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VAGIF SABIR oglu GULIYEV

(to the 65th birthday)

On February 22, 2022 was the 65th birthday of Vagif Sabir oglu Guliyev, editor-in-chief of the Transactions of the Azerbaijan National Academy of Science, Issue Mathematics, Series of physical-technical and mathematics science (Scopus, Q3), deputy editor-in-chief of the Applied and Computational Mathematics (Web of Science, Q1), deputy director of the Institute of Applied Mathematics (IAM) of the Baku State University (BSU), head of the Department of Mathematical Analysis at the Institute of Mathematics and Mechanics (IMM) of the Azerbaijan National Academy of Sciences (ANAS), member of the Editorial Board of the Eurasian Mathematical Journal.

V.S. Guliyev was born in the city of Salyan in Azerbaijan. In 1978 Vagif Guliyev graduated from the Faculty of Mechanics and Mathematics of the Azerbaijan State University (now the Baku State University) with an honors degree and then completed his postgraduate studies at this university. His scientific supervisors were distinguished mathematicians A.A. Babayev and S.K. Abdullayev. In 1983 he defended his PhD thesis at the BSU. From 1983 he continued his scientific activities at the V.A. Steklov Mathematical Institute of the Academy of Sciences of the USSR. In 1987-1991 he was in internship at this institute and in 1994 defended there his DSc thesis.

From 1983 to 1995 he worked as assistant, a senior lecturer, docent and from 1995 to 2018 as a professor of Mathematical Analysis Chair of the BSU. In 1995-2008 he worked on part-time basis at the Institute of the IMM. From 2008 to 2014 he was a chief researcher of the Department of Mathematical Analysis of the IMM, from 2014 to the present day he is the head of this department.

In 2014 V.S. Guliyev was elected a corresponding member of the ANAS.

From 2015 to 2019, he worked as deputy director on science at the IMM. From 2019 to the present day, he has been working as a chief researcher at the IAM. Since May 2021, he has been working as a deputy director on science of the IAM.

Professor Vagif Guliyev has been a member of the Presidium of the Higher Attestation Commission under the President of the Republic of Azerbaijan since 2014 to the present day.

V.S. Guliyev is a world recognized specialist in real and harmonic analysis, function spaces and partial differential equations. He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. He was one of the first to study local Morrey-type spaces, generalized weighted Morrey-type spaces and anisotropic Banach-valued Sobolev spaces, for which appropriate embedding theorems were established.

Some of his results and methods are named after him: the Adams-Guliyev and Spanne-Guliyev conditions for the boundedness of operators in Morrey-type spaces, Guliyev's method of local estimates of integral operators of harmonic analysis, the Burenkov-Guliyevs conditions for the boundedness of operators in general local Morrey-type spaces.

On the whole, the results obtained by V.S. Guliyev have laid a groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations.

Vagif Sabir oglu Guliyev is an author of more than 250 scientific publications including 2 monographs. Among his pupils there are more than 20 candidates of sciences and 5 doctors of sciences. The results obtained by V.S. Guliyev, his pupils, collaborators and followers gained worldwide recognition.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vagif Sabir oglu Guliyev on the occasion of his 65th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ON PROPERTIES OF FUNCTIONS INVERTIBLE IN THE SENSE OF EHRENPREIS IN THE SCHWARTZ ALGEBRA

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Key words: Schwartz algebra, entire function, distribution of zero sets, slowly decreasing function.

AMS Mathematics Subject Classification: 30D15, 30E5, 42A38, 46F05.

Abstract. We consider those elements of the Schwartz algebra of entire functions which are the Fourier-Laplace transforms of invertible distributions with compact supports on the real line. These functions are called invertible in the sense of Ehrenpreis. The first of the presented results is about the properties of zero subsets of invertible in the sense of Ehrenpreis function f. Namely, we establish some properties of the zero subset formed by zeros of f laying not far from the real axis. We also obtain estimates from below for $|f|$ which generalize the corresponding ones in the definition of the notion of sine-type functions.

