ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

VAGIF SABIR oglu GULIYEV

(to the 65th birthday)

On February 22, 2022 was the 65th birthday of Vagif Sabir oglu Guliyev, editor-in-chief of the Transactions of the Azerbaijan National Academy of Science, Issue Mathematics, Series of physical-technical and mathematics science (Scopus, Q3), deputy editor-in-chief of the Applied and Computational Mathematics (Web of Science, Q1), deputy director of the Institute of Applied Mathematics (IAM) of the Baku State University (BSU), head of the Department of Mathematical Analysis at the Institute of Mathematics and Mechanics (IMM) of the Azerbaijan National Academy of Sciences (ANAS), member of the Editorial Board of the Eurasian Mathematical Journal.

V.S. Guliyev was born in the city of Salyan in Azerbaijan. In 1978 Vagif Guliyev graduated from the Faculty of Mechanics and Mathematics of the Azerbaijan State University (now the Baku State University) with an honors degree and then completed his postgraduate studies at this university. His scientific supervisors were distinguished mathematicians A.A. Babayev and S.K. Abdullayev. In 1983 he defended his PhD thesis at the BSU. From 1983 he continued his scientific activities at the V.A. Steklov Mathematical Institute of the Academy of Sciences of the USSR. In 1987-1991 he was in internship at this institute and in 1994 defended there his DSc thesis.

From 1983 to 1995 he worked as assistant, a senior lecturer, docent and from 1995 to 2018 as a professor of Mathematical Analysis Chair of the BSU. In 1995-2008 he worked on part-time basis at the Institute of the IMM. From 2008 to 2014 he was a chief researcher of the Department of Mathematical Analysis of the IMM, from 2014 to the present day he is the head of this department.

In 2014 V.S. Guliyev was elected a corresponding member of the ANAS.

From 2015 to 2019, he worked as deputy director on science at the IMM. From 2019 to the present day, he has been working as a chief researcher at the IAM. Since May 2021, he has been working as a deputy director on science of the IAM.

Professor Vagif Guliyev has been a member of the Presidium of the Higher Attestation Commission under the President of the Republic of Azerbaijan since 2014 to the present day.

V.S. Guliyev is a world recognized specialist in real and harmonic analysis, function spaces and partial differential equations. He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. He was one of the first to study local Morrey-type spaces, generalized weighted Morrey-type spaces and anisotropic Banach-valued Sobolev spaces, for which appropriate embedding theorems were established.

Some of his results and methods are named after him: the Adams-Guliyev and Spanne-Guliyev conditions for the boundedness of operators in Morrey-type spaces, Guliyev's method of local estimates of integral operators of harmonic analysis, the Burenkov-Guliyevs conditions for the boundedness of operators in general local Morrey-type spaces.

On the whole, the results obtained by V.S. Guliyev have laid a groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations.

Vagif Sabir oglu Guliyev is an author of more than 250 scientific publications including 2 monographs. Among his pupils there are more than 20 candidates of sciences and 5 doctors of sciences. The results obtained by V.S. Guliyev, his pupils, collaborators and followers gained worldwide recognition.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vagif Sabir oglu Guliyev on the occasion of his 65th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 1 (2022), 86 – 100

MULTIPERIODIC SOLUTIONS OF QUASILINEAR SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH D_c-OPERATOR AND ϵ-PERIOD OF HEREDITARITY

Zh.A. Sartabanov, G.M. Aitenova, G.A. Abdikalikova

Communicated by R. Oinarov

Key words: integro-differential equation, hereditarity period, quasilinear system, multiperiodic solution, differentiation operator, vector field.

AMS Mathematics Subject Classification: 47G20, 35B10, 65M80.

Abstract. We investigate a quasilinear system of partial integro-differential equations with the operator of differentiation in the direction of a vector field, which describes the process of hereditary propagation with an ϵ-period of heredity. Under some conditions on the input data, conditions for the solvability of the initial problem for a quasilinear system of integro-differential equations are obtained. On this basis, sufficient conditions for the existence of multiperiodic solutions of integro-differential systems are found under the exponential dichotomy additional assumption on the corresponding homogeneous integro-differential system. The unique solvability of an operator equation in the space of smooth multiperiodic functions is proved, to which the main question under consideration reduces. Thus, sufficient conditions are established for the existence of a unique multiperiodical in all time variables solution of a quasilinear system of integro-differential equations with the differentiation operator in the directions of a vector field and a finite period of hereditarity.

DOI: https://doi.org/10.32523/2077-9879-2022-13-1-86-100

1 Introduction

In this paper, we investigate the problem of the existence of (θ, ω)-periodic solutions $u(\tau, t)$, where $(\tau, t) = (\tau, t_1, ..., t_m) \in \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R} \times \mathbb{R}^m$, of the system

$$\begin{gathered} D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,s,t-c\tau+cs)u(s,t-c\tau+cs)ds + \\ +f\left(\tau,t,u(\tau,t),\int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,s,t-c\tau+cs)u(s,t-c\tau+cs)ds\right) \end{gathered} \tag{1.1}$$

with the differentiation operator D_c of the form

$$D_c = \partial/\partial\tau + \langle c, \partial/\partial t\rangle\,, \tag{1.2}$$

that turns into the operator of the total derivative $d/d\tau$ along the characteristics $t = c\tau - cs + \sigma$ with the initial data $(s, \sigma) \in \mathbb{R} \times \mathbb{R}^m$, where $c = (c_1, \ldots, c_m)$ is a constant vector with non-zero coordinates c_j, $j = \overline{1, m}$; $\partial/\partial t = (\partial/\partial t_1, \ldots, \partial/\partial t_m)$; $\langle c, \partial/\partial t\rangle$ is the scalar product of vectors; $A(\tau, t)$

and $K(\tau, t, s, \sigma)$ are given $n \times n$-matrices; $f(\tau, t, u, v)$ is an n-vector-function, $u, v \in \overline{\mathbb{R}}^n_\Delta$, $\overline{\mathbb{R}}^n_\Delta$ is the closure of $\mathbb{R}^n_\Delta = \{w \in \mathbb{R}^n : |w| < \Delta = const\}$, $(\theta, \omega) = (\theta, \omega_1, \ldots, \omega_m)$ is a vector-period with rationally incommensurable coordinates, ϵ is a positive constant.

Partial integro-differential equation (1.1) describes many problems of hydrodynamics, acoustics, transport theory and other branches of continuum mechanics. The research of the theory of integro-differential equations was carried out by many authors. The foundations of the hereditary theory of elasticity are laid in the works of Boltzmann and Volterra.

