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The L.N. Gumilyov Eurasian National University
Building no. 3
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Tel.: +7-7172-709500 extension 33312
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The Moscow Editorial Office
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(RUDN University)
Room 473
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MINIMAX SHRINKAGE ESTIMATORS AND ESTIMATORS DOMINATING THE JAMES-STEIN ESTIMATOR UNDER THE BALANCED LOSS FUNCTION

A. Benkhaled, A. Hamdaoui, M. Terbeche

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Key words: Balanced loss function, James-Stein estimator, minimax estimator, multivariate Gaussian random variable, non-central chi-square distribution, shrinkage estimators.

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Abstract. This paper is dealing with the shrinkage estimators of a multivariate normal mean and their minimaxity properties under the balanced loss function. We present here two different classes of estimators: the first which generalizes the James-Stein estimator, and show that any estimator of this class dominates the maximum likelihood estimator (MLE), consequently it is minimax, and the second dominates the James-Stein estimator and we conclude that any estimator of this class is also minimax.

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1 Introduction

Estimation of several mean parameters in multivariate analysis has a long, rich and influential history and has received a great attention from researchers and practitioners in a variety of fields. Among different methods, the shrinkage estimation is of interest. The latter has become a very important technique for modelling data and provides useful techniques for combining data from various sources. However, these methods 'shrink' the estimate with high bias to an estimate with high variance. In other words, it is the sum of an estimator with high variance and an estimator with high bias, with some weighting between the two. In addition shrinkage estimation strategy attempts to incorporate prior uncertain information in the estimation procedure. Early references concerning the estimation of the mean of a multivariate normal distribution by shrinkage estimation can be found in Stein [18], James and Stein [11] and Yang and Berger [21]. Efron and Morris [6] studied the James-Stein estimators in an empirical Bayes framework and proposed several competing shrinkage estimators. Berger and Strawderman [3] discussed this problem from a hierarchical Bayesian perspective. For applications of shrinkage techniques in practice, see Efron and Morris [5] and Brown [4]. Recently, Tsukuma and Kubokawa [19] addresses the problem of estimating the mean vector of a singular multivariate normal distribution with an unknown singular covariance matrix. Xie et al. [20] introduced a class of semi-parametric/parametric shrinkage estimators and established their asymptotic optimality properties. Benkhaled and Hamdaoui [2], have considered the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown. They studied two different forms of shrinkage estimators of θ: estimators of the form $\delta^{\psi} = (1 - \psi(S^2, \|X\|^2)S^2/\|X\|^2)X$, and estimators of Lindley-Type given by $\delta^{\varphi} = (1 - \varphi(S^2, T^2)S^2/T^2)(X - \overline{X}) + \overline{X}$, that shrink the components of the MLE X to the random variable $\overline{X}$. The authors showed that if the shrinkage function ψ (respectively φ) satisfies the new conditions different from the known results in the literature, then the estimator δ^{ψ} (respectively δ^{φ}) is minimax. When the sample size and the dimension of parameters space tend to infinity, they studied the behaviour of risks ratio of these estimators to the MLE. Hamdaoui et al. [9], have treated the minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the Bayesian case. The authors have considered the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown and have taken the prior law $\theta \sim N_p\left(\upsilon, \tau^2 I_p\right)$. They constructed a modified Bayes estimator δ_B^* and an empirical modified Bayes estimator δ_{EB}^*. When n and p are finite, they showed that the estimators δ_B^* and δ_{EB}^* are minimax. The authors have also been interested in studying the limits of risks ratios of these estimators, to the MLE X, when n and p tend to infinity. The majority of these authors have considered the quadratic loss function for computing the risk.

Zellner [22] proposes a balanced loss function that takes error of estimation and goodness of fit into account. This balanced loss function consists of weighting the predictive loss function and the goodness of fit term. In addition for estimation under the balanced loss function we cite, for example, Guikai et al. [8], Karamikabir et al. [13], Marchand and Strawderman [14]. Sanjari Farsipour and Asgharzadeh [15] have considered the model: $X_1, ..., X_n$ to be a random sample from $N_p\left(\theta, \sigma^2\right)$ with σ^2 known and the aim is to estimate the parameter θ. They studied the admissibility of the estimator of the form $a\overline{X} + b$ under the balanced loss function. Selahattin and Issam [16] introduced and derived the optimal extended balanced loss function (EBLF) estimators and predictors and discussed their performances.

In this work, we deal with the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi_n^2$). Our aim is to estimate the unknown parameter θ by shrinkage estimators deduced from the MLE. The adopted criterion to compare two estimators is the risk associated to the balanced loss function. The paper is organized as follows. In Section 2, we recall some preliminaries that are useful for our main results. In Section 3, we establish the minimaxity of the estimators defined by $\delta_{a,r} = \left(1 - a((S^2)^{r/2}/\|X\|^r\right) X$, where $2 \leq r < (p+2)/2$ and the real constant a may depend on n and p. In Section 4, we consider the estimators of the form $\delta_{b,r} = \delta_{JS} + b\left((S^2)^{r/2}/\|X\|^r\right) X$ with $2 \leq r < (p+2)/2$ and the real constant b may depend on n and p. We show that these estimators dominate the James-Stein estimator δ_{JS} under some condition on the parameter b. In Section 5, we conduct a simulation study that shows the performance of the considered estimators. We end the manuscript by giving an Appendix which contains the proofs of some of our main results.

2 Preliminaries

We recall that if X is a random variable in R^p that follow the multivariate normal distribution with a mean vector θ and identity covariance matrix $\sigma^2 I_p$ (i.e. $X \sim N_p\left(\theta, \sigma^2 I_p\right)$), then $\frac{\|X\|^2}{\sigma^2} \sim \chi_p^2(\lambda)$ where $\chi_p^2(\lambda)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$.

We also recall the following definition given in formula (1.2) by Arnold [1]. It will be used to calculate the expectation of functions of a non-central chi-square law's variable.

Definition 1. *Let $U \sim \chi_p^2(\lambda)$ be non-central chi-square with p degrees of freedom and non-centrality parameter λ. The density function of U is given by*

$$f(x) = \sum_{k=0}^{+\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^k}{k!} \frac{x^{(p/2)+k-1}e^{-x/2}}{\Gamma(\frac{p}{2}+k)2^{(p/2)+k}}, \; 0 < x < +\infty.$$

The right-hand side (RHS) of this equality is none other than the formula

$$\sum_{k=0}^{+\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^k}{k!} \chi_{p+2k}^2,$$

where χ_{p+2k}^2 is the density of the central χ^2 distribution with $p+2k$ degrees of freedom.

To this definition we deduce that if $U \sim \chi_p^2(\lambda)$, then for any function $f : \mathrm{R}_+ \longrightarrow \mathrm{R}$, $\chi_p^2(\lambda)$ integrable, we have

$$
\begin{aligned}
E\left[f(U)\right] &= E_{\chi_p^2(\lambda)}\left[f(U)\right] \\
&= \int_{\mathrm{R}_+} f(x)\chi_p^2\left(\lambda\right)dx \\
&= \sum_{k=0}^{+\infty}\left[\int_{\mathrm{R}_+} f(x)\chi_{p+2k}^2\left(0\right)dx\right]e^{-\frac{\lambda}{2}}\frac{\left(\frac{\lambda}{2}\right)^k}{k!} \\
&= \sum_{k=0}^{+\infty}\left[\int_{\mathrm{R}_+} f(x)\chi_{p+2k}^2 dx\right]P\left(\frac{\lambda}{2};dk\right),
\end{aligned}
\tag{2.1}
$$

where $P\left(\frac{\lambda}{2};dk\right)$ being the Poisson distribution of parameter $\frac{\lambda}{2}$ and χ_{p+2k}^2 is the central chi-square distribution with $p+2k$ degrees of freedom.