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1 Introduction

Let $\mathcal{E}'$ denote the strong dual to the Fréchet space $\mathcal{E} := C^\infty(\mathbb{R})$. Recall that the Fourier-Laplace transform operator acting in $\mathcal{E}$ is defined by the formula

$$\psi(z) = S(e^{-\mathrm{i}tz}), \quad S \in \mathcal{E}',$$

and the image $\mathcal{P}$ of $\mathcal{E}'$ under this transform is a topological algebra (*Schwartz algebra*) if we equip it with the topology and the algebraical structure induced from $\mathcal{E}'$. It is well-known that $\mathcal{P}$ consists of all entire functions of exponential type having at most polynomial growth along the real axis [8, Theorem 7.3.1].

We say that *the division theorem* is valid for $\varphi \in \mathcal{P}$ if the following implication holds:

$$\Phi \in \mathcal{P}, \quad \Phi/\varphi \in H(\mathbb{C}) \Longrightarrow \Phi/\varphi \in \mathcal{P}.$$

Now, we explain why having this property for $\varphi \in \mathcal{P}$ is important for applications.

In [7] L. Ehrenpreis establishes that the validity of the division theorem for $\varphi \in \mathcal{P}$ is equivalent to the invertibility in the spaces $\mathcal{E}$ and $\mathcal{D}' = (C_0^\infty(\mathbb{R}))'$ of the distribution $S = \mathcal{F}^{-1}(\varphi)$, which means

$$S * \mathcal{E} = \mathcal{E},$$

$$S * \mathcal{D}' = \mathcal{D}',$$

where the symbol $*$ denotes the convolution.

We say that $\varphi \in \mathcal{P}$ is *invertible in the sense of Ehrenpreis* if the division theorem is valid for it (see [2], [3]).

Below, we will use the following analytical criterion due to L. Ehrenpreis [7, Theorems I, 2.2, Proposition 2.7]: $\varphi \in \mathcal{P}$ is invertible (in the sense of Ehrenpreis) if and only if it is *slowly decreasing*, i.e. there exists $a > 0$ such that

$$\forall x \in \mathbb{R}\ \exists x' \in \mathbb{R}: \ |x - x'| \le a \ln(2 + |x|),\ |\varphi(x')| \ge (a + |x'|)^{-a}. \tag{1.1}$$

Each $S \in \mathcal{E}'$ generates the convolution operator M_S acting in $\mathcal{E}$:

$$M_S(f) = S * f, \quad f \in \mathcal{E}.$$

It is easy to check that $\varphi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis if and only if the convolution operator M_S generated by $S = \mathcal{F}^{-1}(\varphi)$ is surjective.

Let $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis, Λ be its zero set, $\Lambda' \subset \Lambda$. Then, $(\mathrm{i}\Lambda')$ coincides with the spectrum of the differentiation-invariant subspace $W \subset \mathcal{E}$ which has the following property: each $f \in W$ is represented as a series with grouping of exponential monomials contained in W. This fact is established in [7], [5], [4] for the subspaces of the form

$$W = \{f \in \mathcal{E} : \quad S * f = 0\},$$

where $S \in \mathcal{E}'$ is fixed. And it is also true for general subspaces admitting (weak) spectral synthesis with respect to the differentiation operator.

Summarizing the above, we may conclude that there are enough reasons to study the behavior and zero subsets of functions $\varphi \in \mathcal{P}$ which are invertible in the sense of Ehrenpreis.

In Section 1, we establish some restrictions on the distribution of real parts of zeros lying not far from the real axis (Theorem 2.1). The result generalizes [3, Lemma 2] and [7, Proposition 6.1]).

In Section 2, we study the estimates from below for $\ln|\varphi|$. The start point is the notion of sine-type function. Recall that entire function ψ of exponential type $\sigma > 0$ is called *sine-type function* if

$$m \le |\psi(z)| e^{-\sigma |\mathrm{Im}\, z|} \le M$$

for all $z : |\mathrm{Im}\, z| \ge y_0$, where m, M, y_0 are some positive constants.

