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VAGIF SABIR oglu GULIYEV

(to the 65th birthday)

On February 22, 2022 was the 65th birthday of Vagif Sabir oglu Guliyev, editor-in-chief of the Transactions of the Azerbaijan National Academy of Science, Issue Mathematics, Series of physical-technical and mathematics science (Scopus, Q3), deputy editor-in-chief of the Applied and Computational Mathematics (Web of Science, Q1), deputy director of the Institute of Applied Mathematics (IAM) of the Baku State University (BSU), head of the Department of Mathematical Analysis at the Institute of Mathematics and Mechanics (IMM) of the Azerbaijan National Academy of Sciences (ANAS), member of the Editorial Board of the Eurasian Mathematical Journal.

V.S. Guliyev was born in the city of Salyan in Azerbaijan. In 1978 Vagif Guliyev graduated from the Faculty of Mechanics and Mathematics of the Azerbaijan State University (now the Baku State University) with an honors degree and then completed his postgraduate studies at this university. His scientific supervisors were distinguished mathematicians A.A. Babayev and S.K. Abdullayev. In 1983 he defended his PhD thesis at the BSU. From 1983 he continued his scientific activities at the V.A. Steklov Mathematical Institute of the Academy of Sciences of the USSR. In 1987-1991 he was in internship at this institute and in 1994 defended there his DSc thesis.

From 1983 to 1995 he worked as assistant, a senior lecturer, docent and from 1995 to 2018 as a professor of Mathematical Analysis Chair of the BSU. In 1995-2008 he worked on part-time basis at the Institute of the IMM. From 2008 to 2014 he was a chief researcher of the Department of Mathematical Analysis of the IMM, from 2014 to the present day he is the head of this department.

In 2014 V.S. Guliyev was elected a corresponding member of the ANAS.

From 2015 to 2019, he worked as deputy director on science at the IMM. From 2019 to the present day, he has been working as a chief researcher at the IAM. Since May 2021, he has been working as a deputy director on science of the IAM.

Professor Vagif Guliyev has been a member of the Presidium of the Higher Attestation Commission under the President of the Republic of Azerbaijan since 2014 to the present day.

V.S. Guliyev is a world recognized specialist in real and harmonic analysis, function spaces and partial differential equations. He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. He was one of the first to study local Morrey-type spaces, generalized weighted Morrey-type spaces and anisotropic Banach-valued Sobolev spaces, for which appropriate embedding theorems were established.

Some of his results and methods are named after him: the Adams-Guliyev and Spanne-Guliyev conditions for the boundedness of operators in Morrey-type spaces, Guliyev's method of local estimates of integral operators of harmonic analysis, the Burenkov-Guliyevs conditions for the boundedness of operators in general local Morrey-type spaces.

On the whole, the results obtained by V.S. Guliyev have laid a groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations.

Vagif Sabir oglu Guliyev is an author of more than 250 scientific publications including 2 monographs. Among his pupils there are more than 20 candidates of sciences and 5 doctors of sciences. The results obtained by V.S. Guliyev, his pupils, collaborators and followers gained worldwide recognition.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vagif Sabir oglu Guliyev on the occasion of his 65th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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DETERMINATION OF FRACTIONAL ORDER AND SOURCE TERM IN SUBDIFFUSION EQUATIONS

R.R. Ashurov, Yu.E. Fayziev

Communicated by V.I. Burenkov

Dedicated to Professor Shavkat Alimov on the occasion of his 75th birthday

Key words: subdiffusion equation, Riemann-Liouville derivatives, inverse problem, determination of the fractional derivative's order and source function, Fourier method.

AMS Mathematics Subject Classification: 35R11, 34K29.

Abstract. In this paper, we consider the inverse problem for simultaneously determining the order of the Riemann-Liouville time fractional derivative and the source function in subdiffusion equations. The classical Fourier method is used to prove uniqueness and existence theorems for this inverse problem.

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1 Introduction

In this article, we consider the inverse problem for simultaneously determining the order of the Riemann-Liouville time fractional derivative and the source function in subdiffusion equations. The fractional part of our subdiffusion equation will be defined through the Riemann-Liouville fractional derivative ∂_t^ρ of order $0 < \rho < 1$. To define the Riemann-Liouville fractional derivative, we first define the fractional integration of order $\rho < 0$ of a function $h(t)$ defined on $[0, \infty)$ by the formula

$$J_t^\rho h(t) = \frac{1}{\Gamma(-\rho)} \int\limits_0^t \frac{h(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler's gamma function. Using this definition one can define the Riemann - Liouville fractional derivative of order ρ, $0 < \rho < 1$, as

$$\partial_t^\rho h(t) = \frac{d}{dt} J_t^{\rho-1} h(t).$$

If in this definition we interchange the differentiation and fractional integration, then we get the definition of a regularized derivative, that is, the definition of a fractional derivative in the sense of Caputo:

$$D_t^\rho h(t) = J_t^{\rho-1} \frac{d}{dt} h(t).$$

Let Ω be an arbitrary N-dimensional domain with twice differentiable boundary $\partial\Omega$. Namely, the functions, defining the boundary equation in the local coordinates, are twice continuously differentiable.