Using the last equality, we conclude the following Lemma.

Lemma 2.1. *Let $U\sim\chi_p^2(\lambda)$ be non-central chi-square with p degrees of freedom and non-centrality parameter λ. Then for $0\le r<\frac{p}{2}$,*

$$
\begin{aligned}
E(U^{-r}) &= E[(\chi_p^2(\lambda))^{-r}] \\
&= E[(\chi_{p+2K}^2)^{-r}] \\
&= 2^{-r}E\left(\frac{\Gamma(\frac{p}{2}-r+K)}{\Gamma(\frac{p}{2}+K)}\right),
\end{aligned}
$$

where K has a Poisson distribution with mean $\frac{\lambda}{2}$.

We recall the following lemma given by Stein [17], that we will often use in the sequel.

Lemma 2.2. *Let X be a $N\left(\upsilon,\sigma^2\right)$ real random variable and let $f:\mathrm{R}\longrightarrow\mathrm{R}$ be an indefinite integral of the Lebesgue measurable function, f' be the derivative of f. Suppose also that $E\left(\left|f'\left(X\right)\right|\right)<+\infty$, then*

$$
E\left[\left(\frac{X-\upsilon}{\sigma}\right)f\left(X\right)\right]=E\left(f'\left(X\right)\right).
$$

3 A class of minimax shrinkage estimators

In this section, we consider the model $X\sim N_p\left(\theta,\sigma^2I_p\right)$ where σ^2 is unknown and estimated by S^2 ($S^2\sim\sigma^2\chi_n^2$). Our aim is to estimate the unknown mean parameter θ by the shrinkage estimators under the balanced squared error loss function. It is well known from the literature that the estimators of type James-Stein of the mean of a multivariate normal distribution, namely $\delta_a=\left(1-a(S^2)/\|X\|^2\right)X$ are minimax for a certain range of values of a. Here, we introduce a more general class of estimators depending on another real parameter r and study its minimaxity property according to this parameter.

Definition 2. *Suppose that X is a random vector having a multivariate normal distribution $N_p\left(\theta,\sigma^2I_p\right)$ where the parameters θ and σ^2 is unknown. The balanced squared error loss function is defined as follows:*

$$
L_\omega(\delta,\theta)=\omega\|\delta-\delta_0\|^2+(1-\omega)\|\delta-\theta\|^2,\ 0\le\omega<1,
\tag{3.1}
$$

where δ_0 is the target estimator of θ, ω is the weight given to the proximity of δ to δ_0, $1-\omega$ is the relative weight given to the precision of estimation portion and δ is a given estimator.

For more details about this loss see Jafari Jozani et al. [10], Zinodiny et al. [23] and Karamikabir and Afsahri [12].

We associate with this balanced squared error loss function the risk function defined by $R_\omega(\delta,\theta)=E(L_\omega(\delta,\theta))$.

In this model, it is clear that the MLE is $X := \delta_0$, its risk function is $(1-\omega)p\sigma^2$.
Indeed: We have

$$\begin{aligned} R_\omega(X,\theta) &= \omega E(\|X-X\|^2) + (1-\omega)E(\|X-\theta\|^2) \\ &= (1-\omega)E(\|X-\theta\|^2). \end{aligned}$$

As $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, then $\frac{X-\theta}{\sigma} \sim N_p\left(0, I_p\right)$, thus $\frac{\|X-\theta\|^2}{\sigma^2} \sim \chi^2_p$.
Hence, $E(\|X-\theta\|^2) = E(\sigma^2\chi^2_p) = \sigma^2 p$, and the desired result follows.

It is well known that δ_0 is minimax and inadmissible for $p \geq 3$, thus any estimator it dominates is also minimax. We give the following Lemma, that will be used in our proofs and its proof is postponed to the Appendix.

Lemma 3.1. *Let $U \sim \chi^2_p(\lambda)$ be non-central chi-square with p degrees of freedom and non-centrality parameter λ then,*

i) for any real numbers s and r where $-\frac{p}{2} < s \leq r < 0$, the real-valued function

$$H_{p,r,s}(\lambda) = \frac{E(U^r)}{E(U^s)} = \frac{\int_{R_+} x^r \chi^2_p(\lambda; dx)}{\int_{R_+} x^s \chi^2_p(\lambda; dx)}$$

is nondecreasing in λ.

ii) Furthermore, if $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, we get

$$\sup_{\|\theta\|} \left(\frac{E(\|X\|^{-2r+2})}{E(\|X\|^{-r})} \right) = 2^{\frac{-r+2}{2}} \frac{\Gamma(\frac{p}{2}-r+1)}{\Gamma(\frac{p-r}{2})}.$$

Now, we consider the estimator

$$\delta_{a,r} = \left(1 - a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}\right) X = X - a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r} X, \tag{3.2}$$

where $2 \leq r < \frac{p+2}{2}$ and the real positive constant a may depend on n and p.

Proposition 3.1. *Under the balanced squared error loss function L_ω, the risk function of the estimator $\delta_{a,r}$ given in* (3.2) *is*

$$\begin{aligned} R_\omega(\delta_{a,r},\theta) &= (1-\omega)\sigma^2 \left\{ p - (p-r)a 2^{\frac{r+2}{2}} \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^r}\right) \right\} \\ &+ a^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^{2r-2}}\right), \end{aligned}$$

where $y = \frac{X}{\sigma} = (y_1, ..., y_p)^t$ and for all $i = 1, ..., p$, $y_i = \frac{X_i}{\sigma} \sim N\left(\frac{\theta_i}{\sigma}, 1\right)$.