Clearly, each sine-type function is invertible in the sense of Ehrenpreis element of the algebra $\mathcal{P}$ and its zero set is contained in the horizontal strip $|\mathrm{Im}\, z| < y_0$.

Consider entire function φ of the exponential type σ having the properties:
its indicator diagram is the segment of the imaginary axis $[-\mathrm{i}\sigma; \mathrm{i}\sigma]$;
the estimate

$$\ln|\varphi(z)| \ge \sigma |\mathrm{Im}\, z| - \mathrm{const} \ln(|\mathrm{Re}\, z| + e) \tag{1.2}$$

holds for all z such that

$$|\mathrm{Im}\, z| \ge \mathrm{const} \ln(|\mathrm{Re}\, z| + e)\}. \tag{1.3}$$

It is clear that the zeros λ_j of such function satisfy

$$|\mathrm{Im}\, \lambda_j| \le \mathrm{const} \ln(|\mathrm{Re}\, \lambda_j| + e)\}. \tag{1.4}$$

By the minimum modulus theorem [9, Chapter 1, Section 8, Theorem 11] and analytical criterion (1.1), we can easily derive that φ is invertible in the sense of Ehrenpreis.

It is turned out that the converse is also true: if $\varphi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis and its zeros λ_j satisfy (1.4) then (1.2) is true for all z submitted to (1.3). This fact is the particular case of Theorem 3.1 corresponding to $\omega(x) = \ln(x + e)$. We prove this theorem in Section 2.

Here, we give the short description of the result. Let $\omega : \mathbb{R} \to [1; +\infty)$ be even and subjected some regularity restrictions, $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis. Suppose that the exponential type of φ equals σ and its zero set Z_φ is contained in

$$\{z : \quad |\mathrm{Im}\, z| \ge \omega(|\mathrm{Re}\, z|)\}.$$

Theorem 3.1 asserts that if $|\mathrm{Im}\, z| < \omega(|\mathrm{Re}\, z|)$ then the estimate

$$\ln|\varphi(z)| \geq \sigma|\mathrm{Im}\, z| - \mathrm{const}\,\omega(|\mathrm{Re}\, z|)$$

holds.

2 Zero sets

Let $\mathcal{M} = \{\mu_j\}$, $\mu_j = \alpha_j + \mathrm{i}\beta_j$,

$$0 < |\mu_1| \leq |\mu_2| \leq \dots,$$

be such that $\beta_j = O(\ln|\mu_j|)$ as $j \to \infty$, and the formula

$$\psi(z) = \lim_{R\to\infty} \prod_{|\mu_j|\leq R} \left(1 - \frac{z}{\mu_j}\right) \tag{2.1}$$

defines an entire function of exponential type.

In [3, Lemma 1], we established the following fact.

Lemma A. *The function ψ defined by (2.1) is invertible in the sense of Ehrempreis element of the Schwartz algebra $\mathcal{P}$ if and only if the same is true about the function*

$$\psi_1(z) = \lim_{R\to\infty} \prod_{|\alpha_j|\leq R} \left(1 - \frac{z}{\alpha_j}\right). \tag{2.2}$$

Taking into account Lemma A, we give another formulation of Lemma 2 in [3].

Lemma B. *Let $\psi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis, $\mathcal{M} = \{\mu_k\} \subset \mathbb{C}$ be its zero set.*

Then, for the subset $\mathcal{M}' \subset \mathcal{M}$ defined by

$$\mu_k \in \mathcal{M}' \Longleftrightarrow |\mathrm{Im}\,\mu_k| \leq M_0 \ln|\mathrm{Re}\,\mu_k|$$

with $M_0 > 0$ fixed,

$$\overline{\lim_{|x|\to\infty}} \frac{m_{\mathrm{Re}}(x,1)}{\ln|x|} < \infty,$$

where $m_{\mathrm{Re}}(x,1)$ denotes the number of points of the sequence

$$\mathrm{Re}\,\mathcal{M}' = \{\mathrm{Re}\,\mu_k : \ \mu_k \in \mathcal{M}'\}$$

contained in the segment $[x-1; x+1]$.

