ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2022, Volume 13, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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VAGIF SABIR oglu GULIYEV

(to the 65th birthday)

On February 22, 2022 was the 65th birthday of Vagif Sabir oglu Guliyev, editor-in-chief of the Transactions of the Azerbaijan National Academy of Science, Issue Mathematics, Series of physical-technical and mathematics science (Scopus, Q3), deputy editor-in-chief of the Applied and Computational Mathematics (Web of Science, Q1), deputy director of the Institute of Applied Mathematics (IAM) of the Baku State University (BSU), head of the Department of Mathematical Analysis at the Institute of Mathematics and Mechanics (IMM) of the Azerbaijan National Academy of Sciences (ANAS), member of the Editorial Board of the Eurasian Mathematical Journal.

V.S. Guliyev was born in the city of Salyan in Azerbaijan. In 1978 Vagif Guliyev graduated from the Faculty of Mechanics and Mathematics of the Azerbaijan State University (now the Baku State University) with an honors degree and then completed his postgraduate studies at this university. His scientific supervisors were distinguished mathematicians A.A. Babayev and S.K. Abdullayev. In 1983 he defended his PhD thesis at the BSU. From 1983 he continued his scientific activities at the V.A. Steklov Mathematical Institute of the Academy of Sciences of the USSR. In 1987-1991 he was in internship at this institute and in 1994 defended there his DSc thesis.

From 1983 to 1995 he worked as assistant, a senior lecturer, docent and from 1995 to 2018 as a professor of Mathematical Analysis Chair of the BSU. In 1995-2008 he worked on part-time basis at the Institute of the IMM. From 2008 to 2014 he was a chief researcher of the Department of Mathematical Analysis of the IMM, from 2014 to the present day he is the head of this department.

In 2014 V.S. Guliyev was elected a corresponding member of the ANAS.

From 2015 to 2019, he worked as deputy director on science at the IMM. From 2019 to the present day, he has been working as a chief researcher at the IAM. Since May 2021, he has been working as a deputy director on science of the IAM.

Professor Vagif Guliyev has been a member of the Presidium of the Higher Attestation Commission under the President of the Republic of Azerbaijan since 2014 to the present day.

V.S. Guliyev is a world recognized specialist in real and harmonic analysis, function spaces and partial differential equations. He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. He was one of the first to study local Morrey-type spaces, generalized weighted Morrey-type spaces and anisotropic Banach-valued Sobolev spaces, for which appropriate embedding theorems were established.

Some of his results and methods are named after him: the Adams-Guliyev and Spanne-Guliyev conditions for the boundedness of operators in Morrey-type spaces, Guliyev's method of local estimates of integral operators of harmonic analysis, the Burenkov-Guliyevs conditions for the boundedness of operators in general local Morrey-type spaces.

On the whole, the results obtained by V.S. Guliyev have laid a groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations.

Vagif Sabir oglu Guliyev is an author of more than 250 scientific publications including 2 monographs. Among his pupils there are more than 20 candidates of sciences and 5 doctors of sciences. The results obtained by V.S. Guliyev, his pupils, collaborators and followers gained worldwide recognition.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vagif Sabir oglu Guliyev on the occasion of his 65th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 13, Number 1 (2022), 32 – 43

ZEROS OF LACUNARY TYPE POLYNOMIALS

S. Das

Communicated by V.I. Burenkov

Key words: zeros, lacunary polynomials, annular region.

AMS Mathematics Subject Classification: 30C15, 30C10, 26C10.

Abstract. Using Schwarz's lemma, Mohammad (1965) proved that all zeros of the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

with real or complex coefficients lie in the closed disc

$$|z| \le \frac{M'}{|a_n|} \text{ if } |a_n| \le M',$$

where

$$M' = \max_{|z|=1} \left| a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} \right|.$$

In this paper, we present new results on the location of zeros of the lacunary type polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_p z^p + a_n z^n,\ p < n.$$

In particular, for $p = n - 1$, our first result implies an important corollary which sharpens the above result. Also, we described some regions in which all zeros of $p(z)$ are simple. In many cases, our results give better bounds for the location of polynomial zeros than the known ones.