Proof. Using the risk function associated with the balanced squared error loss function defined in (3.1) we obtain

$$R_\omega(\delta_{a,r},\theta) = \omega E(\|\delta_{a,r} - X\|^2) + (1-\omega)E(\|\delta_{a,r} - \theta\|^2).$$

From the independence between two random variable S^2 and $\|X\|^2$, we obtain

$$\begin{aligned} E(\|\delta_{a,r} - X\|^2) &= E\left(\left\| -a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r} X \right\|^2\right) \\ &= a^2 E((S^2)^r) E\left(\frac{\|X\|^2}{(\|X\|^2)^r}\right) \\ &= a^2 E((\sigma^2\chi^2_n)^r)(\sigma^2)^{1-r} E\left(\frac{1}{\|y\|^{2r-2}}\right) \\ &= a^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^{2r-2}}\right) \end{aligned}$$

and

$$\begin{aligned} E(\|\delta_{a,r}-\theta\|^2) &= E\left(\left\|X-a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X-\theta\right\|^2\right)\\ &= E(\|X-\theta\|^2)+E\left(\left\|a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\|^2\right)-2E\left(\left\langle X-\theta, a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right). \end{aligned}$$

As,

$$\begin{aligned} E\left(\left\langle X-\theta, a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right) &= aE((S^2)^{\frac{r}{2}})\sum_{i=1}^{p}E\left[(X_i-\theta_i)\frac{1}{\|X\|^r}X_i\right]\\ &= a(\sigma^2)^{\frac{r}{2}}2^{\frac{r}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}(\sigma^2)^{1-\frac{r}{2}}\sum_{i=1}^{p}E\left[\left(y_i-\frac{\theta_i}{\sigma}\right)\frac{1}{\|y\|^r}y_i\right], \end{aligned}$$

and using the Lemma 2.2 we get

$$\begin{aligned} E\left(\left\langle X-\theta, a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right) &= a\sigma^2 2^{\frac{r}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}\sum_{i=1}^{p}E\left(\frac{\partial}{\partial y_i}\frac{1}{\|y\|^r}y_i\right)\\ &= a\sigma^2 2^{\frac{r}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}\sum_{i=1}^{p}E\left(\frac{1}{\|y\|^r}-\frac{ry_i^2}{\|y\|^{r+2}}\right)\\ &= a\sigma^2 2^{\frac{r}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}(p-r)E\left(\frac{1}{\|y\|^r}\right). \end{aligned}$$

Then

$$\begin{aligned} R_\omega(\delta_{a,r},\theta) &= \omega a^2\sigma^2 2^r\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^{2r-2}}\right)+(1-\omega)p\sigma^2\\ &+ (1-\omega)\left[a^2\sigma^2 2^r\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^{2r-2}}\right)-2a\sigma^2 2^{\frac{r}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}(p-r)E\left(\frac{1}{\|y\|^r}\right)\right]\\ &= (1-\omega)\sigma^2\left\{p-(p-r)a2^{\frac{r+2}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right)\right\}\\ &+ a^2\sigma^2 2^r\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^{2r-2}}\right), \end{aligned}$$

and the desired result is obtained. □

Theorem 3.1. *Assume that the estimator $\delta_{a,r}$ is defined by (3.2).*

i) A sufficient condition that $\delta_{a,r}$ dominates the MLE (so it is minimax), is

$$0\le a\le(1-\omega)(p-r)\frac{\Gamma(\frac{n+r}{2})\Gamma(\frac{p-r}{2})}{\Gamma(\frac{n+2r}{2})\Gamma(\frac{p-2r+2}{2})},$$

ii) the optimal value for a that minimizes the risk function $R_\omega(\delta_{a,r},\theta)$, is

$$\widehat{a}=\frac{(1-\omega)(p-r)}{2}\frac{\Gamma(\frac{n+r}{2})\Gamma(\frac{p-r}{2})}{\Gamma(\frac{n+2r}{2})\Gamma(\frac{p-2r+2}{2})}.$$

Proof. i) By using Proposition 3.1 we have

$$\begin{aligned} R_\omega(\delta_{a,r},\theta) &= (1-\omega)\sigma^2\left\{p-(p-r)a2^{\frac{r+2}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right)\right\} \\ &+ a^2\sigma^2 2^r\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}\left(\frac{E\left(\frac{1}{\|y\|^{2r-2}}\right)}{E\left(\frac{1}{\|y\|^r}\right)}\right)E\left(\frac{1}{\|y\|^r}\right). \end{aligned}$$

Application of Lemma 3.1 leads to

$$\begin{aligned} R_\omega(\delta_{a,r},\theta) &\le (1-\omega)\sigma^2\left\{p-(p-r)a2^{\frac{r+2}{2}}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right)\right\} \\ &+ a^2\sigma^2 2^r\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}2^{-\frac{r-2}{2}}\frac{\Gamma(\frac{p-2r+2}{2})}{\Gamma(\frac{p-r}{2})}E\left(\frac{1}{\|y\|^r}\right) \\ &= \sigma^2(1-\omega)p-2^{\frac{r+2}{2}}(1-\omega)a(p-r)\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right) \\ &+ 2^{\frac{r+2}{2}}a^2\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{p-2r+2}{2})}{\Gamma(\frac{p-r}{2})}E\left(\frac{1}{\|y\|^r}\right). \end{aligned} \tag{3.3}$$

From the RHS of the last equality, it is easy to show that a sufficient condition for the validity of the inequality $R_\omega(\delta_{a,r},\theta)\le R_\omega(X,\theta)=(1-\omega)p\sigma^2$ which implies that $\delta_{a,r}$ dominates the MLE (so it is minimax), is

$$-2^{\frac{r+2}{2}}(1-\omega)a(p-r)\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right)+2^{\frac{r+2}{2}}a^2\frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{p-2r+2}{2})}{\Gamma(\frac{p-r}{2})}E\left(\frac{1}{\|y\|^r}\right)\le 0,$$

that is equivalent to

$$2^{\frac{r+2}{2}}a\frac{1}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right)\left[-(1-\omega)(p-r)\Gamma(\frac{n+r}{2})+a\Gamma(\frac{n+2r}{2})\frac{\Gamma(\frac{p-2r+2}{2})}{\Gamma(\frac{p-r}{2})}\right]\le 0,$$

which leads to

$$0\le a\le(1-\omega)(p-r)\frac{\Gamma(\frac{n+r}{2})\Gamma(\frac{p-r}{2})}{\Gamma(\frac{n+2r}{2})\Gamma(\frac{p-2r+2}{2})}.$$

ii) Using the convexity on a of the function given in RHS of equality (3.3) one can easily obtain the result. □

For $r=2$, we note $\widehat{a}$ by $d:=\frac{(1-\omega)(p-2)}{n+2}$, then we obtain the James-Stein estimator

$$\delta_{JS}=\delta_{d,2}=\left(1-d\frac{S^2}{\|X\|^2}\right)X. \tag{3.4}$$

From Proposition 3.1 the risk function of δ_{JS} is

$$R_\omega(\delta_{JS},\theta)=(1-\omega)p\sigma^2-(p-2)^2(1-\omega)^2\frac{n}{n+2}E\left(\frac{1}{p-2+2K}\right), \tag{3.5}$$

where $K\sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$.

From formula (3.5) we note that

$$R_\omega(\delta_{JS},\theta)\le(1-\omega)p\sigma^2=R_\omega(X,\theta),$$

then δ_{JS} dominates the MLE X, therefore, it is also minimax.

4 Estimators dominating the James-Stein estimator

Since the estimator $\delta_{a,r} = X - a\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X$ dominates the MLE X for certain values of a and r, we think to add the term $b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X$ to the James-Stein estimator δ_{JS} to obtain an estimator that outperforms δ_{JS}. Namely, we consider

$$\delta_{b,r} = \delta_{JS} + b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X, \tag{4.1}$$

where $2 \le r < \frac{p+2}{2}$ and the real positive constant b may depend on n and p.