As we know, V. Volterra used integro-differential equations in problems of hereditary elasticity [24], developed the theory of hereditary elasticity in the case of hereditary vibrations [25], investigated the phenomena of electric and magnetic hysteresis [26], substantiated the existence of periodic fluctuations in biological associations, and created a general theory of functionals [27].

Numerous researches of the authors are devoted to integro-differential equations, which form the basis of the theory of oscillatory processes in natural science and technology, we note books [14], [18]. It is known that if the oscillation phenomenon is hereditary in nature, then the equation the motion of a string at a certain moment $m(\tau)$ is specified by a change in the torsion angle of a string, and the eriditic biological phenomenon of "predator-prey" related to the law of oscillation [14], [15]. It should be noted that the mathematical model of hereditary phenomena described by the system of equations

$$\frac{dx}{d\tau} = P(\tau)x(\tau) + \int\limits_{\tau-\epsilon}^{\tau} Q(\tau, s)x(s)ds + \psi\left(\tau, x(\tau), \int\limits_{\tau-\epsilon}^{\tau} Q(\tau, s)x(s)ds\right) \tag{1.3}$$

is the first approximation of the desired n-vector-function $x(\tau)$ with given $n \times n$-matrices $P(\tau)$ and $Q(\tau, s)$ and the n-vector-function $\psi(\tau, x, y)$, where ϵ is the hereditary period of the phenomenon. Since the process is oscillatory, as a rule, the matrix $P(\tau)$ and the vector-function $\psi(\tau, x, y)$, in the general case are almost periodic in τ. The kernel $Q(\tau, s)$ has the property of the diagonal periodicity in $(\tau, s) \in \mathbb{R} \times \mathbb{R}$.

Particularly, if the indicated input data of system (1.3) are quasiperiodic in $\tau \in \mathbb{R}$ with the frequency basis $\nu_0 = \theta^{-1}, \nu_1 = \omega_1^{-1}, \ldots, \nu_m = \omega_m^{-1}$, then in the theory of fluctuations, the question of the existence of quasiperiodic solutions $x(\tau)$ of system (1.3) with a modified frequency basis is important, namely with $\widetilde{\nu}_0 = \theta^{-1}, \widetilde{\nu}_1 = c_1\omega_1^{-1}, \ldots, \widetilde{\nu}_m = c_m\omega_m^{-1}$. We assume that $0 < \epsilon < \theta = \omega_0 < \omega_1 < \ldots < \omega_m$.

The well-known theorem of G. Bohr plays an important role in solving this problem. It decribes the deep connection between quasiperiodic functions and periodic functions of many variables (multiperiodic functions). According to this theorem, the matrix and vector functions are defined $A = A(\tau, t)$, $K = K(\tau, t, s, \sigma)$, $\sigma = t - c\tau + cs$, $f = f(\tau, t, u)$, $u = u(\tau, t)$ with the properties: $A|_{t=c\tau} = P(\tau)$, $K|_{t=c\tau} = Q(\tau, s)$, $f|_{t=c\tau} = \psi\left(\tau, x(\tau), \int\limits_{\tau-\epsilon}^{\tau} Q(\tau, s)x(s)ds\right)$, $u|_{t=c\tau} = x(\tau)$ and the operator $d/d\tau$ is replaced by the differentiation operator D_c of form (1.2).

Thus, the problem of quasiperiodic fluctuations in system (1.3) becomes equivalent to the problem on the existence of (θ, ω)-periodic in (τ, t) solutions $u(\tau, t)$ of the system partial integro-differential equations of form (1.1) with differentiation operator (1.2).

In [2], [6], [8] the questions of qualitative theory of integro-differential equations were investigated and in [3] the solution of integro-differential equations via the kernel resolvent is given. The existence of periodic solutions of nonlinear integro-differential systems is considered in [5], [7]. For the systems with aftereffect, the existence of families of forced motions was established, which, under unlimited increase in time, exponentially tend to periodic modes [22].

Integro-differential equations describe rheological processes [9], [17], hereditary elasticity of the model, creep of the metal at high temperatures [17]. Integro-differential equations can be applied in descriptions of the processes with aftereffects [21], are used in the problems of the theory of heredity [15], and arise in the problems of the interaction of waves of electromagnetic fields.

At present, a development of the theory of nonlinear integro-differential equations with partial derivatives causes a certain interest in studying multiperiodic and almost periodic solutions of such equations. The study of multiperiodic and almost periodic oscillations is of theoretical and practical importance in science and technology. The variety of problems in mechanics, physics, and technology that describe periodic and almost periodic processes leads to the study of nonlinear integro-differential equations containing a small parameter. To the researh of such issues works of many authors are devoted. Monograph [11] is devoted to the study of almost periodic solutions of equation systems with quasiperiodic right-hand sides. In monograph [23], multiperiodic and almost periodic solutions of systems of partial differential evolution equations containing various small parameters are investigated both in time and in spatial variables. The existence and construction of multiperiodic and pseudoperiodic solutions of the system of integro-differential equations were studied in [20], [21]. In [4], the existence of multiperiodic in spatial variables solutions of a countable system of quasilinear equations was established. Some results in this direction were obtained in [1], [12], [13], [16].

2 Linear systems of integro-differential equations

We start with recalling necessary information on the zeros of the differential operator D_c.

By a zero of the operator D_c we mean a smooth function $u = u(\tau, t) \in C^{(1,e)}_{\tau,t}(\mathbb{R} \times \mathbb{R}^m)$ satisfying the equation

$$D_c u = 0. \tag{2.1}$$

It is easy to verify that the base zero of the operator D_c is the vector-function

$$h(s, \tau, t) = t - c\tau + cs \tag{2.2}$$

with the parameter $s \in \mathbb{R}$ that has the following properties

$$\begin{gathered} h(s, s, t) = t, \\ h(s, \sigma, h(\sigma, \tau, t)) = h(s, \tau, t), \\ h(s + \theta, \tau + \theta, t + q\omega) = h(s, \tau, t) + q\omega, q \in Z^m, \end{gathered} \tag{2.3}$$

where Z^m is the set of integer m-vectors.

A zero of the operator D_c with the initial data $u|_{\tau=\tau^0} = u^0(t) \in C^{(e)}_t(\mathbb{R}^m)$ is represented by the relation

$$u(\tau^0, \tau, t) = u^0(h(\tau^0, \tau, t)). \tag{2.4}$$

The condition

$$p_0 c\theta - p\omega = 0, \tag{2.5}$$

means commensurability of the vectors $c\theta$ and ω, where $(p_0, p) \in Z \times Z^m$, $p\omega = (p_1\omega_1, \ldots, p_m\omega_m)$.