DOI: https://doi.org/10.32523/2077-9879-2022-13-1-32-43

1 Introduction and main results

For the moduli of the zeros of the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

of degree n with real or complex coefficients, the Lagrange [14] (see [3, Theorem 1.1]) and Cauchy [5] (see also [15, Chapter VII, Section 27, Theorem 27.2]) bounds are well known. In 1964, Guggenheimer [9] generalized the Cauchy result for the following class of lacunary type polynomials [15, Chapter VIII, Section 34, page 156]

$$p(z) = a_0 + a_1 z + \cdots + a_p z^p + a_n z^n,\ p < n,$$

and proved the following

Theorem A. *All zeros of $p(z)$ lie in the disc*

$$|z| < \delta,$$

where $\delta(>1)$ is the only positive root of the equation

$$t^{n-p} - t^{n-p-1} = Q_{p,n}$$

where

$$Q_{p,n} = \max_{0\le k\le p} \left|\frac{a_k}{a_n}\right|.$$

In 1965, Mohammad [16] obtained the following results by using Schwarz's lemma.

Theorem B. *All zeros of $f(z)$ lie in the disc*

$$|z| \le \frac{M'}{|a_n|} \text{ if } |a_n| \le M',$$

where

$$\begin{aligned} M' &= \max_{|z|=1} \left|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\right| \\ &= \max_{|z|=1} \left|a_{n-1} + a_{n-2} z + \cdots + a_0 z^{n-1}\right|. \end{aligned}$$

Theorem C. *All zeros of $f(z)$ lie in the annular region*

$$r\frac{|a_0|}{K'} \le |z| \le r \text{ if} |a_0| \le K',$$

where r is the largest modulus of the zeros of $f(z)$ and

$$K' = \max_{|z|=r} \left|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z\right|.$$

Using a particular lacunary type polynomial, in 2004, Shah and Liman [17] established the following result.

Theorem D. *All zeros of $f(z)$ with $a_{n-1} = 0$, lie in the disc*

$$|z| \le \frac{1}{r}\sqrt{\frac{M_2}{|a_n|}}$$

where $r > 0$ and

$$M_2 = \max_{|z|=r} \left|a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-2}z^2\right|.$$

In the literature, there exist several results [1, 2, 10, 11, 12, 17, 20] on polynomial zeros of lacunary type polynomials.

In this paper, we obtain some new bounds for the moduli of zeros for the considered class of lacunary type polynomials. Some of the results give generalizations of Theorem B, Theorem C and Theorem D, respectively. Moreover, the second, third and fourth results, and all corollaries related to these results do not require any numerical methods. Finally, we describe some regions in which zeros are simple. More precisely, we prove

Theorem 1.1. *All zeros of $p(z)$ lie in the disc*

$$|z| \le \frac{1}{t_0} \text{ if} |a_n| \le M_r,$$

where $r > 0$ and $t_0(\leq r)$ is the only positive root of the equation

$$F_r(t) = 0,$$

$$\begin{aligned} F_r(t) &= (n-p)!M_r^2 t^{n-p+1} + |a_p| M_r r^{n-p+1} t^{n-p} \\ &\quad - |a_p| |a_n| r^{2(n-p)} t - (n-p)! |a_n| M_r r^{n-p+1}, \end{aligned}$$

and

$$M_r = \max_{|z|=r} \left| a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n \right|.$$

Moreover, if $M_r = |a_n|$, all zeros of $p(z)$ lie in the disc

$$|z| \leq \frac{1}{r}.$$

Remark 1. The bound in Theorem 1.1 is attainable by the polynomial

$$Q(z) = -a_n z^n + a_p z^p + \cdots + a_0, \ p < n, \ a_i > 0; \ i = 0, 1, 2, \ldots, p, n,$$

which can be seen by observing that

$$|-a_n| = M_{r_0} \text{ and } Q\left(\frac{1}{r_0}\right) = 0,$$

where r_0 is the unique positive root of the equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_p r^{n-p} - |-a_n| = 0.$$

Here we note that there is no loss of generality if we take

$$M_r \geq \max_{|z|=r} \left| a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n \right|$$

in Theorem 1.1. For $r = 1$, replacing M_1 by M in Theorem 1.1, we have the following corollary.