Proposition 4.1. *Under the balanced squared error loss function L_ω, the risk function of the estimator $\delta_{b,r}$ given in* (4.1) *is*

$$\begin{aligned} R_\omega(\delta_{b,r},\theta) &= R_\omega(\delta_{JS},\theta) + b^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^{2r-2}}\right) \\ &+ b\sigma^2 2^{\frac{r+2}{2}}\left[(1-\omega)(p-r) - d(n+r)\right]\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}E\left(\frac{1}{\|y\|^r}\right), \end{aligned}$$

where $y = \frac{X}{\sigma} = (y_1, ..., y_p)^t$ and for all $i = 1, ..., p$, $\;y_i = \frac{X_i}{\sigma} \sim N\left(\frac{\theta_i}{\sigma}, 1\right)$.

Proof. Using the risk function associated with the balanced loss function defined in (3.1) we obtain

$$\begin{aligned} R_\omega(\delta_{b,r},\theta) &= \omega E\left(\left\|\delta_{JS} + b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X - X\right\|^2\right) + (1-\omega)E\left(\left\|\delta_{JS} + b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X - \theta\right\|^2\right) \\ &= \omega E(\|\delta_{JS} - X\|^2) + \omega E\left(\left\|b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\|^2\right) + 2\omega E\left(\left\langle \delta_{JS} - X, b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right) \\ &+ (1-\omega)E(\|\delta_{JS} - \theta\|^2) + (1-\omega)E\left(\left\|b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\|^2\right) \\ &+ 2(1-\omega)E\left(\left\langle \delta_{JS} - \theta, b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right) \end{aligned}$$

$$
\begin{aligned}
&= R_\omega(\delta_{JS},\theta) + b^2 E(S^2)^r E\left(\frac{1}{\|X\|^{2r-2}}\right) - 2\omega d b E(S^2)^{\frac{r}{2}+1} E\left(\frac{1}{\|X\|^r}\right) \\
&+ 2(1-\omega) E\left(\left\langle X - \theta - d\frac{S^2}{\|X\|^2}X, b\frac{(S^2)^{\frac{r}{2}}}{\|X\|^r}X\right\rangle\right) \\
&= R_\omega(\delta_{JS},\theta) + b^2 E(S^2)^r E\left(\frac{1}{\|X\|^{2r-2}}\right) - 2\omega d b E(S^2)^{\frac{r}{2}+1} E\left(\frac{1}{\|X\|^r}\right) \\
&- 2(1-\omega) d b E(S^2)^{\frac{r}{2}+1} E\left(\frac{1}{\|X\|^r}\right) + 2(1-\omega) b E(S^2)^{\frac{r}{2}} \sum_{i=1}^p E\left[(X_i-\theta_i)\frac{X_i}{\|X\|^r}\right] \\
&= R_\omega(\delta_{JS},\theta) + b^2 E(S^2)^r E\left(\frac{1}{\|X\|^{2r-2}}\right) - 2 d b E(S^2)^{\frac{r}{2}+1} E\left(\frac{1}{\|X\|^r}\right) \\
&+ 2(1-\omega) b E(S^2)^{\frac{r}{2}} (\sigma^2)^{1-\frac{r}{2}} \sum_{i=1}^p E\left[\frac{(X_i-\theta_i)}{\sigma}\frac{1}{(\frac{\|X\|^2}{\sigma^2})^{\frac{r}{2}}}\frac{X_i}{\sigma}\right] \\
&= R_\omega(\delta_{JS},\theta) + b^2 E(\sigma^2\chi_n^2)^r (\sigma^2)^{1-r} E\left(\frac{1}{(\chi^2_{p+2k})^{r-1}}\right) \\
&- 2 d b E(\sigma^2\chi_n^2)^{\frac{r}{2}+1} (\sigma^2)^{-\frac{r}{2}} E\left(\frac{1}{(\chi^2_{p+2k})^{\frac{r}{2}}}\right) \\
&+ 2(1-\omega) b E(\sigma^2\chi_n^2)^{\frac{r}{2}} (\sigma^2)^{1-\frac{r}{2}} \sum_{i=1}^p E\left[\frac{\partial}{\partial y_i}\left(\frac{1}{\|y\|^2} y_i\right)\right] \\
&= R_\omega(\delta_{JS},\theta) + b^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^{2r-2}}\right) - 2 d b \sigma^2 2^{\frac{r}{2}+1} \frac{\Gamma(\frac{n+r+2}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^r}\right) \\
&+ 2(1-\omega) b \sigma^2 2^{\frac{r}{2}} \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})}(p-r) E\left(\frac{1}{\|y\|^r}\right) \\
&= R_\omega(\delta_{JS},\theta) + b^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^{2r-2}}\right) \\
&+ b\sigma^2 2^{\frac{r+2}{2}} \left[(1-\omega)(p-r) - d(n+r)\right] \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^r}\right).
\end{aligned}
$$

□

Theorem 4.1. *Under the balanced squared error loss function L_ω, the estimator $\delta_{b,r}$ with*

$$
b = \frac{(1-\omega)(r-2)}{2}\frac{n+r}{n+2}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n+2r}{2})}\frac{\Gamma(\frac{p-r}{2})}{\Gamma(\frac{p-2r+2}{2})},
$$

dominates the James-Stein estimator δ_{JS}.

Proof. By using Proposition 4.1, we have

$$
\begin{aligned}
R_\omega(\delta_{b,r},\theta) &\leq R_\omega(\delta_{JS},\theta) + b^2\sigma^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} \frac{E\left(\frac{1}{\|y\|^{2r-2}}\right)}{E\left(\frac{1}{\|y\|^r}\right)} E\left(\frac{1}{\|y\|^r}\right) \\
&+ b\sigma^2 2^{\frac{r+2}{2}} \left[(1-\omega)(p-r)\frac{n+r}{n+2} - \frac{(1-\omega)(p-2)}{n+2}(n+r)\right] \\
&\times \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^r}\right).
\end{aligned}
$$

Using Lemma 3.1 we have

$$\begin{aligned} R_\omega(\delta_{b,r},\theta) &\leq R_\omega(\delta_{JS},\theta) + b^2\sigma^2 2^{\frac{r+2}{2}} \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{p-2r+2}{2})}{\Gamma(\frac{p-r}{2})} E\left(\frac{1}{\|y\|^r}\right) \\ &- b\sigma^2 2^{\frac{r+2}{2}}(1-\omega)(r-2)\frac{n+r}{n+2}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} E\left(\frac{1}{\|y\|^r}\right). \end{aligned} \quad (4.2)$$

The optimal value for b that minimizes the RHS of the inequality (4.2) is

$$\widehat{b} = \frac{(1-\omega)(r-2)}{2}\frac{n+r}{n+2}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n+2r}{2})}\frac{\Gamma(\frac{p-r}{2})}{\Gamma(\frac{p-2r+2}{2})}.$$

Thus

$$\begin{aligned} R_\omega(\delta_{\widehat{b},r},\theta) &\leq R_\omega(\delta_{JS},\theta) - 2^{\frac{r-2}{2}}\sigma^2(1-\omega)^2(r-2)^2\left(\frac{n+r}{n+2}\right)^2 \\ &\times \frac{\Gamma^2(\frac{n+r}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{n+2r}{2})}\frac{\Gamma(\frac{p-r}{2})}{\Gamma(\frac{p-2r+2}{2})} E\left(\frac{1}{\|y\|^r}\right) \leq R_\omega(\delta_{JS},\theta). \end{aligned}$$

