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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

Short communications

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BOUNDEDNESS OF THE GENERALIZED RIESZ POTENTIAL IN LOCAL MORREY TYPE SPACES

V.I. Burenkov, M.A. Senouci

Communicated by M.L. Goldman

Key words: generalized Riesz potential operator, boundedness, local Morrey type spaces.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. We generalize the results obtained in [4] on the boundedness of the Riesz potential from one general local Morrey-type space to another one to the case of the generalized Riesz potential.

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1 Introduction

In this paper, we study the boundedness from one general local Morrey-type space to another one of the generalized Riesz potential

$$\left(I_{\rho(\cdot)}f\right)(x) = \int_{\mathbb{R}^n} \rho(|x-y|)f(y)dy,\ x \in \mathbb{R}^n,$$

under certain assumptions on the kernel ρ. Our aim is to generalized the results obtained in [4] for the case of the classical Riesz potential I_α, in which $\rho(t) = t^{\alpha-n}$, $t > 0$, $0 < \alpha < n$.

Let, for a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$, $\mathfrak{M}(\Omega)$ denote the space of all functions $f : \Omega \longrightarrow \mathbb{C}$ Lebesgue measurable on Ω, and $\mathfrak{M}^+(\Omega)$ denote the subset of $\mathfrak{M}(\Omega)$ of all non-negative functions.

Definition 1. Let $0 < p, \theta \leq \infty$ and let $w \in \mathfrak{M}^+\big((0,\infty)\big)$ be not equivalent to 0. We denote by $LM_{p\theta,w(\cdot)}$ the local Morrey-type spaces, the space of all functions $f \in \mathfrak{M}(\mathbb{R}^n)$ with finite quasi-norms

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \|f\|_{LM_{p\theta,w(\cdot)}}(\mathbb{R}^n) = \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)}.$$

If $w(r) = 0$ and $\|f\|_{L_p(B(0,r))} = \infty$, we assume that $w(r)\|f\|_{L_p(B(0,r))} = 0$.

Definition 2. Let $0 < p, \theta \leq \infty$. We denote by Ω_θ the set of all functions $w \in \mathfrak{M}^+((0,\infty))$ which are non-equivalent to 0 and such that

$$\|w\|_{L_\theta(t,\infty)} < \infty$$

for some $t > 0$.

Lemma 1.1. [1], [2], [3]. *Let $0 < p,\ \theta \leq \infty$ and let $w \in \mathfrak{M}^+((0,\infty))$ be not equivalent to 0. Then the space $LM_{p\theta,w(\cdot)}$ is notrivial if and only if $w \in \Omega_\theta$.*

Let $F, G : A \times B \longrightarrow [0, \infty]$. Throughout this paper we say that F is dominated by G uniformly in $x \in A$ and write

$$F \lesssim G \text{ uniformly in } x \in A$$

if there exists $c(B) > 0$ such that

$$F(x, y) \leq c(B)G(x, y) \text{ for all } x \in A.$$

(So $c(B)$ is independent of $x \in A$, but may depend on $y \in B$).

Respectively, we say that G dominates F uniformly in $x \in A$ and write

$$G \gtrsim F \text{ uniformly in } x \in A$$

if there exists $c(B) > 0$ such that

$$F(x, y) \geq c(B)G(x, y) \text{ for all } x \in A.$$

We also say that F is equivalent to G uniformly in $x \in A$ and write

$$F \approx G \text{ uniformly in } x \in A$$

if F and G dominate each other uniformly in $x \in A$. We note recent papers [5], [6] in which some properties of Morrey-type spaces were investigated.

2 L_p and WL_p estimates of generalised Riesz potentials over balls

Let $n \in \mathbb{N}, 0 < p < \infty, \Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. Recall that the weak L_p-space $WL_p(\Omega)$ is the space of all functions $f \in \mathfrak{M}(\Omega)$ for which

$$\| f \|_{WL_p(\Omega)} = \sup_{t \geq 0} t \Big| \{y \in \Omega : |f(y)| > t\} \Big|^{\frac{1}{p}} < \infty.$$

Here $|G|$ denotes the Lebesgue measure of a set $G \subset \mathbb{R}^n$.

Definition 3. Let $n \in \mathbb{N}$. We say that $\rho \in S_n$ if $\rho \in \mathfrak{M}^+((0, \infty))$ and

1) $\int_0^r \rho(t) t^{n-1} dt < \infty$ for all $r > 0$,
2) for some $c_1, c_2 > 0$,

$$c_1 \rho(t) \leq \rho(s) \leq c_2 \rho(t)$$

for all $s, t > 0$ satisfying the inequality $\frac{t}{2} \leq s \leq 2t$.