Corollary 1.1. *All zeros of $p(z)$ lie in the disc*

$$|z| \leq \frac{1}{t_0} \text{ if } |a_n| \leq M,$$

where $t_0\,(\leq 1)$ is the only positive root of the equation

$$F(t) \equiv (n-p)!M^2 t^{n-p+1} + |a_p| M t^{n-p} - |a_p| |a_n| t - (n-p)! M |a_n| = 0$$

and

$$\begin{aligned} M &\geq \max_{|z|=1} \left| a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n \right| \\ &= \max_{|z|=1} \left| a_0 + a_1 z + \cdots + a_p z^p \right|. \end{aligned}$$

Put $p = n-1$ in Corollary 1.1, we can easily obtain the following corollary.

Corollary 1.2. *All zeros of $f(z)$ lie in the disc*

$$|z| \leq \frac{1}{t_0} \text{ if } |a_n| \leq M,$$

where $t_0\,(\leq 1)$ is the only positive root of the equation

$$G(t) \equiv M^2t^2 + |a_{n-1}|\,(M - |a_n|)\,t - M\,|a_n| = 0$$

i.e.,

$$t_0 = \frac{-\,|a_{n-1}|\,(M - |a_n|) + \sqrt{|a_{n-1}|^2\,(M - |a_n|)^2 + 4M^3\,|a_n|}}{2M^2}$$

and

$$\begin{aligned} M &= \max_{|z|=1} \left|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}\right| \\ &= \max_{|z|=1} \left|a_{n-1} + a_{n-2} z + \cdots + a_0 z^{n-1}\right|. \end{aligned}$$

Remark 2. The bound in Corollary 1.2 is sharper than the bound in Theorem B. Indeed

$$M = M' = \max_{|z|=1} \left|a_{n-1} + a_{n-2} z + \cdots + a_0 z^{n-1}\right|$$

and by the Maximum Modulus Principle it follows that

$$M \geq |a_{n-1}|\,,$$

where $|a_{n-1}|$ is the value of $|a_{n-1} + a_{n-2}z + \cdots + a_0 z^{n-1}|$ at the centre of the circle $|z| = 1$. Therefore

$$\begin{aligned} G\left(\frac{|a_n|}{M}\right) &= M^2\left(\frac{|a_n|}{M}\right)^2 + |a_{n-1}|\,(M - |a_n|)\,\frac{|a_n|}{M} - M\,|a_n| \\ &= |a_n|\,(|a_n| - M) - \frac{|a_n|\,|a_{n-1}|}{M}\,(|a_n| - M) \\ &= \frac{|a_n|}{M}\,(M - |a_{n-1}|)\,(|a_n| - M) \leq 0. \end{aligned}$$

Hence

$$t_0 \geq \frac{|a_n|}{M}$$

i.e.,

$$\frac{1}{t_0} \leq \frac{M}{|a_n|}.$$

Remark 3. In some cases Corollary 1.2 gives a better bound than those given by other similar results. To illustrate this, we consider the polynomial

$$f(z) = a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

with $|a_0| = 3, |a_1| = 2, |a_2| = 1, |a_3| = 3, |a_4| = 1, |a_5| = 4$. By Corollary 1.2, all zeros of $f(z)$ lie in the disc