□

5 Simulation results

5.1 On simulated data

We recall the form of the James-Stein estimator δ_{JS} given in (3.4)

$$\delta_{JS} = \left(1 - d\frac{S^2}{\|X\|^2}\right)X = \left(1 - \frac{(1-\omega)(p-2)}{n+2}\frac{S^2}{\|X\|^2}\right)X,$$

its risk function associated with the balanced squared error loss function L_ω is given by the formula (3.5). It is well known that the Positive-part of James-Stein estimator is defined by

$$\delta_{JS}^+ = \left(1 - d\frac{S^2}{\|X\|^2}\right)^+ X = \left(1 - d\frac{S^2}{\|X\|^2}\right) X \mathrm{I}_{d\frac{S^2}{\|X\|^2}\leq 1},$$

where $\left(1 - d\frac{S^2}{\|X\|^2}\right)^+ = max\left(0, 1 - d\frac{S^2}{\|X\|^2}\right)$ and $d = \frac{(1-\omega)(p-2)}{n+2}$, its risk function associated with L_ω is

$$\begin{aligned} R_\omega(\delta_{JS}^+,\theta) &= R_\omega(\delta_{JS},\theta) \\ &+ E\left[\left(\|X\|^2 - d^2\frac{S^4}{\|X\|^2} + 2(1-\omega)\sigma^2(p-2)d\frac{S^2}{\|X\|^2} - p\sigma^2\right)\mathrm{I}_{d\frac{S^2}{\|X\|^2}\geq 1}\right], \end{aligned}$$

where $\mathrm{I}_{d\frac{S^2}{\|X\|^2}\geq 1}$ denotes the indicating function of the set $(d\frac{S^2}{\|X\|^2} \geq 1)$.

We also recall the estimator $\delta_{a,r}$ given in (3.2) where

$$a = \frac{(1-\omega)(p-r)}{2}\frac{\Gamma(\frac{n+r}{2})\Gamma(\frac{p-r}{2})}{\Gamma(\frac{n+2r}{2})\Gamma(\frac{p-2r+2}{2})},$$

its risk function associated with L_ω is given in Proposition 3.1 and the estimator $\delta_{b,r}$ given in (4.1) where

$$b = \frac{(1-\omega)(r-2)}{2}\frac{(n+r)}{(n+2)}\frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n+2r}{2})}\frac{\Gamma(\frac{p-r}{2})}{\Gamma(\frac{p-2r+2}{2})},$$

its risk function associated with L_ω is given in Proposition 4.1.

In this part, we firstly present the graphs of the risks ratios of the estimators δ_{JS}, δ_{JS}^+, $\delta_{a,r}$ and $\delta_{b,r}$, to the MLE X denoted respectively: $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$ and $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ as a function of $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$, for various values of n, p, r and ω. Secondly, we give tables that present the values of risks ratios $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$ and $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ where in this case we fix r and vary the values of n, p and ω.