Consider non-decreasing function $l : [0; +\infty) \to [1; +\infty)$ satisfying

$$\ln t = O(l(t)), \quad t \to \infty, \tag{2.3}$$

$$\overline{\lim_{t\to+\infty}} \frac{\ln l(t)}{\ln t} < \frac{1}{2}, \tag{2.4}$$

and

$$\overline{\lim_{t\to+\infty}} \frac{l(Kt)}{l(t)} < +\infty \tag{2.5}$$

for some $K > 1$.

The following theorem generalizes Lemma B.

Theorem 2.1. *Let $\psi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis with the zero set $\mathcal{M} = \{\mu_k\}$, and $\mathcal{M}' \subset \mathcal{M}$ be defined by*

$$\mu_k \in \mathcal{M}' \Longleftrightarrow |\mathrm{Im}\,\mu_k| \le M_0 \cdot l\,(|\mathrm{Re}\,\mu_k|)$$

for a fixed $M_0 > 0$.

Then,

$$\overline{\lim_{|x|\to\infty}} \frac{m_{\mathrm{Re}}(x,1)}{l(|x|)} < \infty, \tag{2.6}$$

where $m_{\mathrm{Re}}(x,1)$ denotes the number of points of the sequence

$$\mathrm{Re}\,\mathcal{M}' = \{\mathrm{Re}\,\mu_k : \ \mu_k \in \mathcal{M}'\}$$

contained in the segment $[x-1; x+1]$.

First, we prove the following auxiliary proposition.

Proposition 2.1. *Let ψ, $\mathcal{M}$, $\mathcal{M}'$ be the same as in Theorem 2.1,*

$$\mathcal{M}'' = \mathcal{M} \setminus \mathcal{M}', \quad \alpha_j = \mathrm{Re}\,\mu_j.$$

Then, the function

$$\psi_1(z) = \lim_{R\to\infty} \left(\prod_{\substack{|\mu_j|\le R \\ \mu_j \in \mathcal{M}'}} \left(1 - \frac{z}{\alpha_j}\right) \prod_{\substack{|\mu_j|\le R \\ \mu_j \in \mathcal{M}''}} \left(1 - \frac{z}{\mu_j}\right) \right) \tag{2.7}$$

belongs to the algebra $\mathcal{P}$, and there exists $M_1 > 0$ such that

$$\forall x \in \mathbb{R}, |x| > 2, \ \exists z' \in \mathbb{C} : \ |z' - x| \le M_1 l(|x|) \quad \text{and} \quad \ln|\psi_1(z')| \ge -M_1 l(|z'|). \tag{2.8}$$

Proof. We start with estimating the single multiplier $\left(1 - \frac{x}{\alpha_j}\right)$, $x \in \mathbb{R}$, where $\alpha_j = \mathrm{Re}\,\mu_j$, $\mu_j \in \mathcal{M}'$. It is easy to see that

$$\left|1 - \frac{x}{\alpha_j}\right| \le \left|1 - \frac{x}{\mu_j}\right| \left(1 + \frac{M_0^2 l^2(\alpha_j)}{\alpha_j^2}\right)^{1/2} \le \left|1 - \frac{x}{\mu_j}\right| \left(1 + O\left(\frac{l^2(\alpha_j)}{\alpha_j^2}\right)\right).$$

Taking into account (2.4), we get

$$\ln|\psi_1(x)| \le \mathrm{const}\,\ln|\psi(x)|, \quad x \in \mathbb{R}. \tag{2.9}$$

Hence, $\psi_1 \in \mathcal{P}$.