If condition (2.5) is satisfied and the initial function $u^0(t)$ is ω-periodic, then

$$u^0(t + q\omega) = u^0(t) \in C^{(e)}_t(\mathbb{R}^m), q \in Z^m.$$

Solution (2.4) of equation (2.1) is a $(p_0\theta, p\omega)$-periodic zero of D_c.

If the vectors c and (θ, ω) do not satisfy the condition of form (2.5), then the $(q_0\theta, q\omega)$-periodic zeros of the operator D_c are constants: $u = const$, where $(q_0, q) \in Z \times Z^m$.

For the linear homogeneous integro-differential system

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds \tag{2.6}$$

with a certain constant period of heredity $\epsilon > 0$ and under the conditions

$$A(\tau+\theta, t+q\omega) = A(\tau,t) \in C_{\tau,t}^{(1,2e)}(\mathbb{R}\times\mathbb{R}^m), q \in Z^m \tag{2.7}$$

$$\begin{gathered} K(\tau+\theta, t+q\omega, s, \sigma) = K(\tau,t,s+\theta,\sigma+q\omega) \\ = K(\tau,t,s,\sigma) \in C_{\tau,t,s,\sigma}^{(1,2e,1,2e)}(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m), q \in Z^m, \end{gathered} \tag{2.8}$$

by the method of successive approximations, we can determine the resolving operator $U(s,\tau,t)$ with the initial condition $U(s,s,t) = E$ and which has the properties

$$D_c U(s,\tau,t) = A(\tau,t)U(s,\tau,t) + \int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(s,\xi,h(\xi,\tau,t))d\xi, \tag{2.9}$$

$$U(s+\theta,\tau+\theta,t+q\omega) = U(s,\tau,t), q \in Z^m, \tag{2.10}$$

where E is the identity n-matrix.

If for system (2.6) we consider the initial problem with the condition

$$u|_{\tau=\tau^0} = u^0(t) \in C_t^{(e)}(\mathbb{R}^m), \tag{2.11}$$

then under conditions (2.7) and (2.8) we can establish the existence of a unique solution $u = u(\tau^0,\tau,t)$ of the form

$$u(\tau^0,\tau,t) = U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t)), \tag{2.12}$$

where $\tau^0 \in \mathbb{R}$.

For the (θ,ω)-periodicity of a solution $u = u(\tau,t)$ of system (2.6) with the initial condition $u(0,t) = u^0(t)$, based on relations (2.7)–(2.12), we can prove that the initial function $u^0(t)$ is an ω-periodic solution of the linear functional-difference system

$$U(0,\theta,t)u^0(t-c\theta) = u^0(t) \tag{2.13}$$

with the difference $c' = c\theta$ in t belongs to the class of smooth functions: $u^0(t) \in C_t^{(e)}(\mathbb{R}^m)$.

In the case of splitting the resolving operator $U(s,\tau,t)$ into the sum of two matrices $U_-(s,\tau,t)$ and $U_+(s,\tau,t)$, possessing properties similar to (2.9) and (2.10) and satisfying the estimates

$$|U_-(s,\tau,t)| \le ae^{-\alpha(\tau-s)}, \tau \ge s; |U_+(s,\tau,t)| \le ae^{\alpha(\tau-s)}, \tau \le s \tag{2.14}$$

with some constants $a \ge 1$ and $\alpha > 0$. System (2.6) is called a system possessing the property of exponential dichotomy.

By virtue of conditions (2.7), (2.8) and (2.13), we can prove that the system (2.6), which has a property of exponential dichotomy, has no (θ,ω)-periodic solutions except zero.

Under the same conditions (2.7), (2.8) and (2.14), it can be established that a linear inhomogeneous integro-differential system

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds + f(\tau,t) \tag{2.15}$$

with n-vector-function $f(\tau,t)$ satisfying the requirements

$$f(\tau+\theta,t+q\omega)=f(\tau,t)\in C^{(0,e)}_{\tau,t}(\mathbb{R}\times\mathbb{R}^m), q\in Z^m \tag{2.16}$$

has a unique solution

$$u(\tau^0,\tau,t)=U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t))+\int\limits_{\tau^0}^{\tau}U(s,\tau,t)f(s,h(s,\tau,t))ds \tag{2.17}$$

for any initial function $u^0(t)\in C^{(e)}_t(\mathbb{R}^m)$ and has a unique (θ,ω)-periodic solution

$$u^*(\tau,t)=\int\limits_{-\infty}^{+\infty}G(s,\tau,t)f(s,h(s,\tau,t))ds \tag{2.18}$$

in the case of exponential dichotomy of system (2.6).

Here the matrix function $G(s,\tau,t)$ has the structure

$$G(s,\tau,t)=\begin{cases}U_-(s,\tau,t), & \tau\geq s,\\ -U(s,\tau,t), & \tau<s\end{cases} \tag{2.19}$$

and has the following properties

1.
$$\begin{gathered}D_cG(s,\tau,t)=A(\tau,t)G(s,\tau,t)\\ +\int\limits_{\tau-\epsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))G(s,\xi,h(\xi,\tau,t))d\xi, s\neq\tau.\end{gathered} \tag{2.20}$$

2.
$$G(s-0,\tau,t)-G(s+0,\tau,t)=E. \tag{2.21}$$

3.
$$\begin{gathered}|G(s,\tau,t)|\leq ae^{-\alpha|\tau-s|}, \left|\frac{\partial}{\partial t_j}G(s,\tau,t)\right|\leq ae^{-\alpha|\tau-s|},\\ j=\overline{0,m}, a=const\geq 1,\alpha=const>0.\end{gathered} \tag{2.22}$$

4.
$$G(s+\theta,\tau+\theta,t+q\omega)=G(s,\tau,t), q\in Z^m. \tag{2.23}$$

The matrix $G(s,\tau,t)$ defined by (2.19) and satisfying conditions (2.20)–(2.23) is called the Green matrix of the problem on (θ,ω)-periodic solutions for systems of form (2.15) with constant term (2.16).

