ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 2 (2022), 37 – 42

COMPLETENESS OF THE EXPONENTIAL SYSTEM ON A SEGMENT OF THE REAL AXIS

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Communicated by E.D. Nursultanov

Key words: Lebesgue-Stieltjes integral, indicatrix of the growth, Borel adjoint diagram, Beurling-Malliavin multiplier theorem, Paley-Wiener theorem, Cartwright class.

AMS Mathematics Subject Classification: 30D15, 30D20, 46E30.

Abstract. Let $\Lambda = \{\lambda_n\}$ be the sequence of all zeros of the entire function $\Delta(\lambda) = 1 - i\lambda \int_0^1 f(t)e^{i\lambda t}dt$ of exponential type. We consider exponential system of functions $e(\Lambda) = \{t^{p-1}e^{i\lambda_n t}, 1 \leq p \leq m_n\}$, where m_n–is the multiplicity of the zero λ_n. The question is: for which a, b $(a < b)$ is the system $e(\Lambda)$ complete (incomplete) in the space $L^2(a, b)$? Let D be the length of the indicator conjugate diagram of the entire function $\Delta(\lambda)$. Then the following statements are valid:

- when $b - a > D$ the system $e(\Lambda)$ is incomplete in $L^2(a, b)$;
- when $b - a < D$ the system $e(\Lambda)$ is complete in $L^2(a, b)$;
- if we remove from Λ any two points λ and μ, then the system $e(\Omega), \Omega = \Lambda \backslash \{\lambda, \mu\}$ is incomplete in $L^2(a, b)$ also when $b - a = D$.

DOI: https://doi.org/10.32523/2077-9879-2022-13-2-37-42

1 Introduction

Let $\Lambda = \{\lambda_n\}$ be the sequence of all zeros of the entire function of exponential type

$$\Delta(\lambda) = 1 - i\lambda \int_0^1 f(t)e^{i\lambda t}dt, \lambda = re^{i\varphi} = x + iy, \tag{1.1}$$

where $f \in L^2(0, 1)$ (in the sequence Λ, each point λ_n counts as many times as its multiplicity m_n). We assume that it is impossible to reduce the interval of integration without changing the value of the integral itself, that is for any positive number $\epsilon < \frac{1}{2}$ the function f is not equivalent to 0 on the intervals $(0, \epsilon)$ and $(1 - \epsilon, 1)$.

We consider the following exponential system

$$e(\Lambda) = \{t^{p-1}e^{i\lambda_n t}, 1 \leq p \leq m_n\}.$$

We set the following question: for which a, b $(a < b)$ is the system $e(\Lambda)$ complete (incomplete) in the space $L^2(a, b)$ (or, what is the same, in C[a,b])?

It is enough to consider the case $L^2(-\rho, \rho)$ (or $C[-\rho, \rho]$), since the completeness (or incompleteness) of $e(\Lambda)$ is invariant under the shift of the argument. The question posed is reduced to clarifying the quantity $\rho(\Lambda)$, the radius of completeness of the system $e(\Lambda)$. By definition, $\rho(\Lambda)$ is the exact upper bound of the numbers ρ for which the system $e(\Lambda)$ is complete in $C[-\rho, \rho]$ (the radius of completeness of the system $e(\Lambda)$ is the same for the space $L^2(-\rho, \rho)$). An estimate of the radius of completeness of a system of exponentials was investigated in [3]. In terms of entire functions, $\rho(\Lambda)$ can be interpreted as the exact lower bound of the types of entire functions F of exponential type, bounded on R and vanishing on Λ (see [5]). The latter means that at each point $\lambda_n \in \Lambda$ the function F vanishes with a multiplicity of at least m_n. We write this fact as follows: $F(\Lambda) = 0$.

