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VAGIF SABIR oglu GULIYEV

(to the 65th birthday)

On February 22, 2022 was the 65th birthday of Vagif Sabir oglu Guliyev, editor-in-chief of the Transactions of the Azerbaijan National Academy of Science, Issue Mathematics, Series of physical-technical and mathematics science (Scopus, Q3), deputy editor-in-chief of the Applied and Computational Mathematics (Web of Science, Q1), deputy director of the Institute of Applied Mathematics (IAM) of the Baku State University (BSU), head of the Department of Mathematical Analysis at the Institute of Mathematics and Mechanics (IMM) of the Azerbaijan National Academy of Sciences (ANAS), member of the Editorial Board of the Eurasian Mathematical Journal.

V.S. Guliyev was born in the city of Salyan in Azerbaijan. In 1978 Vagif Guliyev graduated from the Faculty of Mechanics and Mathematics of the Azerbaijan State University (now the Baku State University) with an honors degree and then completed his postgraduate studies at this university. His scientific supervisors were distinguished mathematicians A.A. Babayev and S.K. Abdullayev. In 1983 he defended his PhD thesis at the BSU. From 1983 he continued his scientific activities at the V.A. Steklov Mathematical Institute of the Academy of Sciences of the USSR. In 1987-1991 he was in internship at this institute and in 1994 defended there his DSc thesis.

From 1983 to 1995 he worked as assistant, a senior lecturer, docent and from 1995 to 2018 as a professor of Mathematical Analysis Chair of the BSU. In 1995-2008 he worked on part-time basis at the Institute of the IMM. From 2008 to 2014 he was a chief researcher of the Department of Mathematical Analysis of the IMM, from 2014 to the present day he is the head of this department.

In 2014 V.S. Guliyev was elected a corresponding member of the ANAS.

From 2015 to 2019, he worked as deputy director on science at the IMM. From 2019 to the present day, he has been working as a chief researcher at the IAM. Since May 2021, he has been working as a deputy director on science of the IAM.

Professor Vagif Guliyev has been a member of the Presidium of the Higher Attestation Commission under the President of the Republic of Azerbaijan since 2014 to the present day.

V.S. Guliyev is a world recognized specialist in real and harmonic analysis, function spaces and partial differential equations. He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. He was one of the first to study local Morrey-type spaces, generalized weighted Morrey-type spaces and anisotropic Banach-valued Sobolev spaces, for which appropriate embedding theorems were established.

Some of his results and methods are named after him: the Adams-Guliyev and Spanne-Guliyev conditions for the boundedness of operators in Morrey-type spaces, Guliyev's method of local estimates of integral operators of harmonic analysis, the Burenkov-Guliyevs conditions for the boundedness of operators in general local Morrey-type spaces.

On the whole, the results obtained by V.S. Guliyev have laid a groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations.

Vagif Sabir oglu Guliyev is an author of more than 250 scientific publications including 2 monographs. Among his pupils there are more than 20 candidates of sciences and 5 doctors of sciences. The results obtained by V.S. Guliyev, his pupils, collaborators and followers gained worldwide recognition.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vagif Sabir oglu Guliyev on the occasion of his 65th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ON AN ALGORITHM OF FINDING AN APPROXIMATE SOLUTION OF A PERIODIC PROBLEM FOR A THIRD-ORDER DIFFERENTIAL EQUATION

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Key words: partial differential equation, third order boundary value problem, algorithm, approximate solution.

AMS Mathematics Subject Classification: 34A30, 35Q92.

Abstract. In this paper, we study a periodic boundary value problem for a partial differential equation of the third order. An algorithm for finding a solution to this boundary value problem is proposed, and sufficient conditions for the convergence of the proposed algorithm are obtained.

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1 Introduction

On $\Omega = [0, \omega] \times [0, T]$ we consider the periodic boundary value problem

$$\frac{\partial^3 u}{\partial x^2 \partial t} = A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial u}{\partial x} + C(x,t)u + f(x,t), (x,t) \in \Omega, \tag{1.1}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{1.2}$$

$$u(0,t) = \varphi(t), \quad t \in [0,T], \tag{1.3}$$

$$\frac{\partial u(0,t)}{\partial x} = \psi(t), \quad t \in [0,T], \tag{1.4}$$

where $(n \times n)$ - matrix functions $A(x,t), B(x,t), C(x,t)$, n-vector functions $f(x,t)$ are continuous on Ω, n-vector functions $\varphi(t)$, $\psi(t)$ are continuously differentiable on $[0,T]$ satisfying the conditions $\varphi(0) = \varphi(T), \psi(0) = \psi(T)$.

In particular, for $A(x,t) \equiv 1, B(x,t) \equiv C(x,t) \equiv f(x,t) \equiv 0$ the general solution of equation (1.1) satisfying conditions (1.3), (1.4) has the form:

$$u(x,t) = \varphi(t) + \psi(t)x + e^t V(x),$$

where $V(x)$ is an arbitrary twice continuously differentiable function. Substituting it in conditions (1.2) and taking into account the conditions $\varphi(0) = \varphi(T), \psi(0) = \psi(T)$ we obtain that $V(x) = 0$. Then the solution of problem (1.1)-(1.4) for $A(x,t) \equiv 1, B(x,t) \equiv C(x,t) \equiv f(x,t) \equiv 0$ is $u(x,t) = \varphi(t) + \psi(t)x$.

Modeling of various processes of physics, mechanics, biology, and others sciences leads to the study of boundary value problems for partial differential equations of the third order [1], [3], [4], [6], [7], [12] and construction of approximate methods for finding their solutions. Application of different approaches, ideas and methods leads to results formulated in different terms. In this paper we investigate the existence of a solution to problem (1.1)-(1.4) and propose a method for constructing an approximate solution. With the help of additional functions [2], [8]-[10] the considered problem reduces to an equivalent problem consisting of a family of multipoint problems for an ordinary differential equation of the first order with a functional parameter and an integral relation. An algorithm for finding an approximate solution to the problem under study is proposed and its convergence is proved. Sufficient conditions for the existence and uniqueness of the solution of the periodic problem for the system of partial differential equations of the third order are established. Works [5], [11], are devoted to singular differential equations of the third order.