$$|z| \leq 1.6579,$$

whereas all zeros of $f(z)$ lie in the regions

$$
\begin{array}{rrcl}
(i) & |z| & \leq & 1.75, \text{ by Cauchy [5] ([15, Chapter VII, Section 27, Theorem 27.2])} \\
(ii) & |z| & \leq & 2.5, \text{ by Theorem B} \\
(iii) & |z| & < & 2.7825, \text{ by Jain and Tawary [12, Theorem]} \\
(iv) & |z| & < & 1.8101, \text{ by Lagrange [14] ([3, Theorem 1.1])} \\
(v) & |z| & < & 1.7721, \text{ by Batra, Mignotte, Stefanescu [3, Theorem 3.1]} \\
(vi) & |z| & \leq & 1.8464, \text{ by Kojima [13]} \\
(vii) & |z| & < & 1.697, \text{ by Boese and Luther [4, Theorem 1]} \\
(viii) & |z| & \leq & 1.791, \text{ by Dehmer [7, Theorem 3.2]} \\
(ix) & |z| & \leq & 1.704, \text{ by Datt and Govil [6, Theorem 2]} \\
(x) & |z| & < & 2.25, \text{ by Bairagi, Jain, Mishra and Saha [2, Corollary 1.1]} \\
(xi) & |z| & < & 3.5309, \text{ by Walsh [19].}
\end{array}
$$

Theorem 1.2. *For any $r > 0$, all zeros of $p(z)$ lie in the disc*

$$|z| \leq \frac{1}{r}\left(\frac{M_r}{|a_n|}\right)^{\frac{1}{n-p}} \quad \text{if } |a_n| \leq M_r,$$

where

$$M_r = \max_{|z|=r}\left|a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n\right|.$$

For $p = n-2$, the bound in Theorem 1.2 coincides with the bound in Theorem D. So Theorem 1.2 is a generalization of Theorem D when $|a_n| \leq M_r$.

Remark 4. The equality in Theorem 1.2 is attainable by the polynomial

$$p(z) = -\left(a_p r^{n-p} + a_{p-1} r^{n-p+1} + \cdots + a_0 r^n\right) z^n + a_p z^p + \cdots + a_0,$$

where $r > 0$ and $a_i > 0$; $i = 0, 1, 2, \ldots, p$.

For $r = 1$, replacing M_1 by M in Theorem 1.2, we have the following corollary.

Corollary 1.3. *All zeros of $p(z)$ lie in the disc*

$$|z| \leq \left(\frac{M}{|a_n|}\right)^{\frac{1}{n-p}} \quad \text{if } |a_n| \leq M,$$

where

$$
\begin{aligned}
M &= \max_{|z|=1}\left|a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n\right| \\
&= \max_{|z|=1}\left|a_0 + a_1 z + \cdots + a_p z^p\right|.
\end{aligned}
$$

Clearly, Corollary 1.3 is a generalization of Theorem B.

Remark 5. In some cases Corollaries 1.1 and 1.3 give better bounds than those given by other similar results. To illustrate this, we consider the lacunary type polynomial

$$p(z) = a_5 z^5 + a_1 z + a_0$$

with $a_0 = 4, a_1 = 1, a_5 = 1$. Then all zeros of $p(z)$ lie in the regions

$$\begin{array}{rrcl} (i) & |z| & \leq & 3.061, \text{ by Aziz [1] ([20, Theorem B])} \\ (ii) & |z| & < & 1.748, \text{ by Theorem A} \\ (iii) & |z| & < & 5, \text{ by Theorem B} \\ (iv) & |z| & < & 2.2361, \text{ by Theorem D for } r=1 \\ (v) & |z| & < & 1.995, \text{ by Jain [10, Theorem 2]} \\ (vi) & |z| & < & 2.3195, \text{ by Lagrange [14] ([3, Theorem 1.1])} \\ (vii) & |z| & < & 2.1724, \text{ by Batra, Mignotte, Stefanescu [3, Theorem 3.1]} \\ (viii) & |z| & < & 2.556, \text{ by Jain [11, Theorem 1]} \\ (ix) & |z| & < & 4.015, \text{ by Bairagi, Jain, Mishra and Saha [2, Theorem 1.5]} \\ (x) & |z| & \leq & 1.4954, \text{ by corollary 1.3} \\ (xi) & |z| & \leq & 1.3812, \text{ by corollary 1.1.} \end{array}$$

By Corollary 1.3, we can easily obtain the following corollary.