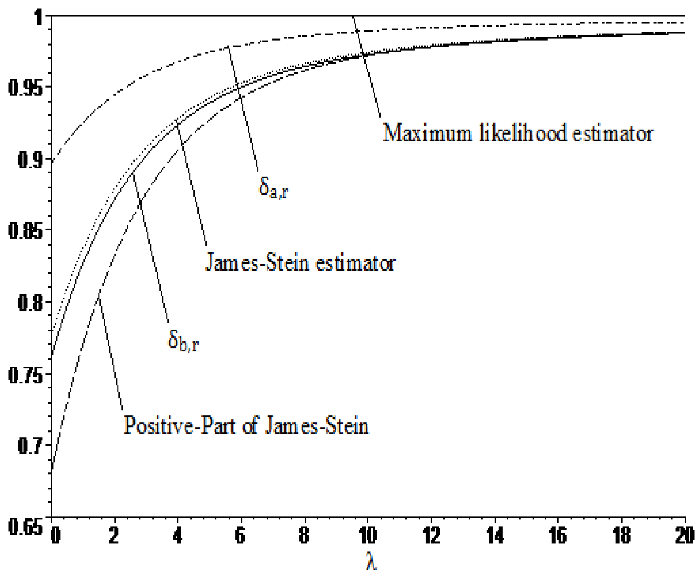

Figure 1: $n = 6, p = 3, r = 2.25$ and $\omega = 0.1$

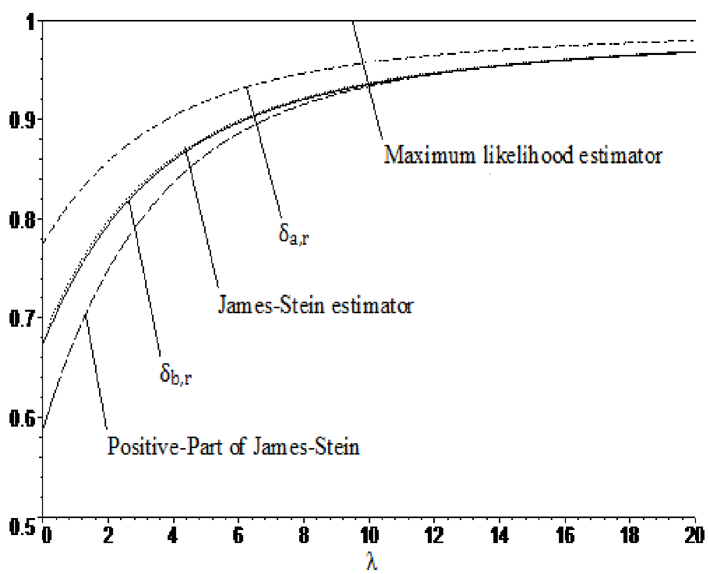

Figure 2: $n = 8, p = 4, r = 2.25$ and $\omega = 0.2$

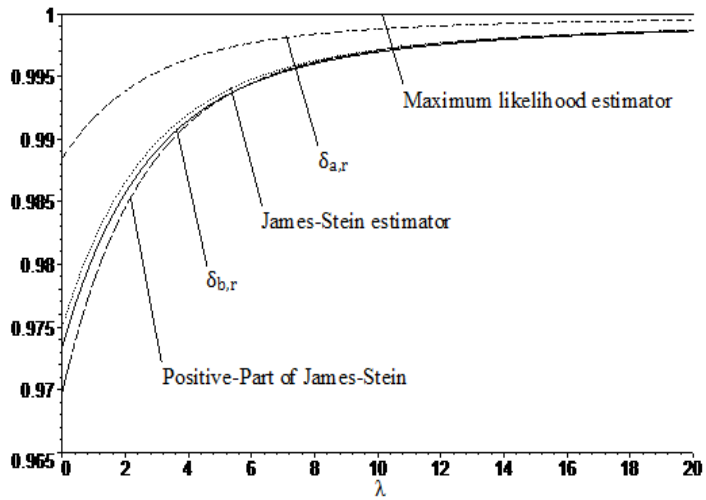

Figure 3: $n = 6, p = 3, r = 2.25$ and $\omega = 0.7$

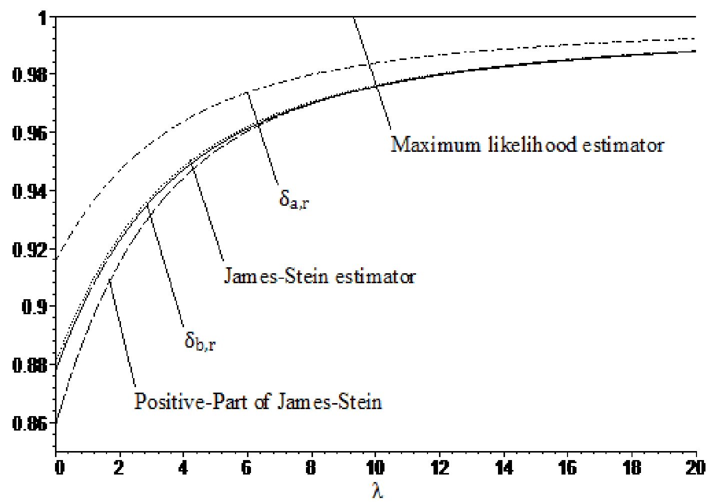

Figure 4: $n = 8, p = 4, r = 2.25$ and $\omega = 0.9$

The previous figures show that the risks ratios $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$ and $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ are less than 1, then the estimators δ_{JS}, δ^+_{JS}, $\delta_{a,r}$ and $\delta_{b,r}$ dominate the MLE X for diverse values of n, p, r and ω, therefore are minimax. We note that the estimator $\delta_{b,r}$ dominates the James-Stein estimator δ_{JS}. We also observe that the gain increases if ω is near to 0 and decreases if ω is near to 1. The following tables illustrate this note. In these tables we give the values of the risks ratios $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$ and $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ for the different values of λ, n, p and ω when $r = 2.25$. The first entry is $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$, the middle entry is $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, and the third entry is $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$.

Table 1: $n = 6, p = 3$ and $r = 2.25$

λ	$\omega = 0.0$	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.5$	$\omega = 0.7$	$\omega = 0.9$
	0.9002	0.9101	0.9201	0.9301	0.9501	0.9700	0.9900
	0.7833	0.8050	0.8267	0.8483	0.8917	0.9350	0.9783
0.4359	0.7694	0.7925	0.8156	0.8386	0.8847	0.9308	0.9769
	0.9226	0.9303	0.9381	0.9458	0.9613	0.9768	0.9923
	0.8318	0.8486	0.8654	0.8822	0.9159	0.9495	0.9832
1.2418	0.8216	0.8389	0.8568	0.8747	0.9105	0.9463	0.9821
	0.9712	0.9741	0.9770	0.9799	0.9856	0.9914	0.9971
	0.9360	0.9424	0.9488	0.9552	0.9680	0.9808	0.9936
5.0019	0.9320	0.9388	0.9456	0.9524	0.9660	0.9796	0.9932
	0.9883	0.9895	0.9907	0.9918	0.9942	0.9965	0.9988
	0.9725	0.9752	0.9780	0.9807	0.9862	0.9917	0.9972
10.4311	0.9709	0.9738	0.9767	0.9796	0.9855	0.9912	0.9971
	0.9928	0.9935	0.9943	0.9950	0.9964	0.9978	0.9993
	0.9824	0.9841	0.9859	0.9877	0.9912	0.9947	0.9982
15.4110	0.9814	0.9833	0.9851	0.9870	0.9907	0.9944	0.9981
	0.9947	0.9953	0.9958	0.9963	0.9974	0.9984	0.9998
	0.9867	0.9881	0.9894	0.9907	0.9934	0.9960	0.9987
20.0000	0.9860	0.9874	0.9888	0.9902	0.9930	0.9958	0.9986

In tables 1-4, we note that: if ω and $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ are small, the gain of the risks ratios $\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$, $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$ and $\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ is very important. Also, if the values of ω and λ increase, the gain decreases and approach to zero, a little improvement is then obtained. We also observe that, if the values of p increase, the gain increases and this for each fixed value of ω. Moreover, the influence of n on the risks ratios is the same as for p, but with a small gain. We also see that, if the values of p and n are large, the gain is large and consequently we obtain more improvement. We conclude that, the gain is important when the parameters p, n and λ are large and ω is near to 0. As seen above, the gain of the risks ratios is influenced by various values of ω, p, n and λ.

Table 2: $n = 6, p = 8$ and $r = 2.25$

λ	$\omega = 0.0$	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.5$	$\omega = 0.7$	$\omega = 0.9$
	0.5441	0.5897	0.6353	0.6809	0.7721	0.8632	0.9544
	0.4669	0.5202	0.5735	0.6268	0.7334	0.8401	0.9467
0.4359	0.4647	0.5182	0.5717	0.6253	0.7323	0.8394	0.9465
	0.5854	0.6268	0.6683	0.7098	0.7927	0.8756	0.9585
	0.5150	0.5635	0.6120	0.6605	0.7575	0.8545	0.9515
1.2418	0.5130	0.5617	0.6104	0.6591	0.7565	0.8539	0.9513
	0.7161	0.7445	0.7729	0.8013	0.8581	0.9148	0.9716
	0.6668	0.7001	0.7334	0.7667	0.8334	0.9000	0.9667
5.0019	0.6655	0.6989	0.7324	0.7658	0.8327	0.8996	0.9665
	0.8115	0.8303	0.8492	0.8680	0.9057	0.9434	0.9811
	0.7769	0.7992	0.8215	0.8438	0.8884	0.9331	0.9777
10.4311	0.7760	0.7984	0.8208	0.8432	0.8880	0.9328	0.9776
	0.8579	0.8721	0.8863	0.9006	0.9290	0.9574	0.9858
	0.8305	0.8474	0.8644	0.8813	0.9152	0.9491	0.9830
15.4110	0.8298	0.7984	0.8639	0.8809	0.9149	0.9490	0.9830
	0.8849	0.8964	0.9079	0.9194	0.9424	0.9655	0.9885
	0.8616	0.8755	0.8893	0.9031	0.9308	0.9585	0.9862
20.0000	0.8611	0.8750	0.8889	0.9028	0.9306	0.9583	0.9861

5.2 Real data application

Here we apply the theoretical results obtained in the previous section to real data. More precisely, we examine the performance of the shrinkage estimators δ_{JS}, $\delta_{a,r}$ and $\delta_{b,r}$ compared to the natural estimator. For this purpose application, we consider the air pollution dataset of USA cities in 1981, from Everitt and Hothorn [7]. We have the following list of variables: SO2 content of air in micrograms per cubic meter (SO2), average annual temperature in degrees Fahrenheit (temp), number of manufacturing enterprises employing 20 or more workers (manu), population size (1970 census) in thousands (popul), average annual wind speed in miles per hour (wind), average annual precipitation in inches (precip), average number of days with precipitation per year (predays). Table 5 lists the values of the risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$, $R_\omega(\delta_{a,r},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{b,r},\theta)/R_\omega(X,\theta)$ for different value of ω when $p = 7$ and $r = 3$.

We note that, all the values in this table are less than 1 and we also observe that $\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)} < \frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} < \frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$ for each value of ω. Thus, the values on the table are compatible with the theoretical results obtained in the previous sections.