We have

$$\Delta(\lambda) = 1 - i\lambda e^{i\frac{\lambda}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\lambda t} g(t)dt, g(t) = f\left(t + \frac{1}{2}\right).$$

Since $g \in L^1(-\frac{1}{2}, \frac{1}{2})$ then we obtain that

$$G(t) = \int_{-\frac{1}{2}}^{t} g(\tau)d\tau$$

is absolutely continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and equation (1.1) may be rewritten in the form

$$\Delta(\lambda) = 1 - i\lambda e^{i\frac{\lambda}{2}} P(\lambda),$$

where

$$P(\lambda) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\lambda t} dG(t)$$

is the Lebesgue-Stieltjes integral (see [4, pp. 337-359]). Thus Λ is the null set of the entire function of exponential type

$$\Phi(\lambda) = e^{-i\frac{\lambda}{2}} - i\lambda P(\lambda). \tag{1.2}$$

The type of this entire function is $\sigma(\Phi) \leq \frac{1}{2}$, however, it is possible that it is strictly less than $\frac{1}{2}$. Then, the indicatrix of the growth of the function $P(\lambda)$ is $h_p(\varphi) = \frac{1}{2}|\sin\varphi|$. This follows from the fact that for almost all φ there exists the following limit (see [6, p.109])

$$\lim_{r\to\infty} \frac{\ln|P(re^{i\varphi})|}{r} = \frac{1}{2}|\sin\varphi|$$

Because the function $h_p(\varphi)$ is continuous, then $h_p(\varphi) = \frac{1}{2}|\sin\varphi|$ for all φ, $0 \leq \varphi \leq 2\pi$. Then, the adjoint diagram of the function $i\lambda P(\lambda)$ is a segment of the imaginary axis $I = [-\frac{1}{2}i, \frac{1}{2}i]$. Therefore, all singularities of the function $\gamma_p(\lambda)$, associated with $P(\lambda)$ in the sense of Borel, lie in this segment, moreover, the points $t = \pm\frac{i}{2}$ are singular for $\gamma_p(t)$. Since

$$\gamma_\Phi(t) = \frac{1}{t + \frac{i}{2}} - \gamma_p(t),$$

singularities of the function $\gamma_\Phi(t)$ also lie in the segment I. The point $t = -\frac{i}{2}$ is obviously singular for $\gamma_\Phi(t)$ but possible removable. Indeed, $t = -\frac{i}{2}$ is singular for $\gamma_p(t)$ and, hence, it is possible that at the point $t = -\frac{i}{2}$ the function $\gamma_\Phi(t)$ is holomorphic. Therefore the length of the adjoint diagram J of the function $\Phi(\lambda)$ is equal to

$$|J| = h_\Phi(\frac{\pi}{2}) + h_\Phi(-\frac{\pi}{2}) = \frac{1}{2} + h_\Phi(-\frac{\pi}{2}) < 1.$$

We give sufficient conditions for the case when the length of the adjoint diagram J of the function $\Phi(\lambda)$ is equal to 1. To do so we rewrite equation (1.2) in the following form

$$\Phi(\lambda) = e^{-i\frac{\lambda}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)d(e^{i\lambda t}).$$

If the function $g(t)$ is of bounded variation, then by integrating by parts, we obtain

$$\Phi(\lambda) = e^{-i\frac{\lambda}{2}}\left(1 + g\left(-\frac{1}{2}\right)\right) - e^{i\frac{\lambda}{2}} g\left(\frac{1}{2}\right) + \Psi(\lambda)$$

where

$$\Psi(\lambda) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\lambda t} d(g(t)).$$

Now it is easily seen that if $g\left(-\frac{1}{2}\right) = -1$, $g\left(\frac{1}{2}\right) = 0$ and the function $g(t)$ not constant in the neighborhood of points $\pm\frac{1}{2}$, then the adjoint diagrams of the function Φ and Ψ coincide with $J = [-\frac{1}{2}i, \frac{1}{2}i]$.

The points $\pm\frac{1}{2}$ are singularities for $\gamma_\Psi(t)$ associated with $\Psi(\lambda)$ in the sense of Borel, hence,

$$|J| = h_\Psi\left(-\frac{\pi}{2}\right) + h_\Psi\left(\frac{\pi}{2}\right) = 1.$$

Thus, if the function f is of bounded variation and not constant near the points $t = 0$, $t = 1$ and $f(0) = -1$, $f(1) = 0$, then $|J| = 1$.

Now we consider another case when $g(t)$ may be constant near the points $\pm\frac{1}{2}$ or in a neighborhood of at least one of these points.