Corollary 1.4. *All zeros of $p(z)$ lie in the disc*

$$|z| \leq \left(\frac{a_0 + a_1 + \cdots + a_p}{|a_n|} \right)^{\frac{1}{n-p}},$$

provide that $a_i \geq 0$; $i = 0, 1, \ldots, p$ and $|a_n| \leq a_0 + a_1 + \cdots + a_p$.

In particular, the bound in Corollary 1.4 is equal to 1 when

$$|a_n| = a_0 + a_1 + \cdots + a_p.$$

Theorem 1.3. *All zeros of*

$$P(z) = a_0 + a_q z^q + a_{q+1} z^{q+1} \cdots + a_n z^n, \; 0 < q \leq n,$$

lie in the annular region

$$r \left(\frac{|a_0|}{M''_r} \right)^{\frac{1}{q}} \leq |z| \leq r \text{ if } |a_0| \leq M''_r,$$

where r is the largest modulus of the zeros of $P(z)$ and

$$M''_r = \max_{|z|=r} \left| a_q z^q + a_{q+1} z^{q+1} + \cdots + a_n z^n \right|.$$

For $q = 1$, this reduces to Theorem C. So Theorem 1.3 is a generalization of Theorem C.

Theorem 1.4. *For any $r > 0$, all zeros of*

$$h(z) = a_0 + a_q z^q + \cdots + a_p z^p + a_n z^n, \; 0 < q \leq p < n,$$

lie in the annular region

$$r \left(\frac{|a_0|}{M''_r} \right)^{\frac{1}{q}} \leq |z| \leq \frac{1}{r} \left(\frac{M_r}{|a_n|} \right)^{\frac{1}{n-p}} \text{ if } |a_n| \leq M_r, \;\; |a_0| \leq M''_r,$$

where

$$\begin{aligned} M_r &= \max_{|z|=r} \left| a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_q z^{n-q} + a_0 z^n \right|, \\ M''_r &= \max_{|z|=r} \left| a_q z^q + a_{q+1} z^{q+1} + \cdots + a_p z^p + a_n z^n \right|. \end{aligned}$$

Remark 6. The bound below in Theorem 1.4 is attainable by the polynomial

$$h(z) = -\left(a_q r^q + a_{q+1} r^{q+1} + \cdots + a_p r^p + a_n r^n\right) + a_q z^q + \cdots + a_p z^p + a_n z^n,$$

$0 < q \leq p < n$, where $r > 0$ and $a_i > 0$ for $i = q,\ q+1,\ \ldots,\ p,\ n$, which can be seen by observing that

$$|a_0| = M_r'' = a_q r^q + a_{q+1} r^{q+1} + \cdots + a_p r^p + a_n r^n \text{ and } h(r) = 0,$$

where

$$a_0 = -\left(a_q r^q + a_{q+1} r^{q+1} + \cdots + a_p r^p + a_n r^n\right).$$

Now we present some regions in which all zeros of a polynomial are simple.

Theorem 1.5. *Let*

$$p(z) = a_0 + a_1 z + \cdots + a_p z^p + a_n z^n,\ \ p < n$$

be a polynomial with complex coefficients satisfying the condition

$$n|a_n| \leq M_r = \max_{|z|=r} \left|p a_p z^{n-p} + (p-1) a_{p-1} z^{n-p+1} + \cdots + a_1 z^{n-1}\right|$$

for some $r > 0$. Then the zeros of $p(z)$ are simple in the region

$$|z| > \frac{1}{t_0},$$

where t_0 is the only positive root of the equation

$$G_r(t) = 0,$$

$$\begin{aligned} G_r(t) &= (n-p)! M_r^2 t^{n-p+1} + p|a_p| M_r r^{n-p+1} t^{n-p} \\ &\quad - np|a_p||a_n| r^{2(n-p)} t - (n-p)! n |a_n| M_r r^{n-p+1}. \end{aligned}$$

Moreover, if $n|a_n| = M_r$, all zeros of $p(z)$ in the region

$$|z| > \frac{1}{r}$$

are simple.