To estimate $|\psi_1|$ from below, we consider the auxiliary function

$$\psi^+(z) = \lim_{R\to\infty} \left(\prod_{\substack{|\mu_j|\le R \\ \mu_j \in \mathcal{M}'_- \bigcup \mathcal{M}''}} \left(1 - \frac{z}{\mu_j}\right) \prod_{\substack{|\mu_j|\le R \\ \mu_j \in \mathcal{M}'_+}} \left(1 - \frac{z}{\alpha_j}\right) \right),$$

where

$$\mathcal{M}'_+ = \{\mu_j \in \mathcal{M}' : \ \beta_j \ge 0\}, \quad \mathcal{M}'_- = \mathcal{M}' \setminus \mathcal{M}'_+.$$

Because of (2.5), we have

$$l(Kt) \leq C_0 l(t), \quad t \geq 0, \tag{2.10}$$

for some $C_0 > 0$. Notice that

$$\left|1 - \frac{z}{\mu_j}\right| \leq \frac{|\mu_j - z|}{|\alpha_j|} = \frac{\left((\alpha_j - x)^2 + (\beta_j - y)^2\right)^{1/2}}{|\alpha_j|},$$

where $\mu_j = \alpha_j + \mathrm{i}\beta_j \in \mathcal{M}'_+$, $z = x + \mathrm{i}y$. Together with (2.10), it gives us the inequality

$$\left|1 - \frac{z}{\mu_j}\right| \leq \left|1 - \frac{z}{\alpha_j}\right| \tag{2.11}$$

if $z = x + 2\mathrm{i}C_0 M_0 l(|x|)$, $\mu_j \in \mathcal{M}'_+$ and $|\mathrm{Re}\,\mu_j| = |\alpha_j| \leq 4K|x|$.

From the other hand, for $z = x + 2\mathrm{i}C_0 M_0 l(|x|)$, $\mu_j = \alpha_j + \mathrm{i}\beta_j \in \mathcal{M}'_+$ we have

$$\left|1 - \frac{z}{\mu_j}\right| \leq \left|1 - \frac{z}{\alpha_j}\right| \cdot \left(1 + \frac{C_1 l^2(\alpha_j)}{\alpha_j^2}\right)^{1/2} \tag{2.12}$$

if $|\alpha_j| > 4K|x|$, where the constant $C_1 > 0$ depends only on l and M_0.

Relations (2.4), (2.11), (2.12) lead to the estimate

$$\ln|\psi(z)| \leq \mathrm{const}\ln|\psi^+(z)| + O(1) \quad \text{as} \quad |x| \to \infty \tag{2.13}$$

if $z = x + 2\mathrm{i}C_0 M_0 l(|x|)$.

In a similar way, we get that

$$\ln|\psi^+(z)| \leq \mathrm{const}\ln|\psi_1(z)| + O(1), \quad \text{as} \quad |x| \to \infty, \tag{2.14}$$

where $z = x - 2\mathrm{i}C_0 M_0 l(|x|)$.

Applying analytical criterion (1.1) and the minimum modulus theorem [9, Chapter 1, Section 8, Theorem 11], we arrive to the estimate

$$\ln|\psi(z)| \geq -C_2 \ln|x|$$

for all $z : |z - x| = C_2 \ln|x|$, $x \in \mathbb{R}$, $|x| > 2$ and some $C_2 > 0$.

Notice that $\psi \in \mathcal{P}$ implies the inequality

$$|\psi(z)| \leq C_\psi (2 + |z|)^{C_\psi} e^{C_\psi |\mathrm{Im}\,z|} \tag{2.15}$$

with some $C_\psi > 0$.

Fix $x \in \mathbb{R}$, $|x| > 2$, set $z_x = x + \mathrm{i}C_2 \ln|x|$ and then apply the minimum modulus theorem to the function $\frac{\psi}{\psi(z_x)}$ in the disc $|z - z_x| \leq 4C_0 M_0 l(|x|)$. Taking into account (2.3), (2.5) and (2.15), we find $M_2 > 0$ and $\theta \in (2;4)$ such that

$$\ln|\psi(z)| \geq -M_2 l(|x|) \quad \text{if} \quad |z - z_x| = \theta C_0 M_0 l(|x|). \tag{2.16}$$