Definition 4. Let $n \in \mathbb{N}, 1 \leq p_1 < \infty, 0 < p_2 \leq \infty$. Then $\rho \in S_{n,p_1,p_2}$ if $\rho \in \mathfrak{M}^+((0, \infty))$ and there exists a positive non-increasing continuously differentiable function function $\tilde{\rho} : (0, \infty) \to (0, \infty)$ such that

1) $\tilde{\rho}(t) \approx \rho(t)$ uniformly in $t > 0$,
2) $\lim\limits_{t \to \infty} \tilde{\rho}(t) = 0$,
3) $\tilde{\rho} \in S_n$,
4) $\int\limits_0^1 \tilde{\rho}(t) t^{\frac{n}{p_1'} - 1} dt = \infty$, $\int\limits_1^\infty \tilde{\rho}(t) t^{\frac{n}{p_1'} - 1} dt < \infty$,
5) $|\tilde{\rho}'(t)| t \gtrsim \tilde{\rho}(t)$ uniformly in $t > 0$,
6) if $0 < p_2 \leq p_1$ and $1 \leq p_1 < \infty$, then

$$\int_0^r \tilde{\rho}(t) t^{n-1} dt \lesssim \tilde{\rho}(r) r^n$$

uniformly in $r>0$,

7) if $1\le p_1<p_2<\infty$, then the function $\phi_{n,\tilde\rho,p_1,p_2}(t)=\tilde\rho(t)t^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)}$ is almost non-decreasing on $(0,\infty)$, that is for some $c>0$

$$\varphi_{n,\rho,p_1,p_2}(t_1)\le c\varphi_{n,\rho,p_1,p_2}(t_2) \text{ for all } 0<t_1<t_2<\infty.$$

Moreover, if $p_1=1$ and $1<p_2<\infty$, then $\rho\in\tilde S_{n,1,p_2}$ if there exists a positive non-increasing continuously differentiable function $\tilde\rho:(0,\infty)\to(0,\infty)$ such that Conditions 1) - 3) and 5) are satisfied, Conditions 4) and 6) are satisfied for $p_1=1$ and instead of Condition 7) the following condition is satisfied

8)

$$\left\|\tilde\rho(t)t^{\frac{n-1}{p_2}}\right\|_{L_{p_2}(0,r)}\lesssim\tilde\rho(r)r^{\frac{n}{p_2}}$$

uniformly in $r>0$.

Remark 1. For the Riesz potential $\rho(t)=t^{\alpha-n}$, $0<\alpha<n$, $\tilde\rho=\rho$. Condition 2) is satisfied because $\alpha<n$, Condition 3) is satisfied because $\alpha>0$, Condition 4) is satisfied if $\alpha<\frac{n}{p_1}$, Condition 5) is obviously satisfied, Condition 6) is satisfied because $\alpha>0$, Condition 7) is satisfied if $\alpha\ge n(1/p_1-1/p_2)$, Condition 8) is satisfied if $\alpha>n(1-1/p_2)$.

Theorem 2.1. *Let $n\in\mathbb{N}, 1\le p_1<\infty, 0<p_2\le\infty$.*

1. *If $1<p_1<p_2<\infty$ or $1\le p_1<\infty$ and $0<p_2\le p_1$, and $\rho\in S_{n,p_1.p_2}$, then*

$$\| I_{\rho(.)}|f| \|_{L_{p_2}(B(x,r))}\lesssim r^{\frac{n}{p_2}}\int_r^\infty \rho(t)t^{\frac{n}{p_1'}-1}\| f \|_{L_{p_1}(B(x,t))}\,dt$$

uniformly in $x\in\mathbb{R}^n, r>0$ and $f\in L_{p_1}^{\rm loc}(\mathbb{R}^n)$.

2. *If $p_1=1$, $0<p_2<\infty$, and $\rho\in\tilde S_{n,1,p_2}$, then*

$$\| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))}\approx\| I_{\rho(.)}|f| \|_{L_{p_2}(B(x,r))}\approx r^{\frac{n}{p_2}}\int_r^\infty \rho(t)t^{-1}\| f \|_{L_1(B(x,t))}\,dt$$

uniformly in $x\in\mathbb{R}^n, r>0$ and $f\in L_1^{\rm loc}(\mathbb{R}^n)$.

3. *If $p_1=1$, $1<p_2<\infty$, and $\rho\in S_{n,1,p_2}$, then*

$$\| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))}\approx r^{\frac{n}{p_2}}\int_r^\infty \rho(t)t^{-1}\| f \|_{L_1(B(x,t))}\,dt$$

uniformly in $x\in\mathbb{R}^n, r>0$ and $f\in L_1^{\rm loc}(\mathbb{R}^n)$.