If $g(t)$ is constant near $-\frac{1}{2}$ and $g\left(-\frac{1}{2}\right) = -1$, then $|J| < 1$. If $g(t)$ is constant near $\frac{1}{2}$ and $g\left(\frac{1}{2}\right) = 0$, then $|J| < 1$. If $g(t)$ is constant in neighborhoods of both points $\pm\frac{1}{2}$ and $g\left(-\frac{1}{2}\right) \neq -1$, $g\left(\frac{1}{2}\right) \neq 0$, then $|J| = 1$ since the adjoint diagram J_Ψ of the entire function Ψ has a length less than 1, while the adjoint diagram of the entire function

$$a(\lambda) = C_1 e^{-i\frac{\lambda}{2}} + C_2 e^{i\frac{\lambda}{2}}, C_1 \neq 0, C_2 \neq 0$$

is a segment $J_a = [-\frac{1}{2}i, \frac{1}{2}i]$, moreover $J_\Psi \subset J_a$.

If $g(t)$ is constant near $-\frac{1}{2}$ and $g\left(-\frac{1}{2}\right) \neq -1$, and $g\left(\frac{1}{2}\right) = 0$, $g(t)$ is not constant near $\frac{1}{2}$, then $|J| = 1$. If $g(t)$ is constant near $\frac{1}{2}$ and $g\left(\frac{1}{2}\right) \neq 0$, and $g(t)$ is not constant near $-\frac{1}{2}$, $g\left(-\frac{1}{2}\right) = -1$, then $|J| = 1$ again. If it is only known that, for example $g(t)$ is not constant near $\frac{1}{2}$ and $g\left(\frac{1}{2}\right) = 0$, then we can only assert that $|J| \leq 1$.

Since $|\Phi(x)| = O(|x|)$ as $x \to \infty$, then the entire function $\Phi(\lambda)$ belongs to the Cartwright class C, i.e.

$$\int\limits_{-\infty}^{\infty} \frac{\ln^+ |\Phi(x)|}{1 + x^2} dx < \infty.$$

By the Beurling-Malliavin multiplier theorem [2], we obtain $\rho(\Lambda) = \sigma(\Lambda)$, where

$$\sigma(\Lambda) = \inf\{\sigma(F) : F \in C, F(\Lambda) = 0, F(z) \not\equiv 0\}$$

(see also [5]). Therefore, the radius of completeness of the system $e(\Lambda)$ satisfies the inequality $\rho(\Lambda) \leq \frac{1}{2}$. In other words, the system $e(\Lambda)$ is not complete in $C[a, b]$ (or $L^2(a, b)$), if $b - a > 1$.

Let us clarify this fact. Consider the function $\Phi_1(\lambda) = \Phi(\lambda)e^{i\alpha\lambda}$, where $\alpha = \frac{1-|J|}{2}$. Then the adjoint diagram of the function $\Phi_1(\lambda)$ is the segment of the imaginary axis $[-i\frac{|J|}{2}, i\frac{|J|}{2}]$. We also have $\Phi_1 \in C$, $\Phi_1(\Lambda) = 0$ (there are no other zeros of this function). Therefore, we obtain that $\rho(\Lambda) = \frac{|J|}{2}$. This means that when $\rho > \frac{|J|}{2}$ the system $e(\Lambda)$ is not complete in $C[-\rho, \rho]$ $(or L^2(-\rho, \rho))$.

Now we show that if $\rho < \frac{|J|}{2}$ the system $e(\Lambda)$ is complete in given spaces. Let $e(\Lambda)$ be incomplete, for example in $C[-\rho, \rho]$ for $\rho < \frac{|J|}{2}$. Then there exists an entire function of exponential type of the following form

$$\Psi(\lambda) = \int\limits_{-\rho}^{\rho} e^{i\lambda t} d\Psi(t), \Psi(\lambda) \neq 0,$$

that at points λ_n has zeros of multiplicities not less than m_n. Let $\rho_1 < \rho$, $\rho_2 < \rho$, such that in any neighborhood of points ρ_1 and ρ_2 the function $\psi(t)$ is non-constant. Then all zeros of the function Ψ, with the possible exception of sets of zero density, lie inside the angles, $S_1 = \{z : |\arg z| < \varepsilon\}$, $S_2 = \{z : |\arg z - \pi| < \varepsilon\}$ ($\varepsilon > 0$ is arbitrary, $\varepsilon < \frac{\pi}{2}$), moreover, the sets of zeros inside each of the angles $S_i (i = 1, 2)$ have densities $\Delta_i = \frac{\rho_1+\rho_2}{2\pi}$ (see [6, pp. 109-110]).