To find the solution, we introduce the function $z(x,t) = \frac{\partial u(x,t)}{\partial x}$, and we rewrite problem (1.1)-(1.4) in the form

$$\frac{\partial^2 z}{\partial x \partial t} = A(x,t)\frac{\partial z}{\partial x} + B(x,t)z + C(x,t)u + f(x,t), \quad (x,t) \in \Omega, \tag{1.5}$$

$$z(x,0) = z(x,T), \quad x \in [0,\omega], \tag{1.6}$$

$$z(0,t) = \psi(t), \quad t \in [0,T], \tag{1.7}$$

$$u(x,t) = \varphi(t) + \int_0^x z(\xi,t)d\xi. \tag{1.8}$$

For a fixed $u(x,t)$ problem (1.5)-(1.7) is a periodic boundary value problem for a system of hyperbolic equations of the second order.

We next introduce the notation $v(x,t) = \frac{\partial z(x,t)}{\partial x}$, and reduce problem (1.5)-(1.8) to the problem

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)z + C(x,t)u + f(x,t), \quad (x,t) \in \Omega, \tag{1.9}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{1.10}$$

and functional relations

$$z(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi, \quad (x,t) \in \Omega, \tag{1.11}$$

$$u(x,t) = \varphi(t) + \int_0^x z(\xi,t)d\xi, \quad (x,t) \in \Omega. \tag{1.12}$$

To solve problem (1.9)-(1.12) we apply the method of a parametrization .

For the step $h > 0 : Nh = T$ we partition $[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh)$, $N = 1, 2, ...$ In this case, Ω is divided into N parts. By $v_r(x,t)$, $z_r(x,t)$, $u_r(x,t)$ we denote, respectively, the restrictions of the functions $v(x,t)$, $z(x,t), u(x,t)$ on $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$. By $\lambda_r(x)$ we denote the value of the function $v_r(x,t)$ at $t = (r-1)h$, i.e. $\lambda_r(x) = v_r(x,(r-1)h)$ and denote $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x), r = \overline{1,N}$. We obtain an equivalent boundary value problem for the unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r(x) + B(x,t)z_r(x,t) + C(x,t)u_r(x,t) + f(x,t), \tag{1.13}$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \tag{1.14}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t \to T-0} \widetilde{v}_N(x,t) = 0, \quad x \in [0,\omega], \tag{1.15}$$

$$\lambda_s(x) + \lim_{t \to sh-0} \widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, \quad x \in [0,\omega], \quad s = \overline{1,N-1}. \tag{1.16}$$

$$z_r(x,t) = \psi(t) + \int\limits_0^x \widetilde{v}_r(\xi,t)d\xi + \int\limits_0^x \lambda_r(\xi)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{1.17}$$

$$u_r(x,t) = \varphi(t) + \int\limits_0^x z_r(\xi,t)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{1.18}$$

where (1.17) is the condition of gluing functions on the internal lines of the partition.

Problem (1.13), (1.14) for fixed $\lambda_r(x), z_r(x,t), u_r(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,\omega]$, which are equivalent to the integral equations

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^t A(x,\tau)\widetilde{v}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t A(x,\tau)d\tau \cdot \lambda_r(x) + \int\limits_{(r-1)h}^t F(x,\tau,z_r,u_r)d\tau, \tag{1.19}$$

where

$$\int\limits_{(r-1)h}^t F(x,\tau,z_r,u_r)d\tau = \int\limits_{(r-1)h}^t B(x,\tau)z_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t C(x,\tau)u_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t f(x,\tau)d\tau.$$

Instead of $\widetilde{v}_r(x,\tau)$ we substitute the corresponding right-hand side of (1.19) and repeating this process ν ($\nu = 1,2,...$) times we obtain

$$\widetilde{v}_r(x,t) = D_{\nu r}(x,t)\lambda_r(x) + F_{\nu r}(x,t,z_r,u_r) + G_{\nu r}(x,t,\widetilde{v}_r), \quad r = \overline{1,N}, \tag{1.20}$$

where

$$D_{\nu r}(x,t) = \int\limits_{(r-1)h}^t A(x,\tau_1)d\tau_1 + ... + \int\limits_{(r-1)h}^t A(x,\tau_1)... \int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)d\tau_\nu...d\tau_1,$$

$$F_{\nu r}(x,t,z_r,u_r) = \int\limits_{(r-1)h}^t \big[B(x,\tau_1)z_r(x,\tau_1) + C(x,\tau_1)u_r(x,\tau_1) + f(x,\tau_1)\big]d\tau_1 +$$

$$+\sum_{j=1}^{\nu-1} \int\limits_{(r-1)h}^t A(x,\tau_1)... \int\limits_{(r-1)h}^{\tau_{j-1}} A(x,\tau_j) \int\limits_{(r-1)h}^{\tau_j} \big[B(x,\tau_{j+1})z_r(x,\tau_{j+1}) +$$

$$+C(x,\tau_{j+1})u_r(x,\tau_{j+1}) + f(x,\tau_{j+1})\big]d\tau_{j+1}d\tau_j...d\tau_1,$$

$$G_{\nu r}(x,t,\widetilde{v}_r) =$$

$$= \int\limits_{(r-1)h}^t A(x,\tau_1)... \int\limits_{(r-1)h}^{\tau_{\nu-2}} A(x,\tau_{\nu-1}) \int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}...d\tau_1,$$

$\tau_0 = t, r = \overline{1,N}$. Passing to the limit as $t \to rh - 0$ in (1.20) we have

$$\lim_{t\to rh-0} \widetilde{v}_r(x,t) = D_{\nu r}(x,rh)\lambda_r(x) + F_{\nu r}(x,rh,z_r,u_r) + G_{\nu r}(x,rh,\widetilde{v}_r),$$