2 Generalized Riesz potential and Hardy operator

We denote by $\mathfrak{M}^+(0,\infty;\downarrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-increasing on $(0,\infty)$ and set

$$\mathbb{A}=\left\{\varphi\in\mathfrak{M}^+(0,\infty;\downarrow):\lim_{t\to\infty}\varphi(t)=0\right\}.$$

Let H be the Hardy operator

$$(Hg)(t):=\int_0^t g(r)dr,\ 0<t<\infty,$$

Moreover, let, for $0<p\le\infty$ and a function $v\in\mathfrak{M}^+((0,\infty))$, $L_{p,v(\cdot)}(0,\infty)$ denote the space of all functions $f\in\mathfrak{M}^+((0,\infty))$ for which

$$\|f\|_{L_{p,v(\cdot)}(0,\infty)}=\|fv\|_{L_p(0,\infty)}<\infty.$$

Theorem 2.1. *Let $n \in \mathbb{N}$, $1 \le p_1 < \infty$, $0 < p_2 \le \infty$, $0 < \theta_2 \le \infty, \omega_2 \in \Omega_{\theta_2}$, ρ be a positive continuous function on $(0, \infty)$ and*

$$\mu_{n,\rho,p_1}(r) = \frac{\int_r^\infty \rho(t) t^{\frac{n}{p_1'}-1} dt}{\int_1^\infty \rho(t) t^{\frac{n}{p_1'}-1} dt}, \ r > 0,$$

$$v_2(r) = w_2\left(\mu_{n,\rho,p_1}^{(-1)}(r)\right)\left(\mu_{n,\rho,p_1}^{(-1)}(r)\right)^{\frac{n}{p_2}} |(\mu_{n,\rho,p_1}^{(-1)}(r))'|^{\frac{1}{\theta_2}}, \ r > 0,$$

$$g_{n,\rho,p_1}(t) = \|f\|_{L_{p_1}\left(B(0,\mu_{n,\rho,p_1}^{(-1)}(t))\right)}, \ t > 0.$$

1. *If $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1.p_2}$, then*

$$\|I_{\rho(\cdot)} f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \lesssim \|H g_{n,\rho,p_1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)}$$

uniformly in $f \in \mathfrak{M}(\mathbb{R}^n)$.

2. *If $p_1 = 1$ and $0 < p_2 < \infty$, and $\rho \in \tilde{S}_{n,1,p_2}$, then*

$$\|I_{\rho(\cdot)} f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \approx \|H g_{n,\rho,1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)}$$

uniformly in all non-negative functions $f \in \mathfrak{M}(\mathbb{R}^n)$.

3. *If $p_1 = 1$ and $1 < p_2 < \infty$, and $\rho \in S_{n,1,p_2}$, then*

$$\|I_{\rho(\cdot)} f\|_{WLM_{p_2\theta_2,w_2(\cdot)}} \approx \|H g_{n,\rho,1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)}$$

uniformly in all non-negative functions $f \in \mathfrak{M}(\mathbb{R}^n)$.

Remark 2. If $\rho(t) = t^{\alpha-n}$, $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $0 < \alpha < \frac{n}{p_1}$, then $\mu_{n,\rho,p_1}(r) = r^{-\sigma}$, where $\sigma = \frac{n}{p_1} - \alpha$, $\mu_{n,\rho,p_1}^{(-1)}(r) = r^{-\frac{1}{\sigma}}$, $v_2(r) = \sigma^{-\frac{1}{\theta_2}} w_2(r^{-\frac{1}{\sigma}}) r^{-\frac{n}{\sigma p_2} - \frac{1}{\sigma\theta_2} - \frac{1}{\theta_2}}$, $g_{n,\rho,p_1}(t) = \| f \|_{L_{p_1}(B(0,t^{-\frac{1}{\sigma}}))}$.