Further, let $0 < \rho < 1$ be the unknown order of the Riemann - Liouville derivative ∂_t^ρ to be determined and $f(x)$ be the unknown source function. Consider the initial-boundary value problem of the form

$$\begin{cases} \partial_t^\rho u(x,t) - \triangle u(x,t) = f(x), \ x \in \Omega, \quad 0 < t < T; \\ Bu(x,t) \equiv \dfrac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \ 0 < t < T; \\ \lim\limits_{t \to 0} J_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\triangle$ is the Laplace operator, n is the unit outward normal vector to $\partial\Omega$ and $\varphi(x)$ is a given function.

If the parameter ρ is known and $\varphi(x)$ and $f(x)$ are sufficiently smooth given functions, then there is a unique solution to initial-boundary value problem (1.1) (see, for example, [2]).

But when considering initial-boundary value problem (1.1) as a model problem in analyzing different processes, unlike differential equations of integer order, an order of the fractional derivative ρ is often unknown and difficult to be directly measured. Determination of this parameter is actually the first step in modeling of any processes and it requires one to discuss inverse problems of identifying these physical quantities from some indirectly observed information of solutions. Obviously, investigations of these inverse problems are not only theoretically interesting, but also necessary for finding solutions to initial-boundary value problems and studying properties of solutions (see the recent survey paper Li, Liu and Yamamoto [21] and references therein; see also [1], [3] - [6] [10], [15]).

On the other hand, when modeling various processes, practical needs lead to problems of determining the right-hand side of a differential equation (source function) from some available data about the solution. These are the so-called inverse problems of determining sources of fractional partial differential equations. These types of inverse problems arise in various fields of human activity, such as seismology, biology, medicine, quality control of industrial goods, etc. All these circumstances place these inverse problems among the important problems of modern mathematics (see the recent survey paper Liu, Li and Yamamoto [23] and references therein; see also [8], [27], [26]).

Many works of specialists are devoted to the study of these two inverse problems. It should be noted that these two inverse problems were usually studied separately.

Obviously, to identify the fractional derivative ρ an additional condition is necessary, and all previous authors considered the following relation as an additional condition

$$u(x_0, t) = d(t), \ 0 < t < T, \tag{1.2}$$

at a monitoring point $x_0 \in \overline{\Omega}$. Specialists mainly studied the uniqueness of a solution to the inverse problem (with additional condition (1.2)) determining the order of the fractional derivative. It should be emphasized that it was only in Janno's paper [14] that both the existence and the uniqueness of a solution to this inverse problem was proved.

In this work, as well as in the paper of Ashurov and Umarov [3], we will consider the following additional information on the solution at a time instant t_0 (t_0 is defined later):

$$U(\rho, t_0) \equiv \frac{1}{|\Omega|^{1/2}} \int\limits_\Omega u(x, t_0) dx = d_0, \tag{1.3}$$

and prove, that extra condition (1.3) ensures not only the uniqueness, but also the existence of the parameter ρ. Note that the solution to problem (1.1) depends on ρ.

It should also be noted that in articles [1], [4] - [6], similar inverse problems were considered to determine the unknown order ρ of the fractional derivative with slightly different additional conditions for subdiffution and wave equations. These conditions also guarantee both the existence and uniqueness of the order ρ.

In the recent paper [15], the inverse problem of the simultaneous determination of some unknown coefficients and the order of the derivative in the multidimensional time-fractional diffusion (wave) equations are considered both in the Euclidean domain and in the Riemannian manifold. The authors managed to prove the uniqueness of the solution to such an inverse problem.

To identify the source function $f(x)$ in problem (1.1), the following equality

$$u(x,T) = \psi(x), \quad x \in \overline{\Omega}, \tag{1.4}$$

(i.e. the final temperature value) is considered as an additional condition. This condition ensures both the existence and uniqueness of the unknown function $f(x)$. Moreover, the final temperature value is easy to measure in specific applications. The inverse problem (1.1), (1.4), of determining the right-hand side of an equation has been studied by many authors. In the articles [11], [16] - [17], [28] the "elliptic part" of the equation is an ordinary differential expression, and in the articles [8], [27], [20], [25], [26] - an elliptic differential operator.

In this article, as noted above, we will study the inverse problem of simultaneously determining the order of the fractional derivative and the right-hand side of the equation in (1.1). In this case, to determine the source function, as in the works listed above, we will use additional information (1.4). Obviously, if we put $t_0 = T$ in condition (1.3), then it does not give any new information (see (1.4)) and, therefore, we assume that $t_0 < T$. The main result of this work shows that with this choice, conditions (1.3) and (1.4) guarantee both the existence and uniqueness of both the unknown order of the derivative and the source function.