3 Solvability of the initial problem for a quasilinear system of integro-differential equations

Representing system (1.1) in the form

$$\begin{gathered}D_cu(\tau,t)=A(\tau,t)u(\tau,t)+(Bu)(\tau,t)+f(\tau,t,u(\tau,t),(Bu)(\tau,t)),\\ (Bu)(\tau,t)=\int\limits_{\tau-\epsilon}^{\tau}K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds\end{gathered} \tag{3.1}$$

we consider the initial problem with the condition

$$u|_{\tau=\tau^0} = u^0(t) \in S^\omega_\rho \tag{3.2}$$

in the space $S^\omega_{\Delta,\delta}$ of ω-periodic in t, continuously differentiable in $t \in \mathbb{R}^m$ and $\tau \in \mathbb{R}_\delta = \{\tau \in \mathbb{R} : |\tau - \tau^0| < \delta\}$ n-vector-functions $u(\tau,t)$:

$$u(\tau, t + q\omega) = u(\tau, t) \in C^{(1,e)}_{\tau,t}(\mathbb{R}_\delta \times \mathbb{R}^m),$$

for which $||u|| \leq \Delta$.

Here $||u|| = ||u||_0 + \sum\limits_{j=0}^{m} \left\|\frac{\partial u}{\partial t_j}\right\|_0$, $||u||_0 = sup|u(\tau,t)|$ for $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$, where $\overline{\mathbb{R}}_\delta$ is the closure of $\mathbb{R}_\delta$, $t_0 = \tau$, $\delta > 0$ and $\Delta > 0$ are constants, S^ω_ρ is the space of all ω-periodic functions $u^0(t)$ continuously differentiable in $t \in \mathbb{R}^m$, for which $||u^0|| \leq \rho$ with the constant ρ from the interval $0 < \rho < \Delta$. Obviously, $S^\omega_\rho \subset S^\omega_{\Delta,\delta}$.

Let D_c, A and B in system (3.1) have the same meanings, and the n-vector-function $f(\tau,t,u,v)$ have the following properties

$$f(\tau+\theta, t+q\omega, u, v) = f(\tau,t,u,v) \in C^{1,2e,2\widetilde{e},2\widetilde{e}}_{\tau,t,u,v}(\mathbb{R} \times \mathbb{R}^m \times \overline{\mathbb{R}}^n_\Delta \times \overline{\mathbb{R}}^n_\Delta), q \in Z^m, \tag{3.3}$$

where the vectors e and $\widetilde{e}$ have the unit components and they differ from each other in their sizes m and n, $\mathbb{R}^n_\Delta = \{u \in \mathbb{R}^n : |u| < \Delta\}$, $\overline{\mathbb{R}}^n_\Delta$ is a closure $\mathbb{R}^n_\Delta$.

It is easy to show that, in accordance with the structure of solution (2.17) of the initial problem for linear system (2.15), problem (3.1)–(3.2) is equivalent to the unique solvability of the integral equation

$$\begin{aligned} u(\tau,t) &= U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t)) \\ &+ \int\limits_{\tau^0}^{\tau} U(s,\tau,t) f(s, h(s,\tau,t), u(s,h(s,\tau,t)), (Bu)(s,h(s,\tau,t)))ds \end{aligned} \tag{3.4}$$

in the space $S^\omega_{\Delta,\delta_0}$ with some constant δ_0 from the interval $0 < \delta_0 \leq \delta$.

In order to investigate the smoothness of the solution $u = u(\tau,t)$ of system (3.4), it is necessary to determine the matrix equation for the Jacobi matrix $\left(\frac{\partial u}{\partial t}\right)(\tau,t)$ of the desired solution $u = u(\tau,t)$ and its initial condition $\frac{\partial u}{\partial t}|_{\tau=\tau_0}$.

Obviously, the Jacobi matrix $J(\tau,t) = \left(\frac{\partial u}{\partial t}\right)(\tau,t)$ of the vector function $u(\tau,t) = \{u_1(\tau,t),...,u_n(\tau,t)\}$ in $t = (t_1,...,t_m)$ can be represented as a row vector of column vectors of the form $\frac{\partial u(\tau,t)}{\partial t} = \left(\frac{\partial u}{\partial t_1},...,\frac{\partial u}{\partial t_m}\right)$. Here $\frac{\partial u}{\partial t_j} = \left[\frac{\partial u_1(\tau,t)}{\partial t_j},...,\frac{\partial u_n(\tau,t)}{\partial t_j}\right]$, and $\left|\frac{\partial u}{\partial t}\right| = \sum\limits_{j=1}^{m}\left|\frac{\partial u}{\partial t_j}\right|$.

When computing the Jacobi matrix of the product of the matrix $T = [T_{ij}]_{n\times n}$ and vector $u = (u_1,...,u_n)$, it is convenient to represent the matrix T in the form of a column vector $T = [(T_1),...,(T_n)]$ of row vectors $(T_i) = (T_{i1},...,T_{in})$, $i = \overline{1,n}$.

Then the Jacobi matrix J_T of the matrix T can be represented as the matrix of the vector elements $\partial(T_i)/\partial t_j$

$$J_T = \left[\frac{\partial(T_i)}{\partial t_j}\right]_{n\times m},$$

and $\partial(Tu)/\partial t$ is calculated according to the rule

$$\frac{\partial}{\partial t}(Tu) = T\frac{\partial u}{\partial t} + \left[\left\langle\frac{\partial T_i}{\partial t_j}, u\right\rangle\right] \equiv T\frac{\partial u}{\partial t} + \left(\dot{T}u\right) \tag{3.5}$$

by using the operation of the $\langle,\rangle$-scalar product, where $\left(\dot{T}u\right) = \left[\left\langle \frac{\partial T_i}{\partial t_j}, u\right\rangle\right]$. Therefore, by virtue of (3.5), we have the following:

$$\frac{\partial}{\partial t}(Bu)(\tau,t) = \int\limits_{\tau-\epsilon}^{\tau} K(\tau,t,s,h(s,\tau,t))\left(\frac{\partial u}{\partial t}(s,h(s,\tau,t))\right)ds$$
$$+\int\limits_{\tau-\epsilon}^{\tau}\left\{\left[\left\langle\frac{\partial K_i}{\partial t_j},u\right\rangle\right](\tau,t,s,h(s,\tau,t)) + \left[\left\langle\frac{\partial K_i}{\partial \sigma_j},u\right\rangle\right](\tau,t,s,h(s,\tau,t))\right\}ds \tag{3.6}$$
$$= \left(B\frac{\partial u}{\partial t}\right)(\tau,t) + \left(\dot{B}u\right)(\tau,t) + \left(B'u\right)(\tau,t),$$

where, when differentiating $\sigma = h(s,\tau,t)$ in t_j, we used the equalities $\partial h_\alpha/\partial t_\beta = 1$ at $\alpha=\beta$ and $\partial h_\alpha/\partial t_\beta = 0$ at $\alpha\neq\beta$,

$$\dot{B}u = \left[\left\langle\frac{\partial K_i}{\partial t_j},u\right\rangle\right], B'u = \left[\left\langle\frac{\partial K_i}{\partial \sigma_j},u\right\rangle\right].$$