We now consider the entire function of exponential type

$$\Psi_1(\lambda) = \frac{\Phi_1(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)}.$$

This function has the same adjoint diagram, i.e. the segment $[-\frac{|J|}{2}i, \frac{|J|}{2}i]$, in addition $\Psi_1(\Lambda_1) = 0$, where $\Lambda_1 = \Lambda \backslash \{\lambda_1, \lambda_2\}$. Since $\Psi_1 \in L^2(R)$, then by the Paley–Wiener theorem there exists a function $\psi_1 \in L^2[-\frac{|J|}{2}, \frac{|J|}{2}]$ such that

$$\Psi_1(\lambda) = \int\limits_{-\frac{|J|}{2}}^{\frac{|J|}{2}} e^{i\lambda t}\psi_1(t)d(t).$$

Thus, by the above-mentioned reasoning we obtain that in each of the corners S_1 and S_2 the corresponding subsequences of the null set Λ_1 of the function $\Psi_1(\lambda)$ have densities equal to $\Delta^{(i)} = \frac{|J|}{2\pi}$ $(i = 1, 2)$. On the other hand, since the function $\Psi_1(\lambda)$ may have other zeros

$$\Delta^{(i)} \leq \Delta_i = \frac{\rho_1 + \rho_2}{2\pi} < \frac{|J|}{2\pi} = \Delta^{(i)},$$

we obtain a contradiction.

Thus, when $\rho < \frac{|J|}{2}$ the system $e(\Lambda)$ is complete in $C[-\rho, \rho]$ (or $L^2(-\rho, \rho)$). Therefore, we proved the following

Theorem 1.1. *Let $\Lambda = \{\lambda_n\}$ be the null set of the entire function* (1.1)*, and $h_\Phi(\varphi)$ – be an indicatrix of the growth of the entire function $\Phi(\lambda)$, given by formula* (1.2)*. Assume $|J| = h_\Phi\left(\frac{\pi}{2}\right) + h_\Phi\left(-\frac{\pi}{2}\right)$. Then the following are valid*
1) *when $|J| < b - a$ the system $e(\Lambda)$ is incomplete in $C[a, b]$ (or $L^2(a, b)$),*
2) *when $|J| > b - a$ the system $e(\Lambda)$ is complete in $C[a, b]$ (or $L^2(a, b)$),*
3) *if we remove from Λ any two points, then the system $e(\Lambda_1)$ of remaining sequence of the points Λ_1 is incomplete in $C[a, b]$ (or$L^2(a, b)$) and when $|J| = b - a$.*

In particular, when $|J| < 1$, the system $e(\Lambda)$ is incomplete in $L^2(0, 1)$, and when $|J| > 1$, the system is complete in $L^2(0, 1)$. In the latter case, for instance, there exists an entire function

$$L(\lambda) = \int\limits_0^1 e^{i\lambda t}d\mu(t), d\mu(t) = \psi(t)dt,$$

where $\psi \in L^2(0, 1)$, such that

$$L(\Lambda) = 0, L(\lambda) \neq 0.$$

Since the function $\mu(t)$ defines on $(0, 1)$ some measure μ, then we can introduce the spectrum of this measure.

A point $t_0 \in (0, 1)$ is called a point of growth of a measure μ, if $\mu(t_0 - \varepsilon, t_0 + \varepsilon) > 0$ for any $\varepsilon > 0$. The set of points of growth of a measure μ is closed and is called the spectrum S_μ of the measure μ. The support Δ_μ of the measure μ is the least segment containing the spectrum S_μ. In our case

$$S_\mu \subset \Delta_\mu = [0, 1].$$

A point $t_0 \in (0, 1)$ is called a mass concentration point, if $\mu(\{t_0\}) > 0$. It is clear that the set of such points is at most countable (see [7]). It is difficult to say anything more exact about the nature of the spectrum of S_μ.

Close questions related to the application of the Beurling-Malliavin type theorem are also considered in [1].

Acknowledgments

This work supported by the Russian Foundation grant No. 21-11-00168. The work was partially supported by the grant No. AP08855402 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

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Received: 21.02.2021