For the best of our knowledge, only in the recent paper of Z. Li and Z. Zhang [22] the authors studied an inverse problem for simultaneously determining the order of time fractional derivative in the sense of Caputo and a source function in a subdiffusion equation. The authors managed to prove only the uniqueness of the solution to the inverse problem. The initial-boundary value problem considered in this paper has the form:

$$\begin{cases} D_t^\rho u(x,t) - \triangle u(x,t) = \sum\limits_{k=1}^{K} p_k(x)\chi_{t\in[c_{k-1},c_k)}, \ x \in \Omega, \quad 0 < t < \infty; \\ u(x,t) = 0, \quad x \in \partial\Omega, \ 0 < t < \infty; \\ u(x,0) = 0, \quad x \in \overline{\Omega}. \end{cases} \tag{1.5}$$

Here Ω is the unit disc in $\mathbb{R}^2$, $K \le \infty$ and $\chi_{t\in[c_{k-1},c_k)}$ are indicator functions. In problem (1.5) ρ, $\{p_k(x)\}_{k=1}^K, \{c_k\}_{k=0}^K$ and K are the unknowns. If $1/2 < \rho < 1$ and $p_k(x), c_k$ and K satisfy some conditions, then specifying additional data $\frac{\partial u}{\partial n}(x,t), \ t \in (0,\infty), \ x \in X_{ab} = \{a,b\} \subset \partial\Omega$, the authors proved the uniqueness theorem.

2 Main result

Now we pass to a rigorous statement of the main result of our paper. If parameter ρ and source function $f(x)$ are known, then problem (1.1) is called *the forward problem*. If both of them are unknown, then problem (1.1) together with extra conditions (1.3) and (1.4) we call *the inverse problem*.

Definition 1. *The triple $\{u(x,t), f(x), \rho\}$ of the functions $u(x,t)$, $f(x)$ and the parameter ρ with the properties*

1. $\rho \in (0,1)$,

2. $f(x) \in C(\overline{\Omega})$,

3. $\partial_t^\rho u(x,t), \triangle u(x,t) \in C(\overline{\Omega} \times (0,T))$,
4. $J_t^{\rho-1} u(x,t) \in C(\overline{\Omega} \times [0,T])$

satisfying all the conditions of problem (1.1) *and* (1.3), (1.4) *in the classical sense, is called a solution of the inverse problem. The function* $u(x,t)$ *with these properties is called a solution of the forward problem.*

Our proposed method for solving the inverse problem is based on the Fourier method. In accordance with the Fourier method, one should consider the following spectral problem

$$-\triangle v(x) = \lambda v(x), \quad x \in \Omega;$$

$$Bv(x) = 0, \quad x \in \partial\Omega.$$

Since the boundary $\partial\Omega$ is twice differentiable, then this problem has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions $\{v_k(x)\}$, $k \geq 1$, and a countable set of nonnegative eigenvalues $\{\lambda_k\}$ (see, for example, [19]). Note, that $\lambda_1 = 0$, $v_1(x) = |\Omega|^{-1/2}$.

Further, suppose, that given functions $\varphi(x)$ and $\psi(x)$ satisfy the following conditions (the character $[a]$ denotes the integer part of the number a):

(a) $\varphi(x) \in C^{\left[\frac{N}{2}\right]}(\Omega), \quad D^\alpha \varphi(x) \in L_2(\Omega), \quad |\alpha| = \left[\frac{N}{2}\right] + 1;$

(b) $B\varphi(x) = B(\Delta\varphi(x)) = ... = B(\Delta^{\left[\frac{N}{2}\right]}\varphi(x)) = 0, \quad x \in \partial\Omega;$

(c) $\psi(x) \in C^{\left[\frac{N}{2}\right]+1}(\Omega), \quad D^\alpha \psi(x) \in L_2(\Omega), \quad |\alpha| = \left[\frac{N}{2}\right] + 2;$

(d) $B\psi(x) = B(\Delta\psi(x)) = ... = B(\Delta^{\left[\frac{N}{2}\right]+1}\psi(x)) = 0, \quad x \in \partial\Omega.$

It should be noted that conditions (a) and (b) ensure the existence and uniqueness of the solution $u(x,t)$ of the forward problem, and if the function $\psi(x)$ from additional condition (1.4) satisfies the conditions (c) and (d), then, as follows by Theorem 2.1 below, the source function $f(x)$ exists and is unique.

