,\ c_2 = 8$, then this equation has the following form

$$u_t = u_{xxxxx} - 20u_x u_{xx} - 10uu_{xx} + 30u^2 u_x. \tag{4.3}$$

Consider equation (4.3) with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{4.4}$$

$$u_0(x) = \frac{4\left(\left(A+2Bx\right)ia + B\right)e^{2iax} - \frac{4B^2}{a^2}e^{4iax} + \frac{B^2}{4a^4}\left(\left(A+2Bx\right)ia - 3B\right)e^{6iax}}{\left(1 + \frac{1}{2a^2}\left(\left(A+2Bx\right)ia - \frac{B^2}{16a^4}e^{4iax}\right)\right)^2},$$

where $\mathrm{Im}a > 0,\ \ A = -4ia(a\chi_1^1 + \chi_0^1),\ \ B = 4a^2\chi_0^1$.

The scattering data of the operator

$$L(0) = -\frac{d^2}{dx^2} + u_0(x),\ \ x \in \mathbb{R},$$

have the form

$$\lambda_{1,2}(0)=a^2,\quad S(k,0)=0,\ \ \chi_0^1(0)=\chi_0\,,\ \chi_1^1(0)=\chi_1.$$

By virtue of Theorem 3.1, we find the scattering data for the operator $L(t)$, $t>0$ with potential $u(x,t)$:

$$\lambda(t)=\lambda(0)=a^2;\quad S(k,t)=0,\quad \chi_0^1(t):=\chi_0(t),\quad \chi_1^1(t):=\chi_1(t).$$

In this case, $\chi_0^1(t)$ and $\chi_1^1(t)$ are determined from the system of equations

$$\frac{d\chi_0^1(t)}{dt}=32ia^5\chi_0^1(t),\quad \chi_0^1(0)=\chi_0^1;$$

$$\frac{d\chi_1^1(t)}{dt}=32ia^5\chi_1^1(t)+160ia^4\chi_0^1(t),\quad \chi_1^1(0)=\chi_1^1.$$

Solving this system of equations, we obtain

$$\chi_0^1(t)=\chi_0^1(0)e^{32ia^5t},\quad \chi_1^1(t)=\left[160ia^4\chi_0^1(0)t+\chi_1^1(0)\right]e^{32ia^5t}.$$

Substituting these data into formula (2.12), we find

$$F_+(x,t)=[A(t)+B(t)x]e^{iax},$$

where

$$A(t)=-4ai\left(a\chi_1^1(t)+\chi_0^1(t)\right),\qquad B(t)=4a^2\chi_0^1(t).$$

Further, solving the integral equation

$$K_+(x,\,y;\,t)+[A(t)+B(t)(x+y)]e^{ia(x+y)}$$
$$+\int_x^\infty K_+(x,\,s;\,t)[A(t)+B(t)(s+y)]e^{ia(s+y)}ds=0,$$

we obtain the solution to the Cauchy problem (4.3)-(4.4)

$$u(x,t)=\Bigg(4\left(\left(A(t)+2B(t)x\right)ia+B(t)\right)e^{iax}$$
$$-\frac{4B^2(t)}{a^2}e^{4iax}+\frac{B^4(t)}{4a^4}\Big(\big(A(t)+2B(t)x\big)ia-3B(t)\Big)e^{6iax}\Bigg)$$
$$\times\left(1+\frac{1}{2a^2}\Big(\big(A(t)+2B(t)x\big)ia-B(t)\Big)e^{2iax}-\frac{B^2(t)}{16a^4}e^{4iax}\right)^{-2}.$$

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