Theorem 1.6. *Let*

$$p(z) = a_0 + a_1 z + \cdots + a_p z^p + z^n,\ \ p < n,\ \ a_i > 0;\ \ i = 0, 1, 2, \ldots, p$$

satisfy the condition

$$p a_p + (p-1) a_{p-1} + \cdots + 2a_2 + a_1 \geq n,$$

then all zeros of $p(z)$ with modulus greater than

$$\left(\frac{p a_p + (p-1) a_{p-1} + \cdots + 2a_2 + a_1}{n}\right)^{\frac{1}{n-p}}$$

are simple.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. The first lemma can be proved similarly to Schwarz's lemma.

Lemma 2.1. *Let $g(z)$ be a complex valued function which is analytic on the disc $K : |z - z_0| < R$ with the following properties,*
(i) $g(z_0) = g'(z_0) = \cdots = g^{(p-1)}(z_0) = 0$,
(ii) $g^{(p)}(z_0) \neq 0$,
(iii) $|g(z)| \leq M < \infty$ for all $z \in \Gamma_K$ (boundary of K).
Then

$$|g(z)| \leq \frac{M}{R^p} |z - z_0|^p \tag{2.1}$$

holds for all $z \in K \cup \Gamma_K$, and moreover

$$\left|g^{(p)}(z_0)\right| \leq \frac{p!M}{R^p}. \tag{2.2}$$

Also the equality in (2.1) holds for some $z\,(\neq z_0) \in K$ if and only if

$$g(z) = \frac{M}{R^p} e^{i\alpha} (z - z_0)^p,$$

for some real α, where $g^{(i)}(z)$ is the derivative of order i of $g(z)$ with respect to z, and p is a natural number.

In particular, for $p = 1$, Lemma 2.1 reduces to Schwarz's lemma.

Lemma 2.2. ([8, Lemma], see also [17, Lemma 1]) *If $f(z)$ is analytic in $|z| \leq 1, f(0) = a, |a| < 1, f'(0) = b, |f(z)| \leq 1$ on $|z| = 1$, then for $|z| \leq 1$*

$$|f(z)| \leq \frac{(1 - |a|)|z|^2 + |b||z| + |a|(1 - |a|)}{|a|(1 - |a|)|z|^2 + |b||z| + (1 - |a|)}.$$

Lemma 2.3. ([17, Lemma 2]) *If $f(z)$ is analytic in $|z| \leq R, f(0) = 0, f'(0) = b, |f(z)| \leq M$ on $|z| = R$, then for $|z| \leq R$*

$$|f(z)| \leq \frac{M|z|}{R^2} . \frac{M|z| + R^2|b|}{M + |b||z|}.$$

Lemma 2.4. ([18, Chapter VI, Example 5, page 212]) *If $f(z)$ is analytic on $|z| \leq R$, $|f(z)| \leq M$ on $|z| = R$, and $f(0) \neq 0$, then for $|z| \leq R$*

$$|f(z)| \leq M \frac{M|z| + R|f(0)|}{MR + |f(0)||z|}.$$

Lemma 2.5. *If $f(z)$ is analytic on $|z| \leq R$, $|f(z)| \leq M$ on $|z| = R$, and $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$, $f^{(m)}(0) \neq 0$, then for $|z| \leq R$*

$$|f(z)| \leq \frac{M|z|^m}{R^{m+1}} . \frac{m!M|z| + R^{m+1}\left|f^{(m)}(0)\right|}{m!M + R^{m-1}|f^{(m)}(0)||z|}.$$

For $m = 1$, it reduces to Lemma 2.3. So Lemma 2.5 is a generalization of Lemma 2.3.