By virtue of (3.1)–(3.2), the Jacobi matrix $\frac{\partial u}{\partial t}$ of the sought solution $u = u(\tau,t)$ satisfies the matrix equation

$$D_c\left(\frac{\partial u}{\partial t}\right)(\tau,t) = A(\tau,t)\left(\frac{\partial u}{\partial t}\right)(\tau,t) + \left(B\frac{\partial u}{\partial t}\right)(\tau,t)$$
$$+F\left(\tau,t,u(\tau,t),(Bu)(\tau,t),\frac{\partial u}{\partial t}(\tau,t)\right) \tag{3.7}$$

and the condition

$$\frac{\partial u}{\partial t}|_{\tau=\tau^0} = \left(\frac{\partial u^0}{\partial t}\right)(t). \tag{3.8}$$

Here the $n\times m$-matrix function $F\left(\tau,t,u,(Bu),\frac{\partial u}{\partial t}\right)$ constructed according to the formulas (3.5) and (3.6), applied to $A,(Bu)$ and $f(\tau,t,u,Bu)$; moreover, it has form

$$F\left(\tau,t,u,(Bu),\frac{\partial u}{\partial t}\right) = \frac{\partial A}{\partial t}u + \left(\dot{B}u\right) + (B'u) + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial t}$$
$$+\frac{\partial f}{\partial v}\left\{\left(B\frac{\partial u}{\partial t}\right) + \left(\dot{B}u\right) + (B'u)\right\}. \tag{3.9}$$

Thus, the matrix $\frac{\partial u}{\partial t}$ is a solution to problem (3.7)–(3.8) with vector-function (3.9).

Now, similarly, we define the initial problem for the partial derivative $\frac{\partial u}{\partial \tau}(\tau,t)$ in τ of the sought solution $u = u(\tau,t)$ of integral system (3.4).

Supposing that $C = diag(c_1,\ldots,c_m)$,

$$K_0(\tau,t) = K(\tau,t,\tau,t),$$

$$K_\epsilon(\tau,t) = K(\tau,t,\tau-\epsilon,h(\tau-\epsilon,\tau,t)),$$

$$(B_0u) = \int\limits_{\tau-\epsilon}^{\tau}\frac{\partial K}{\partial\tau}(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds,$$

$$(B_c'u) = \left[\left\langle\frac{\partial K_i}{\partial\sigma_j}C,u\right\rangle\right], j=\overline{1,m}$$

we have

$$\frac{\partial}{\partial\tau}(Bu)(\tau,t) = K_0(\tau,t)u(\tau,t) - K_\epsilon(\tau,t)u(\tau-\epsilon, h(\tau-\epsilon,\tau,t))$$
$$+ (B_0u)(\tau,t) - \left(B_c^{'}u\right)(\tau,t) - \left(B\frac{\partial u}{\partial t}C\right)(\tau,t).$$

Assuming continuous differentiability of the solution $u = u(\tau,t)$ of system (3.4), taking into account its equivalence with problem (3.1)–(3.2) and differentiating system (3.1) in τ, we obtain

$$D_c\frac{\partial u}{\partial\tau}(\tau,t) = A(\tau,t)\frac{\partial u}{\partial\tau}(\tau,t) + \left(B\frac{\partial u}{\partial\tau}\right)(\tau,t) + \varphi\left(\tau,t,u,(Bu),\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial t}\right), \tag{3.10}$$

where the vector-function $\varphi\left(\tau,t,u,(Bu),\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial t}\right)$ is defined by the relation

$$\begin{aligned}\varphi\left(\tau,t,u,(Bu),\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial t}\right) = \frac{\partial A(\tau,t)}{\partial\tau}u(\tau,t) - \left(B\frac{\partial u}{\partial\tau}\right)(\tau,t) + \frac{\partial}{\partial\tau}(Bu)(\tau,t)+\\ +\frac{\partial f(\tau,t,u,(Bu))}{\partial\tau} + \frac{\partial f(\tau,t,u,(Bu))}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial f(\tau,t,u,(Bu))}{\partial v}\frac{\partial}{\partial\tau}(Bu).\end{aligned} \tag{3.11}$$

The derivative $\frac{\partial u}{\partial\tau}$ of the sought solution $u = u(\tau,t)$ in τ determined by system (3.10), by virtue of condition (3.2), satisfies the initial condition

$$\frac{\partial u}{\partial\tau}|_{\tau=\tau^0} = 0. \tag{3.12}$$

Under conditions (2.7), (2.8) and (3.3), the matrix function $F = F\left(\tau,t,u,(Bu),\frac{\partial u}{\partial t}\right)$ and the vector-function $\varphi = \varphi\left(\tau,t,u,(Bu),\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial t}\right)$ have the following properties

$$\begin{aligned}\varphi(\tau+\theta,t+q\omega,u,v,w,W) = \varphi(\tau,t,u,v,w,W) \in\\ \in C^{(0,e,\widetilde{e},\widetilde{e},\widetilde{e},\hat{e})}_{\tau,t,u,v,w,W}(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n_\Delta\times\mathbb{R}^n_\Delta\times\mathbb{R}^{nm}), q\in Z^m,\end{aligned} \tag{3.13}$$

$$\begin{aligned}F(\tau+\theta,t+q\omega,u,v,W) = F(\tau,t,u,v,W) \in\\ \in C^{(0,e,\widetilde{e},\widetilde{e},\hat{e})}_{\tau,t,u,v,W}(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n_\Delta\times\mathbb{R}^n_\Delta\times\mathbb{R}^{nm}), q\in Z^m,\end{aligned} \tag{3.14}$$

where the smoothness of the functions with the respect to the vectors and matrices means the smoothness with respect to their elements, and the vectors of orders $e,\widetilde{e},\hat{e}$ have unit elements of the dimensions m,n,mn respectively.

Note that the functions φ and F are linear with the respect to the arguments w and W.