Further, from (2.13), (2.16) and (2.4)–(2.5), it follows that there exists $M_3 > 0$ with the property:

$$\forall x \in \mathbb{R},\ |x| > 2, \quad \exists w_x \in \mathbb{C} \quad \text{such that}$$

$$|w_x - x| \leq M_3 l(|x|) \quad \text{and} \quad \ln|\psi^+(w_x)| \geq -M_3 l(|w_x|).$$

Now, we apply the above argument including the minimum modulus theorem to the function $\frac{\psi^+}{\psi^+(w_x)}$ and the disc $|z - w_x| \leq R$, where

$$R = \max\{2M_3 l(|x|),\ 4C_0 M_0 l(|x|)\}.$$

It gives us the estimate for $\ln|\psi^+(z)|$ which is similar to (2.16). This estimate and (2.14), together with (2.4)–(2.5), lead us to the assertion. □

Remark 1. It is not difficult to see that applying the minimum modulus theorem once more, to the function $\frac{\psi_1}{\psi(z')}$, we obtain the following version of Proposition 2.1:
there exists $M_1 > 0$ such that

$$\forall x \in \mathbb{R}, |x| > 2, \ \exists x' \in \mathbb{R}: \\ |x' - x| \le M_1 l(|x|) \quad \text{and} \quad \ln|\psi_1(x')| \ge -M_1 l(|x'|). \tag{2.17}$$

Proof of Theorem 2.1.
Without loss of generality, we assume that ψ is bounded on the real axis and its exponential type equals 1. Because of (2.9), we may also assume that the same is true for the function ψ_1 defined by (2.7). Let $\lambda_k \in \mathcal{M}'$ and $\alpha_k = \operatorname{Re}\lambda_k$.

Further in the proof, we use symbol ψ to denote the function ψ_1 defined in (2.7).

If (2.6) fails then

$$\lim_{j\to\infty} \frac{m_{\mathrm{Re}}(x_j, 1)}{l(|x_j|)} = \infty \tag{2.18}$$

for some x_j, $|x_j| \to \infty$. For clarity, we assume that $x_j > 0$.

Consider entire functions

$$\psi_j(z) = \psi(z)(z - x_j)^{m_j} \cdot \prod_{k:|\alpha_k - x_j| \le 1} (z - \alpha_k)^{-1},$$

where $m_j = m(x_j, 1)$, $j = 1, 2, \ldots$ It is easy to check that the estimates

$$\sup_{x\in\mathbb{R}} |\psi_j(x)| \le C_0 2^{m_j}, \quad j = 1, 2, \ldots,$$

hold with $C_0 = \max\{\sup_{t\in\mathbb{R}} |\psi(t)|, 1\}$. By Bernstein's theorem [6, Chapter 11], the inequalities

$$\sup_{x\in\mathbb{R}} |\psi_j^{(n)}(x)| \le C_0 2^{m_j} \tag{2.19}$$

are also valid for all n, $j \in \mathbb{N}$.

From the Taylor expansion of ψ_j at x_j and the estimates (2.19), it follows that

$$|\psi_j(z)| \le C_0 2^{m_j} (m_j!)^{-1} |z - x_j|^{m_j} e^{|z - x_j|}, \quad z \in \mathbb{C}.$$

Hence, for all $x \in \mathbb{R}$ satisfying the condition

$$\ln C_0 + |x - x_j| + m_j \ln|x - x_j| - \ln(m_j!) \le -n\, l(x_j) - m_j \ln 2, \tag{2.20}$$

where $n \in \mathbb{N}$, we have the inequality

$$|\psi_j(x)| \le x_j^{-n} \cdot 2^{-m_j}. \tag{2.21}$$

By Stirling's formula, relation (2.20) will follow from the inequality

$$|x - x_j| + m_j \ln|x - x_j| - m_j \ln m_j \le -n\, l(x_j) - C_1 m_j,$$

where C_1 is an absolute constant.