Proof of Lemma 2.5. From the Taylor series expansions we have for all $|z| \leq R$,

$$f(z) = \frac{f^{(m)}(0)}{m!} z^m + \frac{f^{(m+1)}(0)}{(m+1)!} z^{m+1} + \frac{f^{(m+2)}(0)}{(m+2)!} z^{m+2} + \cdots$$

So the function $g(z)$ defined by

$$g(z) = \frac{f(z)}{z^m} = \frac{f^{(m)}(0)}{m!} + \frac{f^{(m+1)}(0)}{(m+1)!} z + \frac{f^{(m+2)}(0)}{(m+2)!} z^2 + \cdots$$

is analytic for all $|z| \leq R$ and takes the value $\frac{f^{(m)}(0)}{m!}$ at the origin. Also from the Maximum Modulus Principle that for all $|z| \leq R$,

$$|g(z)| \leq \frac{M}{R^m}.$$

Now, applying Lemma 2.4 to $g(z)$, we have for all $|z| \leq R$,

$$|g(z)| \leq \frac{M}{R^m} . \frac{\frac{M}{R^m}|z| + R|g(0)|}{\frac{M}{R^m} R + |g(0)||z|}$$

or

$$\left|\frac{f(z)}{z^m}\right| \leq \frac{M}{R^m} . \frac{\frac{M}{R^m}|z| + R\left|\frac{f^{(m)}(0)}{m!}\right|}{\frac{M}{R^m} R + \left|\frac{f^{(m)}(0)}{m!}\right||z|}$$

or

$$|f(z)| \leq \frac{M|z|^m}{R^{m+1}} . \frac{m!M|z| + R^{m+1}\left|f^{(m)}(0)\right|}{m!M + R^{m-1}\left|f^{(m)}(0)\right||z|}.$$

□

3 Proofs of theorems

Proof of Theorem 1.1. Consider

$$Q(z) = a_n + R(z),$$

where

$$R(z) = a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n.$$

Clearly

$$R(0) = R'(0) = \cdots = R^{(n-p-1)}(0) = 0 \text{ and } R^{(n-p)}(0) = a_p \neq 0.$$

Now, for $|z| \leq r$, applying Lemma 2.5, we have

$$|R(z)| \leq \frac{M_r|z|^{n-p}}{r^{n-p+1}} . \frac{(n-p)!M_r|z| + |a_p| r^{n-p+1}}{(n-p)!M_r + |a_p| r^{n-p-1}|z|},$$

which implies

$$|Q(z)| \geq |a_n| - \frac{M_r|z|^{n-p}}{r^{n-p+1}} . \frac{(n-p)!M_r|z| + |a_p| r^{n-p+1}}{(n-p)!M_r + |a_p| r^{n-p-1}|z|} \text{ for } |z| \leq r.$$

Therefore $|Q(z)| > 0$ if

$$(n-p)!M_r^2|z|^{n-p+1} + |a_p| M_r r^{n-p+1}|z|^{n-p} - |a_p||a_n| r^{2(n-p)}|z| - (n-p)!M_r|a_n| r^{n-p+1} < 0.$$

We introduce the function

$$F_r(t) = (n-p)!M_r^2 t^{n-p+1} + |a_p| M_r r^{n-p+1} t^{n-p}$$

$$- |a_p| |a_n| r^{2(n-p)} t - (n-p)! |a_n| M_r r^{n-p+1}$$

Clearly,

$$F(0) = -(n-p)!M_r |a_n| r^{n-p+1} < 0$$

and

$$F_r(r) = \left((n-p)!M_r + |a_p| r^{n-p}\right)(M_r - |a_n|) r^{n-p+1} \geq 0 \text{ if } |a_n| \leq M_r.$$