Let g_k stand for the Fourier coefficient of a function $g(x) \in L_2(\Omega)$, i.e.

$$g_k = \int_\Omega g(x) v_k(x) dx,$$

and let

$$\varphi_1^2 + \psi_1^2 \neq 0. \tag{2.1}$$

We choose the parameter t_0 in extra condition (1.3) in the following way. If $\varphi_1 \cdot \psi_1 \leq 0$, then $t_0 \in (1,T)$ and otherwise

$$t_0 \in (1,T) \cap \begin{cases} \left(1, \ \dfrac{\varphi_1}{\psi_1} \cdot T\right) & \text{if} \quad \dfrac{\varphi_1}{\psi_1} \cdot T > 1; \\ \left(2(\ln 2 + 1)\dfrac{\varphi_1}{\psi_1} \cdot T, \ T\right) & \text{if} \quad \dfrac{\varphi_1}{\psi_1} \cdot T \leq 1. \end{cases} \tag{2.2}$$

Let $E_{\rho,\mu}$ be the Mittag-Leffler function of the form

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

Here is the main result of the present paper.

Theorem 2.1. *Let conditions* (a) - (d) *and* (2.1) - (2.2) *be satisfied. Then the inverse problem has a unique solution* $\{u(x,t), f(x), \rho\}$ *if and only if*

$$\min\left\{\psi_1, \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\} < d_0 < \max\left\{\psi_1, \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\}.$$

The unique solutions $u(x,t)$ *and* $f(x)$ *have the form*

$$u(x,t)=\sum_{k=1}^{\infty}[\varphi_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)+f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)]v_k(x), \tag{2.3}$$

$$f(x)=\sum_{k=1}^{\infty}\frac{\psi_k}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)}v_k(x)-\sum_{k=1}^{\infty}\frac{\varphi_k E_{\rho,\rho}(-\lambda_k T^\rho)}{T E_{\rho,\rho+1}(-\lambda_k T^\rho)}v_k(x), \tag{2.4}$$

where the series converge uniformly and absolutely.

In the proof of Theorem 2.1 we only use the fact that elliptic operator $-\triangle$ has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions and that the first eigenvalue is equal to zero: $\lambda_1=0$. Therefore our proposed method is applicable to an arbitrary elliptic differential operator

$$A(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha, \quad D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}},$$

where $|\alpha|=\alpha_1+\alpha_2+...+\alpha_N$, with an arbitrary boundary conditions $Bu(x,t)=0$, provided that the corresponding spectral problem has the above properties. Note, in this general case extra condition (1.3) will have the form

$$\int_\Omega u(x,t_0)v_1(x)dx=d_0.$$

3 Proof of Theorem 2.1

It is not hard to verify, that functions (2.3) and (2.4) are formal solutions to problem (1.1) together with extra condition (1.4) (see, for example [2]). According to Definition 1, now we only need to prove, that one can validly apply the operators D^α with $|\alpha|\le 2$ and ∂_t^ρ to series in (2.3) term-by-term and show that the resulting series and series (2.4), defining $f(x)$, converge uniformly and absolutely.

To do this for an arbitrary real number τ we introduce the following self-adjoint operator, acting in $L_2(\Omega)$ as:

$$\hat{A}^\tau g(x)=\sum_{k=1}^{\infty}\lambda_k^\tau\, g_k\, v_k(x), \quad g_k=(g,v_k),$$

with the domain of definition

$$D(\hat{A}^\tau)=\{g\in L_2(\Omega):\sum_{k=1}^{\infty}\lambda_k^{2\tau}|g_k|^2<\infty\}.$$

Let A stand for the operator acting in $L_2(\Omega)$ as $Ag(x)=-\triangle g(x)$ with the domain of definition $D(A)=\{g\in C^2(\overline{\Omega}): Bg(x)=0, x\in\partial\Omega\}$. Then operator $\hat{A}\equiv\hat{A}^1$ is the self-adjoint extension of A in $L_2(\Omega)$. In the same way one can define the operator $(\hat{A}+I)^\tau$, where I is the identity operator in $L_2(\Omega)$.

Further we borrow some original ideas from the method developed in the work of M.A. Krasnoselskii et al. [18]. The following lemma plays a key role in this method ([18], p. 453).

Lemma 3.1. *Let $\sigma > 1+\frac{N}{4}$. Then for any multi-index α satisfying $|\alpha| \leq 2$ the operator $D^\alpha(\hat{A}+I)^{-\sigma}$ (completely) continuously maps the space $L_2(\Omega)$ into $C(\overline{\Omega})$, and moreover, the following estimate holds*

$$\|D^\alpha(\hat{A}+I)^{-\sigma} g\|_{C(\Omega)} \leq C\|g\|_{L_2(\Omega)}.$$

First we suppose, that for some $\tau > \frac{N}{4}$ the following numerical series

$$\sum_{k=1}^{\infty}(\lambda_k+1)^{2\tau}|\varphi_k|^2 \leq C_\varphi < \infty \tag{3.1}$$

and

$$\sum_{k=1}^{\infty}(\lambda_k+1)^{2(\tau+1)}|\psi_k|^2 \leq C_\psi < \infty \tag{3.2}$$

converge.