Hereinafter, similarly to the transition from problem (3.1)–(3.2) to system (3.4), from problems (3.10)–(3.12) and (3.7)–(3.8), we pass to the equivalent integral systems

$$\begin{aligned}w(\tau,t) = \int_{\tau^0}^{\tau} U(s,\tau,t)\varphi(s,h(s,\tau,t),u(s,h(s,\tau,t)),v(s,h(s,\tau,t)),\\ w(s,h(s,\tau,t),W(s,h(s,\tau,t))))ds,\end{aligned} \tag{3.15}$$

$$\begin{aligned}W(\tau,t) = U(\tau^0,\tau,t)W^0(h(\tau^0,\tau,t)) + \int_{\tau^0}^{\tau} U(s,\tau,t)\times\\ \times F(s,h(s,\tau,t),u(s,h(s,\tau,t)),v(s,h(s,\tau,t)),W(s,h(s,\tau,t)))ds,\end{aligned} \tag{3.16}$$

where $W^0(t) = \frac{\partial u^0(t)}{\partial t}$, $v(\tau,t) = (Bu)(\tau,t)$, $w(\tau,t) = \frac{\partial u(\tau,t)}{\partial \tau}$, $W(\tau,t) = \frac{\partial u(\tau,t)}{\partial t}$. Now we investigate the problem of the existence of a solution $\xi(\tau,t) = (u(\tau,t), w(\tau,t), W(\tau,t))$ continuous in $(\tau,t) \in \mathbb{R}_\delta \times \mathbb{R}^m$ and ω-periodic in $t \in \mathbb{R}^m$, integral systems of integral equations (3.4), (3.15), (3.16) satisfying the inequality $||\xi|| = ||u||_0 + ||w||_0 + ||W||_0 \leq \Delta$ for $(\tau,t) \in \mathbb{R}_\delta \times \mathbb{R}^m$. By virtue of conditions (3.3), (3.13), (3.14), the function $\Phi(\tau,t,\xi) = (f(\tau,t,u,v), \varphi(\tau,t,u,v,w,W), F(\tau,t,u,v,W))$ has the properties

$$\Phi(\tau,t+q\omega,\xi) = \Phi(\tau,t,\xi) \in C^{(0,e,\overline{e})}_{\tau,t,\xi}(\overline{\mathbb{R}}_\delta \times \mathbb{R}^m \times \overline{\mathbb{R}}^{\overline{n}}_\Delta), q \in Z^m, \tag{3.17}$$

where $e, \overline{e}$ are vectors with unit components of the dimensions m and $\overline{n} = n + m + nm$.

Condition (3.17) implies the Lipschitz condition

$$|\Phi(\tau,t,\xi) - \Phi(\tau,t,\eta)| \leq l|\xi - \eta| \tag{3.18}$$

with some constant $l > 0$ for $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$, $\xi, \eta \in \overline{\mathbb{R}}^{\overline{n}}_\Delta$.

Next, suppose that $|\Phi(\tau,t,0)| \leq r$, by virtue of condition (3.18), we have

$$|\Phi(\tau,t,\xi)| \leq r + l\Delta, (\tau,t,\xi) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m \times \overline{\mathbb{R}}^{\overline{n}}_\Delta, \tag{3.19}$$

where $r = const > 0$.

By notation $\xi^0(t) = (u^0(t), 0, W^0(t))$, system $\{(3.4), (3.15), (3.16)\}$ can be represented as a single integral equation

$$\xi(\tau,t) = \overline{U}(\tau^0,\tau,t)\xi^0(h(\tau^0,\tau,t)) + \int\limits_{\tau^0}^{\tau} \overline{U}(s,\tau,t)\Phi(s,h(s,\tau,t),\xi(s,h(s,\tau,t)))ds, \tag{3.20}$$

which we consider in the space $S^\omega_{\Delta,\delta}$ of $\overline{n}$-vector-functions $\xi = \xi(\tau,t)$ continuous in $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$ and ω-periodic in $t \in \mathbb{R}^m$ and bounded with respect to the norm $||\xi|| = sup|\xi(\tau,t)|$ for $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$, where $\overline{U}(\tau^0,\tau,t) = diag\,[U(\tau^0,\tau,t), U(\tau^0,\tau,t), U(\tau^0,\tau,t)]$.

Since $\overline{U} = \overline{U}(\tau^0,\tau,t)$ at $\tau = \tau^0$ turns into the identity matrix $\overline{E}$, $|\xi^0| \leq \rho$, then

$$|\overline{U}\xi^0| \leq |\xi^0 + [\overline{U} - \overline{E}]\xi^0| \leq (1+\alpha)\rho \tag{3.21}$$

where $\alpha \to 0$ at $\delta \to 0, |\tau - \tau^0| \leq \delta$.

Then, by virtue of estimates (3.19) and (3.21) we obtain

$$|\xi(\tau,t)| \leq (1+\alpha)[\rho + (r + l\Delta)\delta].$$

Since the parameters ρ and δ are controlled by us, we can assume that the conditions

$$(1+\alpha)l\delta < 1, (1+\alpha)[\rho + (r + l\Delta)\delta] \leq \Delta. \tag{3.22}$$

are satisfied.

Therefore, the operator Q is defined by the relation

$$\begin{gathered}(Q\xi)(\tau,t) = \overline{U}(\tau^0,\tau,t)\xi^0(h(\tau^0,\tau,t)) + \\ + \int\limits_{\tau^0}^{\tau} \overline{U}(s,\tau,t)\Phi(s,h(s,\tau,t),\xi(s,h(s,\tau,t)))ds,\end{gathered} \tag{3.23}$$

maps space $S^\omega_{\Delta,\delta}$ into itself and is a contraction. Then it has a unique fixed point $\xi^*(\tau,t) = (Q\xi^*)(\tau,t)$ in $S^\omega_{\Delta,\delta}$, which, by virtue of (3.23), is the only solution to equation (3.20).

Obviously, the first component $u^*(\tau,t)$ of the solution is the unique solution $\xi^*(\tau,t) = (u^*(\tau,t), w^*(\tau,t), W^*(\tau,t))$ to integral system (3.4), while the other components $w^*(\tau,t)$ and $W^*(\tau,t)$, being solutions of systems (3.15) and (3.16), are related to the first component by the relations

$$w^*(\tau,t) = \frac{\partial u^*(\tau,t)}{\partial \tau}, W^*(\tau,t) = \frac{\partial u^*(\tau,t)}{\partial t}$$

by virtue of systems (3.10) and (3.7) with initial conditions (3.12) and (3.8) for them.

Thus, the following theorem is proved.