Because of (2.18), for each $n \in \mathbb{N}$ we can find j_n such that

$$-n\, l(x_j) \ge -m_j, \quad j = j_n, j_n + 1, \ldots \tag{2.22}$$

Fix $b \in (0; e^{-C_1 - 2})$. Estimates (2.21) are valid for $j \ge j_n$ and all $x \in \mathbb{R}$ such that $|x - x_j| \le b m_j$. This fact, inequalities (2.22) and the relations

$$|\psi(z)| \le 2^{m_j} |\psi_j(z)|, \quad z \in \mathbb{C}, \quad j = 1, 2, \ldots,$$

imply

$$|\psi(x)| \le e^{-n l(x_j)}$$

for $x \in \mathbb{R}$ satisfying $|x - x_j| \le bn\, l(x_j)$, $\ j \ge j_n$, $\ n \in \mathbb{N}$. It means that ψ does not satisfy (2.17), i.e. the assumption (2.18) leads to the contradiction.

3 Estimates from below

First, recall that $\mathcal{P}$ is included into the class C (Cartwright class) of entire functions. Hence, each $\varphi \in \mathcal{P}$ has completely regular growth and its indicator diagram is the segment of the imaginary axis $\mathrm{i}[-h_\varphi(-\pi/2); h_\varphi(\pi/2)]$, where h_φ is the indicator function (see [9, Chapter V, Section 4]).

Let $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis $\Lambda = \{\lambda_j\}$ be its zero set. Without loss of generality, we may assume that $\varphi(0) = 1$, $h_\varphi(-\pi/2) = h_\varphi(\pi/2) = a_\varphi$.

Consider even function $\omega : \mathbb{R} \to [1; +\infty)$ which is non-decreasing on $[0; +\infty)$ and such that

$$\omega(e^s) \quad \text{is convex on} \quad \mathbb{R}, \tag{3.1}$$

$$\exists K > 1 : \ \varlimsup_{x\to\infty} \frac{\omega(Kx)}{K\omega(x)} < 1. \tag{3.2}$$

Theorem 3.1. *Let $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis with the zero set $\Lambda = \{\lambda_j\}$, and the inequality*

$$|\mathrm{Im}\,\lambda_j| \le \omega(|\mathrm{Re}\,\lambda_j|) \tag{3.3}$$

holds for each $\lambda_j \in \Lambda$ except at most a finite number of points.

Then, there exists $C_1 > 0$ such that for $z \in \mathbb{C}$

$$|\mathrm{Im}\,z| \ge M_1\omega(|\mathrm{Re}\,z|) \Longrightarrow \ln|\varphi(z)| \ge a_\varphi|\mathrm{Im}\,z| - C_1\omega(|z|). \tag{3.4}$$

Proof. The argument is based on the minimum modulus theorem [9, Chapter 1, Section 8, Theorem 11] and two Phragmen-Lindelöf-type assertions (Lemma C and Lemma D).

Lemma C ([9, Theorem 19]). *Let u be subharmonic in a domain $G \subset \mathbb{C}$, and*

$$\varlimsup_{\substack{z\to\zeta\\ z\in G}} (u(z) + \delta v_0(z)) \le A_0, \quad \zeta \in \partial G,$$

for all sufficiently small $\delta > 0$, where v_0 is harmonic in G, $A_0 \in \mathbb{R}$..

Then,

$$u(z) \le A_0, \quad z \in G.$$

Lemma D ([9, Theorem 20]). *Let u be subharmonic and bounded in G, and $\infty \in \partial G$.*

If

$$\varlimsup_{\substack{z\to\zeta\\ z\in G}} u(z) \le A_0, \quad \zeta \in \partial G \bigcap \mathbb{C},$$

then

$$u(z) \le A_0, \quad z \in G.$$

By (3.1), we have

$$\ln|x| = O(\omega(x)), \quad |x| \to \infty.$$

This estimate, (3.2) (3.3) and the minimum modulus theorem imply that

$$\ln|\varphi(z)| \ge -C_0\omega(|\mathrm{Re}\,z|) \tag{3.5}$$

if $|\mathrm{Im}\,z| \ge C_0\omega(|\mathrm{Re}\,z|)$, where C_0 is a positive constant.