In order to investigate series (2.4), we introduce the following notations:

$$f_j^1(x) = \sum_{k=1}^{j} \frac{\psi_k}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)}\, v_k(x), \tag{3.3}$$

$$f_j^2(x) = \sum_{k=1}^{j} \frac{\varphi_k E_{\rho,\rho}(-\lambda_k T^\rho)}{T E_{\rho,\rho+1}(-\lambda_k T^\rho)}\, v_k(x). \tag{3.4}$$

We also need some estimates for the Mittag-Leffler function. For sufficiently large t one has the asymptotic estimate (see, for examples, [24], p. 13, [12], p. 75, note, $0 < \rho \leq 1$)

$$E_{\rho,\rho+1}(-t) = \frac{1}{t} + O\Big(\frac{1}{t^2}\Big), \quad t > 1, \tag{3.5}$$

and for any complex number μ

$$0 < |E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \tag{3.6}$$

Since $(\hat{A}+I)^{-\tau} v_j(x) = (\lambda_j+1)^{-\tau} v_j(x)$, we have for (3.3)

$$||f_j^1||^2_{C(\Omega)} = \left\|(\hat{A}+I)^{-\tau}\sum_{k=1}^{j}\frac{\psi_k\,(\lambda_k+1)^\tau}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)}\, v_k(x)\right\|^2_{C(\Omega)}$$

(by virtue of Lemma 3.1 with $\alpha = 0$)

$$\leq C\left\|\sum_{k=1}^{j}\frac{\psi_k\,(\lambda_k+1)^\tau}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)}\, v_k(x)\right\|^2_{L_2(\Omega)} \tag{3.7}$$

(since the system $\{v_k\}$ is orthonormal, then using estimate (3.5))

$$\leq C\sum_{k=1}^{j}\left|\frac{\psi_k\,(\lambda_k+1)^\tau}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)}\right|^2 \leq C\sum_{k=1}^{\infty}\big|(\lambda_k+1)^{\tau+1}\,\psi_k\big|^2 = C\cdot C_\psi.$$

This implies the uniform convergence on $x \in \overline{\Omega}$ of sums (3.3). On the other hand, sums (3.7) converge for any permutation of their terms, as well, since these terms are mutually orthogonal. This implies the absolute convergence of sums (3.3).

In the same way one can prove uniformly and absolutely convergence with respect to $x \in \overline{\Omega}$ of sums (3.4).

We proceed to consider series (2.3). By virtue of the definition of the Fourier coefficients f_k we can represent the partial sums for the function $u(x,t)$ in the form of the sum of three partial sums:

$$u_j^1(x,t) = \sum_{k=1}^{j} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x), \tag{3.8}$$

$$u_j^2(x,t) = \sum_{k=1}^{j} \frac{\psi_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)}{T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)} v_k(x), \tag{3.9}$$

$$u_j^3(x,t) = \sum_{k=1}^{j} \frac{\varphi_k E_{\rho,\rho}(-\lambda_k T^\rho)}{T E_{\rho,\rho+1}(-\lambda_k T^\rho)}\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) v_k(x). \tag{3.10}$$

Let $|\alpha| \le 2$ and $\delta > 0$. It is required to show, that in the domain $\overline{\Omega} \times [\delta, T]$ each of these three sums absolutely and uniformly converges after applying the operators D^α and ∂_t^ρ term-by-term.

We have

$$u_j^1(x,t) = (\hat{A} + I)^{-\tau-1} \sum_{k=1}^{j} (\lambda_k + 1)^{\tau+1} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x).$$

Therefore, by virtue of Lemma 3.1,

$$||D^\alpha u_j^1||_{C(\Omega)} = ||D^\alpha \hat{A}^{-\tau-1} \sum_{k=1}^{j} \lambda_k^{\tau+1} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x)||_{C(\Omega)}$$

$$\le C|| \sum_{k=1}^{j} \lambda_k^{\tau+1} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x)||_{L_2(\Omega)}. \tag{3.11}$$

Since the system $\{v_j\}$ is orthonormal, we may write

$$||D^\alpha u_j^1||^2_{C(\Omega)} \le C \sum_{k=1}^{j} |\lambda_k^{\tau+1} \varphi_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)|^2.$$

Further, using estimate (3.6),

$$||D^\alpha u_j^1||^2_{C(\Omega)} \le C t^{-2} \sum_{k=1}^{j} \lambda_k^{2\tau} |\varphi_k|^2 \le C\delta^{-2} C_\varphi.$$

This implies the uniform convergence on $(x,t) \in \overline{\Omega} \times [\delta, T]$ of differentiated sum (3.8) with respect to variables x_k, $k = 1, \ldots, N$. On the other hand, sum (3.11) converges for any permutation of its terms, as well, since these terms are mutually orthogonal. This implies the absolute convergence of differentiated sum (3.8) on the same domain $(x,t) \in \overline{\Omega} \times [\delta, T]$.