Theorem 3.1. *Under conditions* (2.7), (2.8), (3.3) *and* (3.22), *initial problem (3.1)–(3.2) is uniquely solvable in the space* $S^{\omega}_{\Delta,\delta}$ *of n-vector-functions* $u(\tau,t)$ *that are* ω*-periodic in* t, *continuously differentiable in* $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$ *and are such that the norm* $||u|| = ||u||_0 + \left\|\frac{\partial u}{\partial \tau}\right\|_0 + \left\|\frac{\partial u}{\partial t}\right\|_0$ *is bounded by the constant* $\Delta > 0$, *where* $||u||_0 = sup|u(\tau,t)|$ *for* $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$.

4 Multiperiodic solution of a quasilinear system of integro-differential equations

We have investigated the problem on the existence of multiperiodic solutions of system (3.1) when condition of exponential dichotomy (2.14) of homogeneous system (2.6) is satisfied.

Then, constructing the Green function $G(s,\tau,t)$ according to formula (2.19) with properties (2.20)–(2.23), in accordance with the structure of a multiperiodic solution (2.18) of linear inhomogeneous system (2.15), we introduced the operator

$$(Tu)(\tau,t) = \int_{-\infty}^{+\infty} G(s,\tau,t) f(s, h(s,\tau,t), u(s,h(s,\tau,t)), (Bu)(s,h(s,\tau,t)))ds \tag{4.1}$$

defined on the space $S^{\theta,\omega}_{\Delta}$ of continuous (θ,ω)-periodic by $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$ n-vector-functions whose norm $||u||_0 = sup|u(\tau,t)|$ for $(\tau,t) \in \overline{\mathbb{R}}_\delta \times \mathbb{R}^m$ is bounded by the number $\Delta > 0$.

Under condition (3.3), the vector function $f = f(\tau,t,u,Bu)$ satisfies the Lipschitz condition with respect to u with some constant l. Therefore, for $u, v \in S^{\theta,\omega}_{\Delta}$ we have the inequality

$$|f(\tau,t,u,Bu) - f(\tau,t,v,Bv)| \leq l|u-v|. \tag{4.2}$$

Then, by condition (4.2), we have the estimate

$$|f(\tau,t,u,Bu)| \leq |f(\tau,t,0,0)| + l|u| \leq r + l\Delta, \tag{4.3}$$

where $r = ||f(\tau,t,0,0)||_0 > 0$.

Let the parameters α, a, l, r, Δ be, such that

$$a(r + l\Delta) < \alpha\Delta. \tag{4.4}$$

Theorem 4.1. *Under conditions* (2.7), (2.8), (2.14), (3.3) *and* (4.4), *the operator* T *defined by formula* (4.1) *has a unique fixed point* $u^*(\tau,t) = Tu^*(\tau,t)$ *in the space* $S^{\theta,\omega}_{\Delta}$.

In virtue of the conditions the theorem and relations (4.2) and (4.3), it is easy to show that operator (4.1) maps the space $S^{\theta,\omega}_{\Delta}$ into itself and is a contraction. Obviously, the space $S^{\theta,\omega}_{\Delta}$ is complete. Therefore, by the Banach theorem, the operator T in $S^{\theta,\omega}_{\Delta}$ has a unique fixed point.

Now, along with the operator T, we introduce the operators T_0 and T_* defined by

$$(T_0w)(\tau,t) = \int\limits_{-\infty}^{+\infty} G(s,\tau,t)\varphi(s,h(s,\tau,t),u(s,h(s,\tau,t)),(Bu)(s,h(s,\tau,t)), \\ w(s,h(s,\tau,t)),W(s,h(s,\tau,t)))ds, \tag{4.5}$$

$$(T_*w)(\tau,t) = \int\limits_{-\infty}^{+\infty} G(s,\tau,t)F(s,h(s,\tau,t),u(s,h(s,\tau,t)), \\ (Bu)(s,h(s,\tau,t)),W(s,h(s,\tau,t)))ds, \tag{4.6}$$

where $w(\tau,t)$ is the n-vector, $W(\tau,t) = (w_1(\tau,t),\dots,w_n(\tau,t))$ is the row vector with the vectors components $w_j(\tau,t) = [w_{1j}(\tau,t),\dots,w_{nj}(\tau,t)]\,, j = \overline{1,n}$. The functions $\varphi(\tau,t,u,v,w,W)$ and $F(\tau,t,u,v,W)$ are defined by the relations

$$\varphi(\tau,t,u,v,w,W) = \frac{\partial A}{\partial \tau}u + v - (Bw) + \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial u}w + \frac{\partial f}{\partial v}\frac{\partial}{\partial \tau}(Bu),$$

$$\frac{\partial}{\partial \tau}(Bu) = K_0u - K_\epsilon u_\epsilon + (B_0u) - (B_c^{'}u) - (BWC), \tag{4.7}$$

where $u_\epsilon = u(\tau,t,\tau-\epsilon,h(\tau-\epsilon,\tau,t))$, $C = diag[c_1,\dots,c_m]$, and

$$F(\tau,t,u,v,W) = \frac{\partial A}{\partial t}u + \left(\dot{B}u\right) + (B'u) + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}W + \\ + \frac{\partial f}{\partial v}\left\{(BW) + \left(\dot{B}u\right) + (B'u)\right\} \tag{4.8}$$

in accordance with expressions (3.11) and (3.9), $(\tau,t) \in \mathbb{R}_\delta \times \mathbb{R}^m$, $u \in S_\Delta^{\theta,\omega}$, $v \in S_\Delta^{\theta,\omega}$, $W \in S_\Delta^{\theta,\omega} \times \dots \times S_\Delta^{\theta,\omega} = S_\Delta^{m,\theta,\omega}$.

Obviously, by virtue of conditions (2.7), (2.8) and (3.3), these functions satisfy the Lipschitz condition with respect to $\xi = (u,w,W)$. Moreover, we take l as the Lipschitz constant and assume that the norms of the functions $\varphi(\tau,t,0,0,0,0)$ and $F(\tau,t,0,0,0)$ are bounded by a constant $r > 0$.

We note that the operator

$$Q^*\xi = (Tu,T_0w,T_*W), \tag{4.9}$$

is obtained by combining operator (4.1) and additional operators (4.5) and (4.6). It is defined for $(\tau,t) \in \mathbb{R}_\delta \times \mathbb{R}^m$, $\xi = (u,w,W) \in S_\Delta^{\theta,\omega} \times S_\Delta^{\theta,\omega} \times S_\Delta^{m,\theta,\omega} = S_\Delta^{m+2,\theta,\omega}$. Therefore, operator (4.9) satisfies the conditions

$$||Q^*\xi|| \le \frac{a}{\alpha}(r + l\Delta) < \Delta, ||Q^*\xi - Q^*\eta|| \le \frac{al}{\alpha}||\xi - \eta||, \tag{4.10}$$

for $\xi,\eta \in S_\Delta^{m+2,\theta,\omega}$, $||\xi|| = ||u||_0 + ||w||_0 + ||W||_0$, $||W||_0 = \sum\limits_{j=1}^{m}||w_j||_0$.