Let $t_0 (\leq r)$ be the only positive root of the equation $F_r(t) = 0$. As for $|a_n| \leq M_r$,

$$F_r(t) < 0 \text{ if } 0 < t < t_0,$$

which implies

$$|Q(z)| > 0 \text{ if } |z| < t_0.$$

In particular, for $|a_n| = M_r$,

$$F(0) < 0 \text{ and } F(r) = 0,$$

which implies

$$|Q(z)| > 0 \text{ if } |z| < r.$$

Also

$$Q(z) = z^n p\left(\frac{1}{z}\right)$$

and this leads us to the desired result. □

Proof of Theorem 1.2. Consider

$$Q(z) = a_n + R(z),$$

where

$$R(z) = a_p z^{n-p} + a_{p-1} z^{n-p+1} + \cdots + a_0 z^n.$$

Applying Lemma 2.1 to $R(z)$, we get

$$|R(z)| \leq \frac{M_r}{r^{n-p}} |z|^{n-p} \text{ for } |z| \leq r,$$

which implies

$$|Q(z)| \geq |a_n| - \frac{M_r}{r^{n-p}} |z|^{n-p} \text{ for } |z| \leq r.$$

Hence, if $|z| < r\left(\frac{|a_n|}{M_r}\right)^{\frac{1}{n-p}}$, then $|Q(z)| > 0$, and $Q(z) = z^n p\left(\frac{1}{z}\right)$. Thus it follows that all zeros of $p(z)$ lie in the disc

$$|z| \leq \frac{1}{r}\left(\frac{M_r}{|a_n|}\right)^{\frac{1}{n-p}}.$$

□

Proof of Theorem 1.3. Consider
$$P(z) = a_0 + T(z),$$
where
$$T(z) = a_q z^q + a_{q+1} z^{q+1} + \cdots + a_n z^n.$$
Applying Lemma 2.1 to $T(z)$, we can write for $|z| \leq r$,
$$|T(z)| \leq \frac{M_r''}{r^q} |z|^q,$$
which implies
$$|P(z)| \geq |a_0| - \frac{M_r''}{r^q} |z|^q \text{ for } |z| \leq r.$$
Hence, $|P(z)| > 0$ if $|z| < r \left(\frac{|a_0|}{M_r''} \right)^{\frac{1}{q}}$, and this leads us to the desired result. □

Proof of Theorem 1.4. The proof of Theorem 1.4 is not given because it is similar to the proofs of Theorem 1.2 and Theorem 1.3. □

Proof of Theorem 1.5. Clearly
$$p'(z) = n a_n z^{n-1} + p a_p z^{p-1} + (p-1) a_{p-1} z^{p-2} + \cdots + 2 a_2 z + a_1$$
is a polynomial of degree $(n-1)$. Therefore the inequality
$$n |a_n| \leq M_r = \max_{|z|=r} \left| p a_p z^{n-p} + (p-1) a_{p-1} z^{n-p+1} + \cdots + a_1 z^{n-1} \right|$$
holds for some $r > 0$ and can also be written as
$$|n a_n| \quad \leq \quad M_r = \max_{|z|=r} \left| p a_p z^{(n-1)-(p-1)} + (p-1) a_{p-1} z^{(n-1)-(p-1)+1} + \cdots + a_1 z^{n-1} \right|.$$
Applying Theorem 1.1 to the polynomial $p'(z)$ we can easily say that all zeros of $p'(z)$ lie in the region $|z| \leq \frac{1}{t_0}$, and in particular, for $n |a_n| = M_r$. Then the zeros of $p'(z)$ lie in $|z| \leq \frac{1}{r}$. Therefore, $p(z)$ cannot have any multiple zeros in the region $|z| > \frac{1}{t_0}$, and in particular for $n |a_n| = M_r$, the region $|z| > \frac{1}{r}$ has no multiple zeros of $p(z)$, and this proves the desired result. □

Proof of Theorem 1.6. The proof of Theorem 1.6 is not presented as it can be easily obtained by applying Corollary 1.4 to the polynomial $p'(z)$. □

Acknowledgments

The author would like to thank the anonymous referee for his/her helpful comments and suggestions.

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Received: 04.08.2020
Revised: 07.06.2021