$x \in [0,\omega], r=\overline{1,N}$. Substituting in (1.15), (1.16) instead of $\lim\limits_{t\to rh-0} \widetilde{v}_r(x,t), r=\overline{1,N}$, the corresponding to them right-hand sides, for the unknown functions $\lambda_r(x), r=\overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x) = -F_\nu(x,h,z,u) - G_\nu(x,h,\widetilde{v}), \tag{1.21}$$

where $$Q_\nu(x,h) =$$

$$= \begin{bmatrix} I & 0 & \dots & 0 & -[I+D_{\nu N}(x,Nh)] \\ I+D_{\nu 1}(x,h) & -I & \dots & 0 & 0 \\ 0 & I+D_{\nu 2}(x,2h) & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I+D_{\nu,N-1}(x,(N-1)h) & -I \end{bmatrix},$$

$$F_\nu(x,h,z,u) = (-F_{\nu N}(x,Nh,z_N,u_N), F_{\nu 1}(x,h,z_1,u_1), ..., F_{\nu,N-1}(x,(N-1)h,z_{N-1},u_{N-1})),$$
$$G_\nu(x,h,\widetilde{v}) = (-G_{\nu N}(x,Nh,\widetilde{v}_N), G_{\nu 1}(x,h,\widetilde{v}_1), ..., G_{\nu,N-1}(x,(N-1)h,\widetilde{v}_{N-1})),$$

and I is the unit matrix of dimension n.

For finding a system of four functions $\{\lambda_r(x), \widetilde{v}_r(x,t), z_r(x,t), u_r(x,t)\}, r=\overline{1,N}$, we have a closed system consisting of equations (1.21), (1.20), (1.18) and (1.17).

Assuming the invertibility of the matrix $Q_\nu(x,h)$ for all $x\in[0,\omega]$, from equation (1.21), where $\widetilde{v}_r(x,t)=0, \quad z_r(x,t)=\psi(t), \quad u_r(x,t)=\varphi(t)$, we find $\lambda^{(0)}(x) =$
$= (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), ..., \lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x) = -[Q_\nu(x,h)]^{-1}\{F_\nu(x,h,\psi,\varphi) + G_\nu(x,h,0)\}.$$

Using equation (1.20), at $\lambda_r(x)=\lambda_r^{(0)}(x)$ we find the functions $\{\widetilde{v}_r^{(0)}(x,t)\}, r=\overline{1,N}$, i.e.

$$\widetilde{v}_r^{(0)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(0)}(x) + F_{\nu r}(x,t,\psi,\varphi) + G_{\nu r}(x,t,0).$$

The functions $z_r^{(0)}(x,t), u_r^{(0)}(x,t), r=\overline{1,N}$, are defined from the relations

$$z_r^{(0)}(x,t) = \psi(t) + \int_0^x \widetilde{v}_r^{(0)}(\xi,t)d\xi + \int_0^x \lambda_r^{(0)}(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$u_r^{(0)}(x,t) = \varphi(t) + \int_0^x z_r^{(0)}(\xi,t)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}.$$

2 Main results

For the initial approximation of problem (1.13)-(1.18) we take the system $(\lambda_r^{(0)}(x), \widetilde{v}_r^{(0)}(x,t), z_r^{(0)}(x,t),$ $u_r^{(0)}(x,t)), r=\overline{1,N}$ and construct successive approximations by using the following algorithm.

Step 1. A) Assuming that

$$z_r(x,t) = z_r^{(0)}(x,t), \quad u_r(x,t) = u_r^{(0)}(x,t), \quad r=\overline{1,N},$$

we find the first approximations of $\lambda_r(x), \widetilde{v}_r(x,t), r=\overline{1,N}$, by solving problem (1.13)-(1.16). Taking

$$\lambda_r^{(1,0)}(x) = \lambda_r^{(0)}(x), \quad \widetilde{v}_r^{(1,0)}(x,t) = \widetilde{v}_r^{(0)}(x,t),$$

we find the system of couples $\{\lambda_r^{(1)}(x), \widetilde{v}_r^{(1)}(x,t)\}, r = \overline{1,N}$, as the limit of the sequence $\lambda_r^{(1,m)}(x), \widetilde{v}_r^{(1,m)}(x,t)$, defined in the following way.

Step 1.1. Assuming the invertibility of the matrix $Q_\nu(x,h)$, $x \in [0,\omega]$, from equation (1.21), where

$$\widetilde{v}_r(x,t) = \widetilde{v}_r^{(1,0)}(x,t),$$

we find $\lambda^{(1,1)}(x) = (\lambda_1^{(1,1)}(x), \lambda_2^{(1,1)}(x), ..., \lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(0)},u^{(0)}) + G_\nu(x,h,\widetilde{v}^{(1,0)})\Big\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)$, $r = \overline{1,N}$, in (1.20) we find

$$\widetilde{v}_r^{(1,1)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(1,1)}(x) + F_{\nu r}(x,t,z^{(0)},u^{(0)}) + G_{\nu r}(x,t,\widetilde{v}^{(1,0)}).$$

Step 1.2. From equation (1.21), where $\widetilde{v}_r(x,t) = \widetilde{v}_r^{(1,1)}(x,t)$, we define
$\lambda^{(1,2)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(0)},u^{(0)}) + G_\nu(x,h,\widetilde{v}^{(1,1)})\Big\}$.
Using expression (20) again, we find the functions $\{\widetilde{v}_r^{(1,2)}(x,t)\}$, $r = \overline{1,N}$,
$\widetilde{v}_r^{(1,2)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(1,2)}(x) + F_{\nu r}(x,t,z^{(0)},u^{(0)}) + G_{\nu r}(x,t,\widetilde{v}^{(1,1)})$.
On step $(1,m)$ we obtain the system of couples

$$\{\lambda_r^{(1,m)}(x), \widetilde{v}_r^{(1,m)}(x,t)\}, r = \overline{1,N}.$$

Suppose that the solution of problem (1.13)-(1.16) is a sequence of systems of couples $\{\lambda_r^{(1,m)}(x)$, $\widetilde{v}_r^{(1,m)}(x,t)\}$ which are defined for $x \in [0,\omega]$, respectively, and converge as $m \to \infty$ to continuous functions $\lambda_r^{(1)}(x)$, $\widetilde{v}_r^{(1)}(x,t)$, $r = \overline{1,N}$.