Obviously, the space $S_\Delta^{m+2,\theta,\omega}$ is complete. Under condition (4.4), the operator Q^* defined by formula (4.9), by virtue of (4.10), maps the space $S_\Delta^{m+2,\theta,\omega}$ into itself and is contractive. Therefore, there is a unique fixed point $\xi^* = Q^*\xi^* \in S_\Delta^{m+2,\theta,\omega}$ for which we have the componentwise system of identities

$$u^*(\tau,t) = (Tu^*)(\tau,t), w^*(\tau,t) = (T_0w^*)(\tau,t), W^*(\tau,t) = (T_*W^*)(\tau,t), \tag{4.11}$$

where $(u^*(\tau,t),w^*(\tau,t),W^*(\tau,t)) = \xi^*(\tau,t)$.

Then, in accordance with the general theory of differential equations in Banach spaces, relations (4.11) imply the continuous differentiability of the fixed point $u^*(\tau, t)$ of the operator T with respect to τ and t, and

$$\begin{aligned}\frac{\partial u^*(\tau,t)}{\partial\tau} &= w^*(\tau,t)\\ \frac{\partial u^*(\tau,t)}{\partial t} &= W^*(\tau,t)\end{aligned} \tag{4.12}$$

by (3.7)–(3.9), (3.10)–(3.11), (4.7) and (4.8).

Thus, the following theorem on the solvability of the operator equation

$$u(\tau,t) = (Tu)(\tau,t) \tag{4.13}$$

in the space of smooth multiperiodic functions $S_\Delta^{\theta,\omega}$ is proved.

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *the fixed point* $u^*(\tau, t)$ *of the operator* T *is continuously differentiable with respect to* $(\tau, t) \in \mathbb{R}_\delta \times \mathbb{R}^m$, *and relations* (4.12) *are valid.*

Now we can prove the following theorem on the existence of a unique multiperiodic solution to system of integro-differential equations (1.1).

Theorem 4.3. *Suppose that, under conditions* (2.7) *and* (2.8), *linear homogeneous system of integro-differential equations* (2.6) *possess exponential dichotomy property* (2.14). *Then quasilinear system of integro-differential equations* (1.1) *with nonlinearity possessing property* (3.3) *under condition* (4.4) *has a unique* (θ,ω)*-periodic solution whose norm is bounded by the number* $\Delta > 0$.

The problem of the existence of a unique (θ,ω)-periodic solution for the system (1.1) is equivalent to the problem of unique solvability of operator equation (4.13) with operator (4.1) in the space $S_\Delta^{\theta,\omega}$ of smooth (θ,ω)-periodic vector-functions.

Under the assumptions of Theorem 4.3, Theorem 4.1 implies the unique solvability of system (4.13) in the space $S_\Delta^{\theta,\omega}$ of functions continuous in (τ, t) and ω-periodic in $t \in \mathbb{R}^m$, and Theorem 4.2, together with the unique solvability, implies the differentiability of its solution in $(\tau, t) \in \mathbb{R}_\delta \times \mathbb{R}^m$. Therefore, system (1.1) has a unique (θ,ω)-periodic solution whose norm is bounded by the number Δ.

In conclusion, we consider system (1.1) along the characteristic $t = h(\tau, \tau^0, t^0)$ with a fixed initial point (τ^0, t^0). Then the operator D_c acting on the function $u(\tau, t)$ converts to the operator of the full total derivative of $d/d\tau$ of the function $u(\tau, h(\tau, \tau^0, t^0)) = \widetilde{u}(\tau)$. Furthemore, we suppose that

$$A(\tau, h(\tau,\tau^0,t^0)) = \widetilde{A}(\tau),$$

$$K(\tau, h(\tau,\tau^0,t^0), s, h(s,\tau^0,t^0)) = \widetilde{K}(\tau,s),$$

$$f\left(\tau, h(\tau,\tau^0,t^0), \widetilde{u}(\tau), \int\limits_{\tau-\epsilon}^{\tau} \widetilde{K}(\tau,s)\widetilde{u}(s)ds\right) = \widetilde{f}\left(\tau, \widetilde{u}(\tau), \int\limits_{\tau-\epsilon}^{\tau} \widetilde{K}(\tau,s)\widetilde{u}(s)ds\right).$$

Then from system (1.1), we have the following system of ordinary integro-differential equations of the form

$$\frac{d\widetilde{u}(\tau)}{d\tau} = \widetilde{A}(\tau)\widetilde{u}(\tau) + \int\limits_{\tau-\epsilon}^{\tau} \widetilde{K}(\tau,s)\widetilde{u}(s)ds + \widetilde{f}\left(\tau, \widetilde{u}(\tau), \int\limits_{\tau-\epsilon}^{\tau} \widetilde{K}(\tau,s)\widetilde{u}(s)ds\right). \tag{4.14}$$

Then for system (4.14) by Theorem 4.3 we obtain the following conclusion.

Corollary 4.1. *Let the assumptions of Theorem* 4.3 *be satisfied. Then system of ordinary integro-differential equations* (4.14)*, under the assumption of a rational incommensurability of the frequencies* $\widetilde{\nu}_0 = \theta^{-1}, \widetilde{\nu}_j = c_j\omega_j^{-1}, j = \overline{1,m}$*, has a unique quasiperiodic solution* $\widetilde{u}^*(\tau) = u^*(\tau, h(\tau, \tau^0, t^0))$ *with the same frequencies, whose norm is bounded by the number* $\Delta > 0$*.*

Corollary 4.1 can be verified on the basis of G. Bohr's well-known theorem on the connection of multiperiodic and quasiperiodic functions.

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Zhaishylyk Almaganbetovich Sartabanov, Gulsezim Muratovna Aitenova, Galiya Amirgalievna Abdikalikova
Department of Mathematics
K. Zhubanov Aktobe Regional State University
34 A. Moldagulova St,
030000 Aktobe, Kazakhstan
E-mails: sartabanov42@mail.ru, gulsezim-88@mail.ru, agalliya@mail.ru

Received: 22.10.2019