Using completely similar reasoning, it can be shown that sums (3.9) and (3.10) have the same properties as sum (3.8).

By virtue of equation $\partial_t^\rho u(x,t) = \triangle u(x,t) + f(x)$ one may prove the uniform and absolute convergence of series (2.3) after applying operator ∂_t^ρ term-by-term.

It is not hard to verify, that functions (2.3) and (2.4) satisfy all the conditions of problem (1.1) together with (1.4).

Hence, if the functions $\varphi(x)$ and $\psi(x)$ satisfy conditions (3.1) and (3.2), then (2.3) and (2.4) will be solutions of problem (1.1) together with (1.4). As shown in work [13] by V.A. Il'in (see also [19] page 111) the fulfillment of conditions (a)-(d) guarantees the convergence of the series in (3.1) and (3.2).

Lemma 3.2. *Let conditions* (2.1) *and* (2.2) *be satisfied. Then function* $U(\rho;t_0)$*, as a function of* $\rho\in(0,1)$ *is strictly monotone and*

$$\lim_{\rho\to 0}U(\rho;t_0)=\psi_1,\quad U(1;t_0)=\varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}.$$

Proof. Since the system of eigenfunctions $\{v_k(x)\}$ are orthonormal and $v_1(x)=|\Omega|^{-1/2}$, then from (2.3) and (1.3) one has

$$U(\rho;t_0)=\varphi_1 t_0^{\rho-1}E_{\rho,\rho}(0)+f_1 t_0^{\rho}E_{\rho,\rho+1}(0).$$

Using the definition of the Mittag-Leffler function and equality

$$f_1=\frac{\psi_1}{T^\rho E_{\rho,\rho+1}(0)}-\frac{\varphi_1 E_{\rho,\rho}(0)}{TE_{\rho,\rho+1}(0)},$$

we have

$$U(\rho;t_0)=\frac{\varphi_1 t_0^{\rho-1}}{\Gamma(\rho)}\left[1-\frac{t_0}{T}\right]+\frac{\psi_1 t_0^\rho}{T^\rho}.$$

Let $\Psi(\rho)$ be the logarithmic derivative of the gamma function $\Gamma(\rho)$ (for the definition and properties of Ψ see [9]). Then $\Gamma'(\rho)=\Gamma(\rho)\Psi(\rho)$, and therefore,

$$\left(\frac{t_0^{\rho-1}}{\Gamma(\rho)}\right)'=\frac{t_0^{\rho-1}}{\Gamma(\rho)}\left[\ln t_0-\Psi(\rho)\right],\quad \left(\left(\frac{t_0}{T}\right)^\rho\right)'=\left(\frac{t_0}{T}\right)^\rho\ln\frac{t_0}{T}.$$

Then the function $U'(\rho;t_0)$ can be represented as follows

$$U'(\rho;t_0)=\varphi_1\frac{t_0^{\rho-1}}{\Gamma(\rho)}\left[1-\frac{t_0}{T}\right][\ln t_0-\Psi(\rho)]+\psi_1\left(\frac{t_0}{T}\right)^\rho\ln\frac{t_0}{T}.$$

Observe that for all $\rho\in(0,1)$ one has $\Psi(\rho)<0$ and if $t_0\ge 1$ then $\ln t_0\ge 0$. Hence $\ln t_0-\Psi(\rho)>0$.

To show that the function $U(\rho;t_0)$ is monotone, we consider two cases: the Fourier coefficients φ_1 and ψ_1 have different signs, and both of them have the same sign.

Case 1. Let φ_1 and ψ_1 have different signs, i.e. $\varphi_1\cdot\psi_1\le 0$, while $\varphi_1^2+\psi_1^2\ne 0$.

If $1\le t_0<T$, then $1-\frac{t_0}{T}>0$ and $\ln\frac{t_0}{T}<0$. Therefore, if $\varphi_1>0$ and $\psi_1<0$, then $U'(\rho;t_0)>0$. The converse if $\varphi_1<0$ and $\psi_1>0$, then $U'(\rho;t_0)<0$. Hence, if $1\le t_0<T$ and $\varphi_1\cdot\psi_1<0$, then $U(\rho;t_0)$ is a monotone function.

If one of the Fourier coefficients equal to zero, for example, $\varphi_1=0$, then

$$U'(\rho;t_0)=\psi_1\left(\frac{t_0}{T}\right)^\rho\ln\frac{t_0}{T}$$

and if $\psi_1>0$, then $U'(\rho;t_0)<0$, otherwise, i.e. if $\psi_1<0$, then $U'(\rho;t_0)>0$. This implies the monotonicity of the function $U(\rho;t_0)$.