B) The functions $z_r^{(1)}(x,t), u_r^{(1)}(x,t), r = \overline{1,N}$, are defined from the relations

$$z_r^{(1)}(x,t) = \psi(t) + \int_0^x \widetilde{v}_r^{(1)}(\xi,t)d\xi + \int_0^x \lambda_r^{(1)}(\xi)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N},$$

$$u_r^{(1)}(x,t) = \varphi(t) + \int_0^x z_r^{(1)}(\xi,t)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}.$$

Step 2. A) Assuming that

$$z_r(x,t) = z_r^{(1)}(x,t), \quad u_r(x,t) = u_r^{(1)}(x,t), \quad r = \overline{1,N},$$

we find the second approximations of $\lambda_r(x), \widetilde{v}_r(x,t), r = \overline{1,N}$, by solving problem (1.13)-(1.16). Taking

$$\lambda_r^{(2,0)}(x) = \lambda_r^{(1)}(x), \quad \widetilde{v}_r^{(2,0)}(x,t) = \widetilde{v}_r^{(1)}(x,t),$$

we find the system of couples $\{\lambda_r^{(2)}(x), \widetilde{v}_r^{(2)}(x,t)\}, r = \overline{1,N}$, as the limit of the sequence $\lambda_r^{(2,m)}(x), \widetilde{v}_r^{(2,m)}(x,t)$, defined in the following way:

Step 2.1. Assuming the invertibility of the matrix $Q_\nu(x,h), x \in [0,\omega]$, from equation (1.21), where

$$\widetilde{v}_r(x,t) = \widetilde{v}_r^{(2,0)}(x,t),$$

we find $\lambda^{(2,1)}(x) = (\lambda_1^{(2,1)}(x), \lambda_2^{(2,1)}(x), ..., \lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(1)},u^{(1)}) + G_\nu(x,h,\widetilde{v}^{(2,0)})\Big\}.$$

Substituting the found $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$, in (1.20) we find

$$\widetilde{v}_r^{(2,1)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(2,1)}(x) + F_{\nu r}(x,t,z^{(1)},u^{(1)}) + G_{\nu r}(x,t,\widetilde{v}^{(2,0)}).$$

Step 2.2. From equation (1.21), where

$$\widetilde{v}_r(x,t) = \widetilde{v}_r^{(2,1)}(x,t),$$

we define

$$\lambda^{(2,2)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(1)},u^{(1)}) + G_\nu(x,h,\widetilde{v}^{(2,1)})\Big\}.$$

Using expression (1.20), we find the functions $\{\widetilde{v}_r^{(2,2)}(x,t)\}$, $r=\overline{1,N}$:

$$\widetilde{v}_r^{(2,2)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(2,2)}(x) + F_{\nu r}(x,t,z^{(1)},u^{(1)}) + G_{\nu r}(x,t,\widetilde{v}^{(2,1)}).$$

On step $(2,m)$ we obtain the system of couples

$$\{\lambda_r^{(2,m)}(x), \widetilde{v}_r^{(2,m)}(x,t)\}, r=\overline{1,N}.$$

Suppose that the solution of problem (1.13)-(1.16) is a sequence of systems of couples $\{\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)\}$ which as $m\to\infty$ converges to $\{\lambda_r^{(2)}(x)$, $\widetilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$.

B) The functions $z_r^{(2)}(x,t), u_r^{(2)}(x,t), r=\overline{1,N}$, are defined from the relations

$$z_r^{(2)}(x,t) = \psi(t) + \int_0^x \widetilde{v}_r^{(2)}(\xi,t)d\xi + \int_0^x \lambda_r^{(2)}(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$u_r^{(2)}(x,t) = \varphi(t) + \int_0^x z_r^{(2)}(\xi,t)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}.$$

Continuing the process, at the k-th step we obtain the system $\{\lambda_r^{(k)}(x)$, $\widetilde{v}_r^{(k)}(x,t)$, $z_r^{(k)}(x,t)$, $u_r^{(k)}(x,t)\}$, $r=\overline{1,N}$.

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of problem (1.13)-(1.18).

Theorem 2.1. *Let for some* $0<\mu<1, h>0: Nh=T, N=1,2,...,$ *and* ν, $\nu\in\mathbb{N}$, $(nN\times nN)$ *the matrix* $Q_\nu(x,h)$ *be invertible at all* $x\in[0,\omega]$ *and let the following inequalities be satisfied*
1) $\|[Q_\nu(x,h)]^{-1}\|\le\gamma_\nu(x,h)$;
2) $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\le\mu<1$,
where

$$q_\nu(x,h) = 1+\gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}, \quad \alpha(x)=\max_{t\in[0,T]}\|A(x,t)\|,$$

$$\|A(x,t)\| = \max_{i=\overline{1,n}}\sum_{j=1}^{n}|a_{ij}(x,t)|.$$

Then there exists a unique solution $(\lambda_r^*(x),\widetilde{v}^*(x,t),z^*(x,t),u^*(x,t))$ *to problem (1.13)-(1.18) and the following estimates are valid*

$$a)\ \max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{(k)}(x,t)\|$$

$$\leq \frac{\rho_\nu(x,h)}{(k-1)!}\left(\int_0^x \rho_\nu(\xi,h)d\xi\right)^{k-1} e^{\int_0^x \rho_\nu(\xi,h)d\xi} \int_0^x d_\nu(\xi,h)d\xi \max\left\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\right\},$$

$$b)\ \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|z_r^*(x,t) - z_r^{(k)}(x,t)\|$$