Theorem 2.2. *Assume that $n \in \mathbb{N}$, $1 \le p_1 < \infty$, $0 < p_2 \le \infty$, $0 < \theta_1$, $\theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.*

1. *Let $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1,p_2}$. If the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0, \infty)$ to $L_{\theta_2,v_2(\cdot)}(0, \infty)$ on the cone $\mathbb{A}$, that is*

$$\|Hg\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \lesssim ||g||_{L_{\theta_1,v_1(\cdot)}}(0, \infty)$$

uniformly in $g \in \mathbb{A}$, where

$$v_1(r) = w_1\left(\mu_{n,\rho,p_1}^{(-1)}(r)\right) \left| \left(\mu_{n,\rho,p_1}^{(-1)}(r)\right)' \right|^{\frac{1}{\theta_1}}, \ r > 0,$$

then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *Let $p_1 = 1$, $0 < p_2 < \infty$ and $\rho \in \tilde{S}_{n,1,p_2}$. Then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0, \infty)$ to $L_{\theta_2,v_2(\cdot)}(0, \infty)$ on the cone $\mathbb{A}$.*

3. *Let $p_1 = 1$, $1 < p_2 < \infty$ and $\rho \in S_{n,1,p_2}$. Then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $WLM_{p_2\theta_2,w_2(\cdot)}$ if and only if the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0, \infty)$ to $L_{\theta_2,v_2(\cdot)}(0, \infty)$ on the cone $\mathbb{A}$.*

In order to obtain sufficient conditions on the weight functions ensuring boundedness of $I_{\rho(.)}$ we shall apply Theorem 3.2 and the known necessary and sufficient conditions ensuring boundedness of the Hardy operator H from one wegheted Lebesgue space to another one on the cone $\mathbb{A}$ (see, for example, [9], [7] and [8]).

Theorem 2.3. $0<\theta_1,\theta_2\le\infty, \omega_1\in\Omega_{\theta_1}, \omega_2\in\Omega_{\theta_2}$.

1. *Let* $1<p_1<p_2<\infty$ *or* $1\le p_1<\infty$ *and* $0<p_2\le\infty$, *and* $\rho\in S_{n,p_1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{p_1,\theta_1,\omega_1}$ *to* $LM_{p_2,\theta_2,\omega_2}$ *if the following conditions are satistied.*

(*a*) *If* $1<\theta_1\le\theta_2<\infty$, *then*

$$B_1^1:=\sup_{t>0}\left(\int_t^\infty \omega_2^{\theta_2}(r)\mu_{n,\rho,p_1}^{\theta_2}(r)r^{\frac{n}{p_2}\theta_2}dr\right)^{\frac{1}{\theta_2}}\left(\int_r^\infty \omega_1^{\theta_1}(r)dr\right)^{-\frac{1}{\theta_2}}<\infty,$$

and

$$B_2^1:=\sup_{t>0}\left(\int_0^t \omega_2^{\theta_2}(r)\ r^{\theta_2\frac{n}{p_2}}dr\right)^{\frac{1}{\theta_2}}\left(\int_t^\infty \frac{\omega_1^{\theta_1}(r)\mu_{n,\rho,p_1}^{\theta_1'}(r)}{(\int_r^\infty \omega_1^{\theta_1}(\rho)d\rho)^{\theta_1'}}dr\right)^{\frac{1}{\theta_1}}<\infty.$$

(*b*) *If* $0<\theta_1\le 1, 0<\theta_1\le\theta_2<\infty$, *then* $B_1^1<\infty$ *and*

$$B_2^2:=\sup_{t>0}t^{\alpha-\frac{n}{p_1}}\left(\int_0^t \omega_2^{\theta_2}(r)\ r^{\theta_2\frac{n}{p_2}}dr\right)^{\frac{1}{\theta_2}}\left(\int_r^\infty \omega_1^{\theta_1}(r)dr\right)^{-\frac{1}{\theta_1}}<\infty.$$

(*c*) *If* $1<\theta_1<\infty, 0<\theta_2<\theta_1<\infty, \theta_2\ne1$, *then*

$$B_1^3:=\left(\int_0^\infty\left(\frac{\int_t^\infty \omega_2^{\theta_2}(r)\mu_{n,\rho,p_1}^{\theta_2}(r)r^{\frac{n\theta_2}{p_2}}dr}{\int_t^\infty\omega_1^{\theta_1}(r)dr}\right)^{\frac{\theta_2}{\theta_1-\theta_2}}\omega_2^{\theta_2}(t)\mu_{n,\rho,p_1}^{\theta_2}(t)t^{\frac{n\theta_2}{p_2}}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty,$$

and

$$B_2^3:=\left(\int_0^\infty\left[\left(\int_0^t\omega_2^{\theta_2}(r)r^{\theta_2\frac{n}{p_2}}dr\right)^{\frac{1}{\theta_2}}\left(\int_t^\infty\frac{\omega_1^{\theta_1}(r)\mu_{n,\rho,p_1}^{\theta_1'}(r)}{\left(\int_r^\infty\omega_1^{\theta_1}(\rho)d\rho\right)^{\theta_1'}}dr\right)^{\frac{\theta_2-1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\times$$

$$\times\frac{\omega_1^{\theta_1}(t)\mu_{n,\rho,p_1}^{\theta_1'}(t)}{\left(\int_t^\infty\omega_1^{\theta_1}(\rho)d\rho\right)^{\theta_1'}}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty.$$