Now let $\psi_1=0$, then

$$U'(\rho;t_0)=\varphi_1\frac{t_0^{\rho-1}}{\Gamma(\rho)}\left[1-\frac{t_0}{T}\right][\ln t_0-\Psi(\rho)].$$

Therefore, for all $t_0 \geq 1$ one has $U'(\rho; t_0) > 0$ provided $\varphi_1 > 0$, and if $\varphi_1 < 0$, then $U'(\rho; t_0) < 0$. Again, this also implies the monotonicity of the function $U(\rho; t_0)$.

Case 2. Let φ_1 and ψ_1 have the same sign, i.e. $\varphi_1 \cdot \psi_1 > 0$.

First consider the case $\varphi_1 > 0$ and $\psi_1 > 0$, while $\frac{\varphi_1 T}{\psi_1} \leq 1$. According to the Lagrange theorem, there exists such ξ $(t_0 \leq \xi \leq T)$, that

$$\ln \frac{t_0}{T} = \ln t_0 - \ln T = \frac{1}{\xi}(t_0 - T).$$

Then the function $U'(\rho; t_0)$ can be rewritten as:

$$U'(\rho; t_0) = \varphi_1 \frac{t_0^{\rho-1}}{\Gamma(\rho)} \left[1 - \frac{t_0}{T}\right] [\ln t_0 - \Psi(\rho)] + \psi_1 \left(\frac{t_0}{T}\right)^{\rho} \frac{1}{\xi}[t_0 - T]$$

or

$$U'(\rho; t_0) = t_0^{\rho-1} \left[1 - \frac{t_0}{T}\right] \left[\frac{\varphi_1}{\Gamma(\rho)}[\ln t_0 - \Psi(\rho)] - \psi_1 \frac{t_0}{\xi T^{\rho-1}}\right].$$

Since

$$\frac{1}{\Gamma(\rho)} = \frac{\rho}{\Gamma(\rho+1)}, \quad \Psi(\rho) = \Psi(\rho+1) - \frac{1}{\rho},$$

then function U' can be represented as:

$$U'(\rho; t_0) = t_0^{\rho-1} \left[1 - \frac{t_0}{T}\right] \left[\frac{\rho\varphi_1}{\Gamma(\rho+1)} \left[\ln t_0 - \Psi(\rho+1) + \frac{1}{\rho}\right] - \psi_1 \frac{t_0}{\xi T^{\rho-1}}\right]$$

or

$$U'(\rho; t_0) = t_0^{\rho-1} \left[1 - \frac{t_0}{T}\right] \left[\frac{\rho\varphi_1}{\Gamma(\rho+1)} \ln t_0 - \frac{\rho\varphi_1}{\Gamma(\rho+1)}\Psi(\rho+1) + \frac{\varphi_1}{\Gamma(\rho+1)} - \psi_1 \frac{t_0}{\xi T^{\rho-1}}\right].$$

Simple calculations show that

$$0 < \frac{\rho}{\Gamma(\rho+1)} \ln t_0 < 2\ln T; \ \ -\frac{\rho}{\Gamma(\rho+1)}\Psi(\rho+1) < 0.8; \ \ 0 < \frac{1}{\Gamma(\rho+1)} \leq 1.2.$$

On the other hand, if $0 < \rho \leq 1$ then $-\frac{1}{T^{\rho-1}} \leq -1$. Therefore,

$$U'(\rho; t_0) \leq t_0^{\rho-1} \left[1 - \frac{t_0}{T}\right] \left[2\varphi_1(\ln T + 1) - \psi_1 \frac{t_0}{\xi}\right].$$

If $\frac{2\varphi_1(\ln T+1)T}{\psi_1} < t_0 < T$, then $2\varphi_1(\ln T + 1) - \psi_1 \frac{t_0}{\xi} < 0$. Therefore, $U'(\rho; t_0) < 0$ and this implies, that $U(\rho; t_0)$ is monotone in the considered case.

Now let $\varphi_1 > 0$ and $\psi_1 > 0$, but $\frac{\varphi_1 T}{\psi_1} > 1$. Then one has

$$U'(\rho; t_0) \geq t_0^{\rho-1} \left[1 - \frac{t_0}{T}\right] \left[\varphi_1 - \psi_1 \frac{t_0}{T}\right].$$

Therefore, if we choose t_0 as $t_0 > \frac{\varphi_1 T}{\psi_1}$, then $U'(\rho; t_0) > 0$, which implies the monotonicity of the function $U(\rho; t_0)$.

Case $\varphi_1 < 0$ and $\psi_1 < 0$ is considered similarly.

□

From Lemma 3.2 it follows immediately, that if

$$\min\left\{\psi_1, \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\} < d_0 < \max\left\{\psi_1, \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\},$$

then there exists ρ, which satisfies condition (1.3) and this number is unique and if the opposite inequalities hold true, then such a ρ does not exist.