$$\leq \int_0^x \max_{r=\overline{1,N}} \|\lambda_r^*(\xi) - \lambda_r^{(k)}(\xi)\|d\xi + \int_0^x \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(\xi,t) - \widetilde{v}_r^{(k)}(\xi,t)\|d\xi,$$

$$c)\ \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - u_r^{(k)}(x,t)\|$$

$$\leq \int_0^x \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|z_r^*(\xi,t) - z_r^{(k)}(\xi,t)\|d\xi, \quad k = 1,2,...,$$

where $\beta(x) = \max_{t\in[0,T]}\|B(x,t)\|$, $\sigma(x) = \max_{t\in[0,T]}\|C(x,t)\|$, $\|f\|_0 = \max_{(x,t)\in\Omega}\|f(x,t)\|$,

$$\|u(x,t)\| = \max_{i=\overline{1,n}}|u_i(x,t)|, \quad \rho_\nu(x,h) = \frac{\theta_\nu(x,h)[\beta(x)+x\sigma(x)]}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}},$$

$$\theta_\nu(x,h) = [\gamma_\nu(x,h) + q_\nu(x,h)]h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!},$$

$$d_\nu(x,h) = \frac{\theta_\nu(x,h)\beta(x)}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\int_0^x [\beta(\xi)+\sigma(\xi)+1]\theta_\nu(\xi,h)d\xi$$

$$+\frac{\theta_\nu(x,h)\sigma(x)}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\int_0^x\int_0^\xi [\beta(\xi_1)+\sigma(\xi_1)+1]\theta_\nu(\xi_1,h)d\xi_1 d\xi$$

$$+\frac{(\alpha(x)h)^\nu}{\nu!}q_\nu(x,h)[\beta(x)+\sigma(x)+1]\theta_\nu(x,h).$$

Proof. The following inequalities take place

$$\|F_\nu(x,h,z,u)\| \leq h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\big[\beta(x)\|z_r(x,t)\| + \sigma(x)\|u_r(x,t)\| + \|f(x,t)\|\big],$$

$$\|G_\nu(x,h,\widetilde{v})\| \leq \frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|D_{\nu r}(x,t)\| \leq \sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}.$$

From the zero step of the algorithm, the following estimates follow:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(0)}(x)\|$$

$$\leq [\beta(x)+\sigma(x)+1]\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\left\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\right\},$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\|$$

$$\leq [\beta(x)+\sigma(x)+1]q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|z_r^{(0)}(x,t)-\psi(t)\|$$

$$\leq \int_0^x [\beta(\xi)+\sigma(\xi)+1]\theta_\nu(\xi,h)d\xi \max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)-\varphi(t)\|$$

$$\leq \int_0^x\int_0^\xi [\beta(\xi_1)+\sigma(\xi_1)+1]\theta_\nu(\xi_1,h)d\xi_1 d\xi \max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\Big\}.$$

The following estimates are valid:

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|$$

$$\leq \gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|z_r^{(0)}(x,t)-\psi(t)\|$$

$$+\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)-\varphi(t)\|$$

$$+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|$$

$$\leq q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|z_r^{(0)}(x,t)-\psi(t)\|$$

$$+q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(0)}(x,t)-\varphi(t)\|$$

$$+q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(0)}(x,t)\|.$$

Next we establish the inequality

$$\Delta^{(1,1)}(x)=\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|$$

$$\leq \theta_\nu(x,h)\beta(x)\int_0^x [\beta(\xi)+\sigma(\xi)+1]\theta_\nu(\xi,h)d\xi \max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|, \max_{t\in[0,T]}\|\psi(t)\|, \|f\|_0\Big\}$$

$$+\theta_\nu(x,h)\sigma(x)\int\limits_0^x\int\limits_0^\xi[\beta(\xi_1)+\sigma(\xi_1)+1]\theta_\nu(\xi_1,h)d\xi_1 d\xi\max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|,\max_{t\in[0,T]}\|\psi(t)\|,\|f\|_0\Big\}$$

$$+\theta_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}q_\nu(x,h)[\beta(x)+\sigma(x)+1]\max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|,\max_{t\in[0,T]}\|\psi(t)\|,\|f\|_0\Big\}.$$

Thus,

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,m)}(x)\|$$

$$\le\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|, \tag{2.1}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,m)}(x,t)\|$$

$$\le q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|. \tag{2.2}$$

Owing to the inequality $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}<1$ follows the uniform convergence of the sequence $v_r^{(1,m+1)}(x,t)$, at $(x,t)\in\Omega_r$, to $v_r^{(1)}(x,t)$ and the convergence of the sequence of systems of functions $\lambda_r^{(1,m+1)}(x)$ to continuous on $x\in[0,\omega]$ functions $\lambda_r^{(1)}(x)$ for all $r=\overline{1,N}$:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|$$

$$\le\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|$$

$$\le\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\left[1+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|.$$

Passing to the limit at $m\to\infty$, we obtain the estimates:

$$\Delta^{(1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1)}(x)-\lambda_r{}^{(0)}(x)\|$$

$$\le d_\nu(x,h)\max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|,\max_{t\in[0,T]}\|\psi(t)\|,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(1)}(x,t)-z_r^{(0)}(x,t)\|\le\int\limits_0^x\Delta^{(1)}(\xi)d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(1)}(x,t)-u_r^{(0)}(x,t)\|\le\int\limits_0^x\int\limits_0^\xi\Delta^{(1)}(\xi_1)d\xi_1 d\xi.$$

For the systems of the of differences $\lambda_r{}^{(k+1)}(x) - \lambda_r{}^{(k)}(x)$, $\widetilde{v}_r^{(k+1)}(x,t) - \widetilde{v}_r^{(k)}(x,t)$, $z_r^{(k+1)}(x,t) - z_r^{(k)}(x,t)$, $u_r^{(k+1)}(x,t) - u_r^{(k)}(x,t)$, $r = \overline{1,N}$, $k = 1, 2, ...$ the following estimates are valid:

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,1)}(x) - \lambda_r{}^{(k+1,0)}(x)\|$$

$$\leq \gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k)}(x,t) - z_r^{(k-1)}(x,t)\|$$

$$+\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t) - \widetilde{v}_r^{(k+1,0)}(x,t)\|$$

$$\leq q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k)}(x,t) - z_r^{(k-1)}(x,t)\|$$

$$+q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,m+1)}(x) - \lambda_r{}^{(k+1,m)}(x)\|$$

$$\leq \gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t) - \widetilde{v}_r^{(k+1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t) - \widetilde{v}_r^{(k+1,m)}(x,t)\|$$

$$\leq q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t) - \widetilde{v}_r^{(k+1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t) - \widetilde{v}_r^{(k+1,0)}(x,t)\|$$

$$\leq \sum_{j=0}^{m}\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t) - \widetilde{v}_r^{(k+1,0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,m+1)}(x) - \lambda_r{}^{(k+1,0)}(x)\|$$

$$\leq \sum_{j=0}^{m-1}\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t) - \widetilde{v}_r^{(k+1,0)}(x,t)\|$$

$$+\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,1)}(x) - \lambda_r{}^{(k+1,0)}(x)\|.$$

Passing to the limit as $m \to \infty$, we obtain the following estimates:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\|$$

$$\leq \frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1 - q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k)}(x,t) - z_r^{(k-1)}(x,t)\|$$

$$+\frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|, \tag{2.3}$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|$$

$$\le\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k)}(x,t)-z_r^{(k-1)}(x,t)\|$$

$$+\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|, \tag{2.4}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k+1)}(x,t)-z_r^{(k)}(x,t)\|$$

$$\le\int\limits_0^x\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(\xi)-\lambda_r{}^{(k)}(\xi)\|d\xi+\int\limits_0^x\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)\|d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|$$

$$\le\int\limits_0^x\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k+1)}(x,t)-z_r^{(k)}(x,t)\|d\xi.$$

Summing, respectively, the left and right sides of inequalities (2.3), (2.4) we have

$$\Delta^{(k+1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|$$

$$\le\frac{\theta_\nu(x,h)\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k)}(x,t)-z_r^{(k-1)}(x,t)\|$$

$$+\frac{\theta_\nu(x,h)\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|, \tag{2.5}$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k+1)}(x,t)-z_r^{(k)}(x,t)\|\le\int\limits_0^x\Delta^{(k+1)}(\xi)d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|\le\int\limits_0^x\int\limits_0^\xi\Delta^{(k+1)}(\xi_1)d\xi_1d\xi.$$

For the function $\Delta^{(k+1)}(x)$ on the basis of (2.5) we establish the inequality

$$\Delta^{(k+1)}(x)\le\rho_\nu(x,h)\int\limits_0^x\Delta^{(k)}(\xi)d\xi, \tag{2.6}$$

$$\Delta^{(k+1)}(x) \leq \frac{\rho_\nu(x,h)}{(k-1)!}\Big(\int\limits_0^x \rho_\nu(\xi,h)d\xi\Big)^{k-1}\int\limits_0^x \Delta^{(1)}(\xi)d\xi.$$

Next we establish the inequalities

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k+p)}(x)-\lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|$$

$$\leq \rho_\nu(x,h)\sum_{j=k-1}^{k+p-2}\frac{1}{j!}\Big(\int\limits_0^x \rho_\nu(\xi,h)d\xi\Big)^j\int\limits_0^x d_\nu(\xi,h)d\xi\,\max\Big\{\max_{t\in[0,T]}\|\varphi(t)\|,\max_{t\in[0,T]}\|\psi(t)\|,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k+p)}(x,t)-z_r^{(k)}(x,t)\|$$

$$\leq \int\limits_0^x \max_{r=\overline{1,N}}\|{\lambda_r}^{(k+p)}(\xi)-{\lambda_r}^{(k)}(\xi)\|d\xi + \int\limits_0^x \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)\|d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+p)}(x,t)-u_r^{(k)}(x,t)\|$$

$$\leq \int\limits_0^x \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^{(k+p)}(x,t)-z_r^{(k)}(x,t)\|d\xi.$$

By passing to the limit as $p\to\infty$, for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$, we obtain the estimates of Theorem 2.1. Finally we show that the uniqueness of a solution to problem (1.13)-(1.18). Let the quadruples $\{\lambda_r^*(x),\widetilde{v}_r^*(x,t),z_r^*(x,t),u_r^*(x,t)\}$ and $\{\lambda_r^{**}(x),\widetilde{v}_r^{**}(x,t),z_r^{**}(x,t),u_r^{**}(x,t)\}$ be solutions to problem (1.13)-(1.18). Using inequality (2.6) for the differences ${\lambda_r}^*(x)-{\lambda_r}^{**}(x)$, $\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)$, we obtain

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\|$$

$$\leq \rho_\nu(x,h)\int\limits_0^x \max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^*(\xi)\|d\xi + \int\limits_0^x \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{**}(\xi,t)\|d\xi, \tag{2.7}$$

By applying the Gronwall - Bellman inequality to integral equations (2.7), we get

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\| + \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\| = \rho_\nu(x,h)\cdot 0. \tag{2.8}$$

From (2.8) it follows $\widetilde{v}_r^*(x,t)=\widetilde{v}_r^{**}(x,t)$ and ${\lambda_r}^*(x)={\lambda_r}^{**}(x)$, for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$. Then on the inequality

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|z_r^*(x,t)-z_r^{**}(x,t)\|$$

$$\leq \int\limits_0^x (\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{**}(\xi,t)\| + \max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^{**}(\xi)\|)d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t)-u_r^{**}(x,t)\|$$

$$\leq \int\limits_0^x\int\limits_0^\xi (\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi_1,t)-\widetilde{v}_r^{**}(\xi_1,t)\| + \max_{r=\overline{1,N}}\|\lambda_r^*(\xi_1)-\lambda_r^{**}(\xi_1)\|)d\xi_1 d\xi$$

we have $z_r^*(x,t) = z_r^{**}(x,t), u_r^*(x,t) = u_r^{**}(x,t), r = \overline{1,N}$, for all $(x,t) \in \Omega_r$, $r = \overline{1,N}$. This contradicts with our assumption that problem (1.13)-(1.18) has two solution. Therefore, solution to problem (1.13)-(1.18) is unique. □