Table 3: $n = 20, p = 3$ and $r = 2.25$

λ	$\omega = 0.0$	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.5$	$\omega = 0.7$	$\omega = 0.9$
	0.8739	0.8865	0.8991	0.9117	0.9370	0.9622	0.9874
	0.7374	0.76365	0.7899	0.8162	0.8687	0.9212	0.9737
0.4359	0.7221	0.7499	0.7777	0.8055	0.8611	0.9166	0.9722
	0.9022	0.9120	0.9218	0.9316	0.9511	0.9707	0.9902
	0.7961	0.8165	0.8369	0.8573	0.8980	0.9388	0.9796
1.2418	0.7843	0.8058	0.8274	0.8490	0.8921	0.9353	0.9784
	0.9637	0.9673	0.9709	0.9746	0.9818	0.9891	0.9964
	0.9224	0.9301	0.9379	0.9457	0.9612	0.9767	0.9922
5.0019	0.9180	0.9262	0.9344	0.9426	0.9590	0.9754	0.9918
	0.9852	0.9867	0.9882	0.9897	0.9926	0.9956	0.9985
	0.9666	0.9700	0.9733	0.9766	0.9833	0.9900	0.9967
10.4311	0.9649	0.9684	0.9719	0.9754	0.9824	0.9895	0.9965
	0.9909	0.9918	0.9928	0.9937	0.9955	0.9973	0.9991
	0.9786	0.9808	0.9829	0.9851	0.9893	0.9936	0.9979
15.4110	0.9776	0.9798	0.9821	0.9843	0.9888	0.9933	0.9978
	0.9934	0.9940	0.9947	0.9954	0.9967	0.9980	0.9993
	0.9839	0.9855	0.9871	0.9887	0.9920	0.9952	0.9984
20.0000	0.9831	0.9848	0.9865	0.9882	0.9916	0.99497	0.9983

6 Appendix

Proof. (Proof of Lemma 3.1) i) First, we show that, for any real υ

$$\frac{\partial}{\partial\lambda}E(U^{\upsilon}) = \frac{\partial}{\partial\lambda}\int_{R_+} x^{\upsilon}\chi_p^2(\lambda; dx) = \upsilon 2^{\upsilon-1}\sum_{k=0}^{+\infty}\frac{\Gamma(\frac{p}{2}+\upsilon+k)}{\Gamma(\frac{p}{2}+1+k)}P\left(\frac{\lambda}{2}; dk\right),$$

where $P(\frac{\lambda}{2})$ is the Poisson distribution of parameter $\frac{\lambda}{2}$.

Using the formula (2.1) we have, for any real υ

$$E(U^{\upsilon}) = E[(\chi_p^2(\lambda))^{\upsilon}] = E[(\chi_{p+2K}^2)^{\upsilon}] = 2^{\upsilon}E\left[\frac{\Gamma(\frac{p}{2}+K+\upsilon)}{\Gamma(\frac{p}{2}+K)}\right], \tag{6.1}$$

where $K \sim P(\frac{\lambda}{2})$ is the Poisson distribution of parameter $\frac{\lambda}{2}$. Then

$$\begin{aligned}
\frac{\partial}{\partial\lambda}E(U^{\upsilon}) &= \frac{\partial}{\partial\lambda}\int_{R_+} x^{\upsilon}\chi_p^2(\lambda; dx) \\
&= 2^{\upsilon}\sum_{k=0}^{+\infty}\left[\frac{\Gamma(\frac{p}{2}+k+\upsilon)}{\Gamma(\frac{p}{2}+k)}\right]\frac{1}{k!}\frac{\partial}{\partial\lambda}\left[\left(\frac{\lambda}{2}\right)^k exp\left(-\frac{\lambda}{2}\right)\right] \\
&= 2^{\upsilon-1}\sum_{k=0}^{+\infty}\left[\frac{\Gamma(\frac{p}{2}+k+\upsilon)}{\Gamma(\frac{p}{2}+k)}\right]\frac{1}{k!}exp\left(-\frac{\lambda}{2}\right)\left[-\left(\frac{\lambda}{2}\right)^k + k\left(\frac{\lambda}{2}\right)^{k-1}\right] \\
&= 2^{\upsilon-1}exp\left(-\frac{\lambda}{2}\right)\left\{-\sum_{k=0}^{+\infty}\left[\frac{\Gamma(\frac{p}{2}+k+\upsilon)}{\Gamma(\frac{p}{2}+k)}\right]\frac{1}{k!}\left(\frac{\lambda}{2}\right)^k\right\} \\
&+ 2^{\upsilon-1}exp\left(-\frac{\lambda}{2}\right)\left\{\sum_{k=0}^{+\infty}\left[\frac{\Gamma(\frac{p}{2}+k+\upsilon+1)}{\Gamma(\frac{p}{2}+k+1)}\right]\frac{1}{k!}\left(\frac{\lambda}{2}\right)^k\right\}
\end{aligned}$$

Table 4: $n = 20, p = 8$ and $r = 2.25$

λ	$\omega = 0.0$	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.5$	$\omega = 0.7$	$\omega = 0.9$
	0.4243	0.4819	0.5395	0.5970	0.7122	0.8273	0.9424
	0.3538	0.4184	0.4830	0.5476	0.6769	0.8061	0.9354
0.4359	0.3521	0.4169	0.4817	0.5465	0.6760	0.8056	0.9352
	0.4764	0.5288	0.5811	0.6335	0.7382	0.8429	0.9476
	0.4121	0.4709	0.5297	0.5885	0.7061	0.8236	0.9412
1.2418	0.4106	0.4695	0.5285	0.5874	0.7053	0.8232	0.9411
	0.6415	0.6774	0.7132	0.7491	0.8208	0.8925	0.9642
	0.5961	0.6365	0.6769	0.7173	0.7981	0.8788	0.9596
5.0019	0.5951	0.6356	0.6761	0.7166	0.7975	0.8785	0.9595
	0.7619	0.7857	0.8095	0.8333	0.8810	0.9286	0.9762
	0.7295	0.7566	0.7836	0.8107	0.8648	0.9189	0.9730
10.4311	0.7289	0.7560	0.7831	0.8102	0.8644	0.9187	0.9729
	0.8206	0.8385	0.8565	0.8744	0.9103	0.9462	0.9821
	0.7945	0.8151	0.8356	0.8562	0.8973	0.9384	0.9795
15.4110	0.7940	0.8146	0.8352	0.8558	0.8970	0.9382	0.9794
	0.8546	0.8692	0.8837	0.8982	0.9273	0.9564	0.9855
	0.8323	0.8491	0.8658	0.8826	0.9161	0.9497	0.9832
20.0000	0.8319	0.8487	0.8655	0.8823	0.9159	0.9496	0.9832