Hence we have proved the existence of a trey $\{u(x,t), f(x), \rho\}$ which satisfies all the conditions of inverse problem (1.1), (1.3) and (1.4).

To prove the uniqueness of a solution of the inverse problem we suppose that there exist two trays of solutions $\{u^1,\ f^1,\ \rho_1\}$ and $\{u^2,\ f^2,\ \rho_2\}$ such, that $0<\rho_k<1$ and

$$\begin{cases} \partial_t^{\rho_k} u^k(x,t) - \triangle u^k(x,t) = f^k(x), \ x\in\Omega, \quad 0<t<T; \\ Bu^k(x,t) = 0, \quad x\in\partial\Omega,\ 0<t<T; \\ \lim\limits_{t\to 0} J_t^{\rho_k-1} u^k(x,t) = \varphi(x), \quad x\in\overline{\Omega}, \end{cases} \tag{3.12}$$

where $k=1,2$.

Consider the following functions

$$w_j^k(t) = \int\limits_\Omega u^k(x,t)v_j(x)dx, \quad k=1,2.$$

Then equation (3.12) implies

$$\partial_t^{\rho_k} w_j^k(t) + \lambda_j w_j^k(t) = f_j^k, \quad \lim_{t\to 0} J_t^{\rho_k-1} w_j^k(t) = \varphi_j.$$

Solutions of these Cauchy type problems have the form (see, for examples [12], p. 173, [7])

$$w_j^k(t) = \varphi_j t^{\rho_k-1} E_{\rho_k,\rho_k}(-\lambda_j t^{\rho_k}) + f_j^k\, t^{\rho_k} E_{\rho_k,\rho_k+1}(-\lambda_j t^{\rho_k}). \tag{3.13}$$

Therefore, from (1.3) one has $w_1^1(t_0) = w_1^2(t_0) = d_0$, or

$$\frac{\varphi_1 t_0^{\rho_1-1}}{\Gamma(\rho_1)}\left[1-\frac{t_0}{T}\right] + \frac{\psi_1 t_0^{\rho_1}}{T^{\rho_1}} = \frac{\varphi_1 t_0^{\rho_2-1}}{\Gamma(\rho_2)}\left[1-\frac{t_0}{T}\right] + \frac{\psi_1 t_0^{\rho_2}}{T^{\rho_2}} = d_0.$$

As we have seen above (see Lemma 3.2), this equation gives $\rho_1=\rho_2$.

Further, let $\rho_1=\rho_2=\rho$ and denote $u(x,t)=u^1(x,t)-u^2(x,t)$ and $f(x)=f^1(x)-f^2(x)$. Then, since problem (1.1) is linear, we have

$$\begin{cases} \partial_t^{\rho} u(x,t) - \triangle u(x,t) = f(x), \ x\in\Omega, \quad 0<t<T; \\ Bu(x,t) = 0, \quad x\in\partial\Omega,\ 0<t<T; \\ \lim\limits_{t\to 0} J_t^{\rho-1} u(x,t) = 0, \quad x\in\overline{\Omega}, \\ u(x,T)=0, \quad x\in\overline{\Omega}. \end{cases} \tag{3.14}$$

Consider the function

$$w_j(t) = \int\limits_\Omega u(x,t)v_j(x)dx.$$

Then from (3.14) one has

$$\partial_t^\rho w_j(t) = \int\limits_\Omega \partial_t^\rho u(x,t)v_j(x)dx = \int\limits_\Omega \triangle u(x,t)v_j(x)dx + \int\limits_\Omega f(x)v_j(x)dx, \quad t>0,$$

or

$$\partial_t^\rho w_j(t) = \int\limits_\Omega u(x,t)\triangle v_j(x)dx + f_j = -\lambda_j w_j(t) + f_j, \quad t > 0.$$

Hence we have the following Cauchy type problem

$$\partial_t^\rho w_j(t) + \lambda_j w_j(t) = f_j, \quad t > 0; \quad \lim_{t\to 0} J_t^{\rho-1} w_j(t) = 0.$$

The solution of this problem has the form (see (3.13))

$$w_j(t) = f_j t^\rho E_{\rho,\rho+1}(-\lambda_j t^\rho).$$

By virtue of (1.4) we obtain $w_j(T) = f_j T^\rho E_{\rho,\rho+1}(-\lambda_j T^\rho) = 0$. Since $0 < E_{\rho,\rho+1}(-\lambda_j T^\rho) \le 1$, then for all j one has $f_j = 0$ or $f(x) = 0$. Hence, for all j the equality $w_j(t) \equiv 0$ holds. But since the system $\{v_j(x)\}$ is complete in $L_2(\Omega)$, then for all $x \in \overline{\Omega}$ and $t > 0$ we finally have $u(x,t) = 0$. This implies, that $u_1(x,t) = u_2(x,t)$ and $f_1(x) = f_2(x)$. □

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