Consider the function

$$P_\omega(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(x+ty)}{1+t^2}\mathrm{d}t, \quad z = x + \mathrm{i}y \in \mathbb{C}.$$

Because of (3.1), (3.2), P_ω is continious and subharmonic in $\mathbb{C}$. It is also harmonic and satisfies the relations

$$P_\omega(z) \geq \omega(|z|) \tag{3.6}$$

$$\overline{\lim_{z\to\infty}} \frac{P_\omega(z)}{\omega(|z|)} < \infty \tag{3.7}$$

in both half-planes $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, z < 0$ (see [1], [10]).

Set

$$D = \{z : \quad \mathrm{Im}\, z > C_0\omega(|\mathrm{Re}\, z|)\}.$$

Inequality (3.5) and the properties of the function P_ω imply that

$$\ln |e^{-\mathrm{i}(a_\varphi-\varepsilon)z}| - \ln|\varphi(z)| - (a_\varphi+1)C_0P_\omega(z) \leq 0 \tag{3.8}$$

for all $z \in \partial D$ and $\varepsilon \in [0; a_\varphi)$.

Taking into account the completely regular growth of φ, (3.3), (3.6) and the inequality $\omega(x) \geq 1$, $x \in \mathbb{R}$, from (3.8), we get the estimate

$$\ln e^{(a_\varphi-\varepsilon)y} - \ln|\varphi(\mathrm{i}y)| - (a_\varphi+1)C_0P_\omega(\mathrm{i}y) \leq C(\varepsilon), \quad y \geq 1,$$

where $C(\varepsilon) > 0$ depends only on $\varepsilon \in (0; a_\varphi)$, φ and ω.

Fix $\sigma \in (0; 1/2]$ and set

$$v^\pm(z) = -|z|^{1+\sigma} \cos\left((\arg z \mp \pi/4)(1+\sigma)\right), \quad z \in D^\pm,$$

where $D^\pm = D \bigcap \{z : \ \mathrm{Re}\, z \gtrless 0\}$. Applying Lemma C to

$$G = D^\pm, \quad v_0 = v^\pm$$

and

$$u(z) = \ln|e^{-\mathrm{i}(a_\varphi-\varepsilon)z}| - \ln|\varphi(z)| - (a_\varphi+1)C_0P_\omega(z),$$

we obtain that u is bounded in D. Together with (3.8) and Lemma D, it implies that

$$\ln|e^{-\mathrm{i}(a_\varphi-\varepsilon)z}| - \ln|\varphi(z)| - (a_\varphi+1)C_0P_\omega(z) \leq 0$$

for $z \in D$ and for all $\varepsilon > 0$ which are sufficiently small. It follows that

$$\ln|\varphi(z)| \geq a_\varphi \mathrm{Im}\, z - (a_\varphi+1)C_0P_\omega(z), \qquad z \in D. \tag{3.9}$$

Taking into account (3.7) and (3.3), we derive from (3.9) that

$$\ln|\varphi(z)| \geq a_\varphi \mathrm{Im}\, z - C_1\omega(|z|) \tag{3.10}$$

if $\mathrm{Im}\, z \geq C_1\omega(|\mathrm{Re}\, z|)$, where C_1 is a positive constant.

Setting

$$D = \{z : \quad \mathrm{Im}\, z < -C_0\omega(|\mathrm{Re}\, z|)\}$$

and arguing by the similar way, we obtain that (3.10) holds for all z satisfying

$$\mathrm{Im}\, z \leq -C_1\omega(|\mathrm{Re}\, z|).$$

□

Remark 2. Condition (3.2) implies

$$\overline{\lim_{x\to\infty}} \frac{\ln\omega(x)}{\ln|x|} < 1 \tag{3.11}$$

(see [10]). It means that we may use $\omega(|\lambda_j|)$ instead of $\omega(|\mathrm{Re}\, \lambda_j|)$ in (3.3).

Remark 3. If $\omega(x) = \mathrm{const} \ln(|x| + e)$ then (3.3) and (3.4) take the form (1.3) and (1.2), correspondingly.

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