Table 5: $p = 7$ and $r = 3$

Risk ratios	$\omega = 0.2$	$\omega = 0.5$	$\omega = 0.9$
$\frac{R_\omega(\delta_{a,r},\theta)}{R_\omega(X,\theta)}$	0.9999998780	0.9999999230	0.9999999840
$\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}$	0.9999833589	0.9999895994	0.9999979199
$\frac{R_\omega(\delta_{b,r},\theta)}{R_\omega(X,\theta)}$	0.9999833502	0.9999895940	0.9999979180

$$
\begin{aligned}
&= 2^{\upsilon-1} exp\left(-\frac{\lambda}{2}\right)\left\{\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{\lambda}{2}\right)^k\left[\frac{\Gamma(\frac{p}{2}+k+\upsilon)}{\Gamma(\frac{p}{2}+k+1)}\right]\left[-\left(\frac{p}{2}+k\right)+\left(\frac{p}{2}+\upsilon+k\right)\right]\right\} \\
&= \upsilon 2^{\upsilon-1}\sum_{k=0}^{+\infty}\frac{\Gamma(\frac{p}{2}+\upsilon+k)}{\Gamma(\frac{p}{2}+1+k)}P\left(\frac{\lambda}{2};dk\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
K_{p,r,s}(\lambda) &= \left(\frac{\partial}{\partial\lambda}\int_{R_+} x^r\chi_p^2(\lambda;dx)\right)\left(\int_{R_+} x^s\chi_p^2(\lambda;dx)\right) \\
&- \left(\frac{\partial}{\partial\lambda}\int_{R_+} x^s\chi_p^2(\lambda;dx)\right)\left(\int_{R_+} x^r\chi_p^2(\lambda;dx)\right).
\end{aligned}
$$

For the function $H_{p,r,s}$ to be strictly increasing, it suffices that the function $K_{p,r,s}$ takes positive values. From equality (6.1), we obtain

$$
\begin{aligned}
K_{p,r,s}(\lambda) &= 2^{r+s-1}r\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma(\frac{p}{2}+r+i)}{\Gamma(\frac{p}{2}+i+1)}\frac{\Gamma(\frac{p}{2}+s+j)}{\Gamma(\frac{p}{2}+j)}P\left(\frac{\lambda}{2};di\right)P\left(\frac{\lambda}{2};dj\right) \\
&- 2^{r+s-1}s\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{\Gamma(\frac{p}{2}+r+j)}{\Gamma(\frac{p}{2}+j)}\frac{\Gamma(\frac{p}{2}+s+i)}{\Gamma(\frac{p}{2}+i+1)}P\left(\frac{\lambda}{2};dj\right)P\left(\frac{\lambda}{2};di\right).
\end{aligned}
$$

As, $r > s$ then

$$K_{p,r,s}(\lambda) \geq r2^{r+s-1}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} l_{p,r,s}(i,j) P\left(\frac{\lambda}{2};di\right) P\left(\frac{\lambda}{2};dj\right),$$

where

$$l_{p,r,s}(i,j) = \frac{\Gamma(\frac{p}{2}+r+i)\Gamma(\frac{p}{2}+s+j) - \Gamma(\frac{p}{2}+r+j)\Gamma(\frac{p}{2}+s+i)}{\Gamma(\frac{p}{2}+i+1)\Gamma(\frac{p}{2}+j)}.$$

We note that, for any i, $l_{p,r,s}(i,j) = 0$; then we have

$$K_{p,r,s}(i,j) \geq r2^{r+s-1}\sum_{i=0}^{+\infty}\sum_{j>i}^{+\infty}(l_{p,r,s}(i,j) + l_{p,r,s}(j,i)) P\left(\frac{\lambda}{2};di\right) P\left(\frac{\lambda}{2};dj\right).$$

But if $i < j$, we get

$$\begin{aligned} l_{p,r,s}(i,j) + l_{p,r,s}(j,i) &= \left(\Gamma\left(\frac{p}{2}+r+i\right)\Gamma\left(\frac{p}{2}+s+j\right) - \Gamma\left(\frac{p}{2}+r+j\right)\Gamma\left(\frac{p}{2}+s+i\right)\right) \\ &\times \left[\frac{1}{\Gamma(\frac{p}{2}+i+1)\Gamma(\frac{p}{2}+j)} - \frac{1}{\Gamma(\frac{p}{2}+j+1)\Gamma(\frac{p}{2}+i)}\right] \\ &= \frac{\Gamma(\frac{p}{2}+r+i)\Gamma(\frac{p}{2}+s+i)}{\Gamma(\frac{p}{2}+i)\Gamma(\frac{p}{2}+j)}\left[\frac{1}{\frac{p}{2}+i} - \frac{1}{\frac{p}{2}+j}\right] \\ &\times \left[\prod_{t=0}^{j-i-1}\left(\frac{p}{2}+s+i+t\right) - \prod_{t=0}^{j+i-1}\left(\frac{p}{2}+r+i+t\right)\right] \\ &\leq 0, \end{aligned}$$

because for any t, $\frac{p}{2}+s+i+t < \frac{p}{2}+r+i+t$. As in hypothesis $r < 0$, we have $K_{p,r,s}(\lambda) > 0$. Thus, we obtain the desired result.

ii) Using i) it is clear that the function $H^1_{p,r}(\lambda) = \frac{E(\|X\|^{-r})}{E(\|X\|^{-2r+2})}$ is non-decreasing on λ, then the function $\frac{1}{H^1_{p,r}(\lambda)}$ is non-increasing on λ, thus

$$\begin{aligned} \sup_{\|\theta\|}\left(\frac{E(\|X\|^{-2r+2})}{E(\|X\|^{-r})}\right) &= \sup_{\|\theta\|}\left(\frac{1}{H^1_{p,r}(\lambda)}\right) \\ &= \frac{1}{H^1_{p,r}(0)} \\ &= 2^{\frac{-r+2}{2}}\frac{\Gamma(\frac{p}{2}-r+1)}{\Gamma(\frac{p-r}{2})}. \end{aligned}$$

□

Conclusion

In this work, we studied the estimating of the the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown. The criterion adopted for comparing two estimators is the risk associated with the balanced loss function. First, we established the minimaxity of the estimators defined by $\delta_{a,r} = \left(1 - a((S^2)^{r/2}/\|X\|^r)\right) X$, where $2 \leq r < (p+2)/2$ and the real constant a may depend on n and p. Secondly, we showed that the estimator $\delta_{b,r} = \delta_{JS} + b\left((S^2)^{r/2}/\|X\|^r\right) X$ with $2 \leq r < (p+2)/2$ and the real constant b may depend on n and p, dominates the James-Stein estimator δ_{JS}, thus it is also minimax. In the future, we will study the behaviour of risks ratios of our considered estimators to the MLE when the sample size n and the dimension of parameters space p tend to infinity. An extension of this work is to obtain the similar results in the case where the model has a symmetrical spherical distribution.

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Abdelkader Benkhaled
Department of Biology
Mascara University
Laboratory of Stochastic Models, Statistics and Applications, University Tahar Moulay of Saida
Bp 305, Route de Mamounia 29000, Mascara, Algerie
E-mail: benkhaled08@yahoo.fr

Abdenour Hamdaoui
Department of Mathematics
University of Sciences and Technology, Mohamed Boudiaf, Oran
Laboratory of Statistics and Random Modelisations of University Abou Bekr Belkaid (LSMA), Tlemcen
El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria
E-mails: abdenour.hamdaoui@yahoo.fr, abdenour.hamdaoui@univ-usto.dz

Mekki Terbeche
Department of Mathematics
University of Sciences and Technology, Mohamed Boudiaf, Oran
Laboratory of Analysis and Application of Radiation (LAAR), USTO-MB
El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria
E-mail: mekki.terbeche@gmail.com

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