By virtue of the equivalence of problems (1.1)-(1.4) and (1.13)-(1.18) from Theorem 2.1 follows

Theorem 2.2. *Let the conditions of Theorem 2.1 be satisfied. Then problem (1.1)-(1.4) has a unique solution $u^*(x,t)$ and the following estimates are valid*

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - u_r^{(k)}(x,t)\|$$

$$\leq \int_0^x \int_0^\xi \frac{\rho_\nu(\xi_1,h)}{(k-1)!} \Big(\int_0^{\xi_1} \rho_\nu(\xi_2,h) d\xi_2 \Big)^{k-1} e^{\int_0^{\xi_1} \rho_\nu(\xi_2,h)d\xi_2} \int_0^{\xi_1} d_\nu(\xi_2,h) d\xi_2 d\xi_1 d\xi$$

$$\times \max\Big\{ \max_{t\in[0,T]} \|\varphi(t)\|, \max_{t\in[0,T]} \|\psi(t)\|, \|f\|_0 \Big\}, \qquad k = 1, 2, ...$$

The main condition for the unique solvability of the problem under study is the existence of numbers $h > 0 : Nh = T$ and $\nu \in \mathbb{N}$, for which the matrix $Q_\nu(x,h)$ is invertible for all $x \in [0,\omega]$. Since the $(nN \times nN)$ matrix $Q_\nu(x,h)$, for $N \geq 2$, has a special block-band structure, then

Lemma 2.1. *The $(nN \times nN)$ matrix $Q_\nu(x,h)$ for $x \in [0,\omega]$ is invertible if and only if the $(n \times n)$ matrix*

$$M_\nu(x) = I - \prod_{s=N}^{1} [I + D_{\nu s}(x,h)].$$

Proof. It suffices to prove that the equation

$$Q_\nu(x,h) \cdot y = 0, \quad y \in \mathbb{R}^{nN}, \tag{2.9}$$

has a nonzero solution if and only if a nonzero solution has the equation

$$M_\nu(x) \cdot y_1 = 0, \quad y_1 \in \mathbb{R}^n, \tag{2.10}$$

Let $y \in \mathbb{R}^{nN}, y(x) = (y_1(x), ..., y_N(x))'$, be a solution to equation (2.9). Then, block-wise writing equation (2.9), we have the following relations

$$y_1(x) - [I + D_{\nu N}(x,h)]y_N(x) = 0, \tag{2.11}$$

$$[I + D_{\nu j}(x,h)]y_j(x) - y_{j+1}(x) = 0, \quad j = 1, ..., N-1, \tag{2.12}$$

i.e. the components of the vector $y(x)$ satisfy (2.11) - (2.12). Hence, all components $y \in \mathbb{R}^{nN}$ can be expressed in terms of $y_1 \in \mathbb{R}^n$:

$$y_{j+1}(x) = \prod_{s=j}^{1} [I + D_{\nu s}(x,h)]y_1(x), \quad j = 1, ..., N-1, \tag{2.13}$$

From (2.13) for $j = N-1$ we find that $y_N(x)$ has the form:

$$y_N(x) = \prod_{s=N-1}^{1} [I + D_{\nu s}(x,h)]y_1(x). \tag{2.14}$$

Substituting the right-hand side of (2.14) into (2.11) instead of $y_N(x)$ we obtain that $y_1(x) \in \mathbb{R}^n$ is a solution to equation (2.10). Since relation (2.13) is valid for any solution of equation (2.9), it follows from (2.13) that homogeneous equation (2.9) has a nonzero solution if and only if $\|y_1(x)\| \neq 0, y_1(x) \in \mathbb{R}^n$. In view of the fact that $y_1(x)$ satisfies equation (2.10), it follows that equation (2.9) has a nonzero solution if and only if equation (2.10) has a nonzero solution.

Let the matrix $Q_\nu(x,h)$ be invertible. Consider equation (2.10). If it has a nonzero solution $y_1(x) \in \mathbb{R}^n$, then this contradicts the invertibility of $Q_\nu(x,h)$. Indeed, if $y_1(x) \in \mathbb{R}^n$- is a solution to equation (2.10) and $\|y_1(x)\| \neq 0$, then, based on the aforementioned, the vector $y(x) = (y_1(x),...,y_N(x))'$, where $y_{j+1}(x)$, $j = 1,...,N-1$, is determined by formulas (2.13), will be a nonzero solution to equation (2.9), and this contradicts the invertibility of the matrix $Q_\nu(x,h) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ for every $x \in [0,w]$.

Now, let the matrix $M_\nu(x)$ be invertible and equation (2.9) has a nonzero solution $y(x) \in \mathbb{R}^{nN}$. As shown above, $y(x) \in R^{nN}$ will be a nonzero solution to equation (2.9) if and only if the vector $y_1(x) \in \mathbb{R}^n$, is nonzero, which is the first component of the vector $y(x)$ and satisfies equation (2.10). This contradicts our assumption that the matrix $M_\nu(x)$ is invertible. □

Lemma 2.2. *If the matrix $M_\nu(x)$ is invertible, then*

$$[Q_\nu(x,h)]^{-1} = \{q_{rj}(x)\}, r,j = \overline{1,N},$$

where

$$q_{11}(x) = [M_\nu(x)]^{-1},$$

$$q_{1k}(x) = -[M_\nu(x)]^{-1}\prod_{s=N}^{k}[I + D_{\nu s}(x,h)], 1 < k \leq N,$$

$$q_{rj}(x) = [I + D_{\nu,r-1}(x,h)]g_{r-1,j}(x), j \neq r, r = \overline{2,N},$$

$$q_{rr}(x) = [I + D_{\nu,r-1}(x,h)]g_{r-1,j}(x) - I, r = \overline{2,N}.$$

Proof. Consider the system of equations

$$Q_\nu(x,h)\lambda(x) = g(x), \tag{2.15}$$

where $\lambda(x), g(x), \in C([0,\omega], R^{nN})$, can be written block by block in the following form:

$$\lambda_1(x) - [I + D_{\nu N}(x,h)]\lambda_N(x) = g_1(x), \tag{2.16}$$

$$[I + D_{\nu s}(x,h)]\lambda_s(x) - \lambda_{s+1}(x) = g_{s+1}(x), \quad s = 1,...,N-1. \tag{2.17}$$

In system (2.17), sweeping downward, we have

$$\lambda_2(x) = [I + D_{\nu 1}(x,h)]\lambda_1(x) - g_2(x),$$

$$\lambda_3(x) = [I + D_{\nu 2}(x,h)]\lambda_2(x) - g_3(x)$$
$$= [I + D_{\nu 2}(x,h)][I + D_{\nu 1}(x,h)]\lambda_1(x) - [I + D_{\nu 2}(x,h)]g_2(x) - g_3(x), ...$$

$$\lambda_r(x) = [I + D_{\nu,r-1}(x,h)] \cdot ... \cdot [I + D_{\nu,1}(x,h)]\lambda_1(x)$$
$$-[I + D_{\nu,r-1}(x,h)] \cdot ... \cdot [I + D_{\nu 2}(x,h)]g_2(x) - ... - [I + D_{\nu 1}(x,h)]g_{r-1}(x) - g_r(x), \tag{2.18}$$

$r = 2,3,...N$. From here we find $\lambda_N(x)$ and substitute it into equation (2.16):

$$\lambda_1(x) - [I + D_{\nu N}(x,h)][I + D_{\nu,N-1}(x,h)] \cdot ... \cdot [I + D_{\nu 1}(x,h)]\lambda_1(x)$$

$$+[I+D_{\nu N}(x,h)][I+D_{\nu,N-1}(x,h)]\cdot\ldots\cdot[I+D_{\nu 2}(x,h)]g_2(x)$$
$$+\ldots+[I+D_{\nu N}(x,h)][I+D_{\nu 1}(x,h)]g_{N-1}(x)+[I+D_{\nu N}(x,h)]g_N(x)=g_1(x)$$

or

$$\Big\{I-\prod_{s=N}^{1}[I+D_{\nu s}(x,h)]\Big\}\lambda_1(x)=g_1(x)-\sum_{j=2}^{N}\prod_{s=N}^{j}[I+D_{\nu s}(x,h)]g_j(x),$$

that is

$$M_\nu(x)\lambda_1(x)=g_1(x)-\sum_{j=2}^{N}\prod_{s=N}^{j}[I+D_{\nu s}(x,h)]g_j(x). \tag{2.19}$$

Let the matrix $M_\nu(x)$ be invertible for all $x\in[0,\omega]$, then from (2.19) we find $\lambda_1(x)$:

$$\lambda_1(x)=[M_\nu(x)]^{-1}\Big\{g_1(x)-\sum_{j=2}^{N}\prod_{s=N}^{j}[I+D_{\nu s}(x,h)]g_j(x)\Big\}=\sum_{j=2}^{N}q_{1i}^{-1}(x)g_j(x). \tag{2.20}$$

Substituting (2.20) into (2.18), we find the remaining $\lambda_r(x), r=\overline{1,N}$, in the form of an expression in terms of the right-hand sides of $g_r(x)$ where $r=\overline{2,N}$:

$$\lambda_r(x)=\prod_{s=r-1}^{1}[I+D_{\nu s}(x,h)][M_\nu(x)]^{-1}g_1(x)$$
$$-\prod_{s=r-1}^{1}[I+D_{\nu s}(x,h)][M_\nu(x)]^{-1}\sum_{j=2}^{N}\prod_{s=N}^{j}[I+D_{\nu s}(x,h)]g_j(x)$$
$$-\prod_{s=r-1}^{2}[I+D_{\nu s}(x,h)]g_2(x)-\ldots-[I+D_{\nu,r-1}(x,h)]g_{r-1}(x)-g_r(x)$$
$$=\sum_{i=1}^{N}q_{ri}(x)g_i(x), r=\overline{1,N}.$$

On the other hand, using expression (2.20) and equations, starting from the second, of system (2.15), specifically

$$\lambda_{r+1}(x)=\sum_{i=1}^{N}q_{r+1,i}(x)g_i(x)=[I+D_{\nu r}(x,h)]\lambda_r(x)-g_{r+1}(x)$$
$$=[I+D_{\nu r}(x,h)]\sum_{i=1}^{N}q_{ri}(x)g_i(x)-g_{r+1}(x), r=\overline{1,N-1},$$

we find more convenient recurrent formulas for finding the matrix elements $[Q_\nu(x,h)]^{-1}=\{q_{rj}(x)\}, r,j=\overline{1,N}$:

$$q_{11}(x)=[M_\nu(x)]^{-1},$$
$$q_{1k}(x)=-[M_\nu(x)]^{-1}\prod_{s=N}^{k}[I+D_{\nu s}(x,h)], 2\le k\le N,$$
$$q_{rj}(x)=[I+D_{\nu,r-1}(x,h)]g_{r-1,j}(x), j\ne r, r=\overline{2,N},$$
$$q_{rr}(x)=[I+D_{\nu,r-1}(x,h)]g_{r-1,j}(x)-I, r=\overline{2,N}.$$

□

Recurrent formulas allow one to determine $[Q_\nu(x,h)]^{-1}$ via the elements of the matrix $A(x,t)$. Therefore, Theorems 2.1, 2.2 establish a coefficient criterion for the unique solvability of boundary value problem (1.1) - (1.4).

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