(*d*) *If* $1=\theta_2<\theta_1<\infty$, *then*

$$B_1^4:=\left(\int_0^\infty\left(\frac{\int_t^\infty\omega_2(r)\mu_{n,\rho,p_1}(r)r^{\frac{n}{p_2}}}{\int_t^\infty\omega_1^{\theta_1}(r)dr}dr\right)^{\frac{1}{\theta_1-1}}\omega_2(t)\mu_{n,\rho,p_1}(t)t^{\frac{n}{p_2}}dt\right)^{\frac{\theta_1-1}{\theta_1}}<\infty,$$

and

$$B_2^4:=\Big(\int_0^\infty\left(\frac{\int_t^\infty\omega_2(r)\mu_{n,\rho,p_1}(r)r^{\frac{n}{p_2}}dr+\mu_{n,\rho,p_1}(t)\int_0^t\omega_2(r)r^{\frac{n}{p_2}}dr}{\int_t^\infty\omega_1^{\theta_1}(r)dr}\right)^{\theta_1'-1}\times$$

$$\times\mu_{n,\rho,p_1}(t)\left(\int_0^t\omega_2^{\theta_2}(r)r^{\frac{n}{p_2}}dr\right)\frac{dt}{t}\Big)^{\theta_1'}<\infty.$$

(*e*) *If* $0<\theta_2<\theta_1=1$, *then*

$$B_1^5:=\left(\int_0^\infty\left(\frac{\int_t^\infty\omega_2^{\theta_2}(r)\mu_{n,\rho,p_1}^{\theta_2}(r)r^{\frac{\theta_2n}{p_2}}dr}{\int_t^\infty\omega_1(r)dr}\right)^{\frac{\theta_2}{1-\theta_2}}\omega_2^{\theta_2}(t)\mu_{n,\rho,p_1}^{\theta_2}(t)t^{\frac{\theta_2n}{p_2}}dt\right)^{\frac{1-\theta_2}{\theta_2}}<\infty,$$

and

$$B_2^5 := \left(\int_0^\infty \left(\int_0^t \omega_2^{\theta_2}(r) r^{\frac{\theta_2 n}{p_2}} dr \right)^{\frac{\theta_2}{1-\theta_2}} \left(\inf_{t<s<\infty} (\mu_{n,\rho,p_1}(s))^{-1} \int_s^\infty \omega_1(\rho) d\rho \right)^{\frac{\theta_2}{\theta_2 - 1}} \times \right.$$

$$\left. \times \omega_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty.$$

(f) *If* $0 < \theta_2 < \theta_1 < 1$, *then* $B_3^1 < \infty$ *and*

$$B_2^6 := \left(\int_0^\infty \sup_{t \le s < \infty} \frac{\mu_{n,\rho,p_1}(s)^{\frac{\theta_1 \theta_2}{\theta_1 - \theta_2}}}{\left(\int_s^\infty \omega_1^{\theta_1}(\rho) d\rho \right)^{\frac{\theta_2}{\theta_1 - \theta_2}}} \left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{\theta_1 - \theta_2}} \times \right.$$

$$\left. \times \omega_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty.$$

(g) *If* $0 < \theta_1 \le 1, \theta_2 = \infty$, *then*

$$B^7 := \operatorname*{ess\,sup}_{0<t \le s < \infty} \frac{\omega_2(t) t^{\frac{n}{p_2}}}{(\mu_{n,\rho,p_1}(s))^{-1} \left(\int_s^\infty \omega_1^{\theta_1}(r) dr \right)^{\frac{1}{\theta_1}}} < \infty.$$

(h) *If* $1 < \theta_1 < \infty, \theta_2 = \infty$, *then*

$$B^8 := \operatorname*{ess\,sup}_{t>0} \omega_2(t) t^{\frac{n}{p_2}} \left(\int_t^\infty \frac{\mu_{n,\rho,p_1}^{\theta_1'}(r)}{\left(\int_r^\infty \omega_1^{\theta_1}(s) ds \right)^{\theta_1' - 1}} \frac{dr}{r} \right)^{\frac{1}{\theta_1'}} < \infty.$$

(i) *If* $\theta_1 = \infty, 0 < \theta_2 < \infty$, *then*

$$B^9 := \left(\int_0^\infty \left((\mu_{n,\rho,p_1}(t))^{-1} \int_t^\infty \frac{\mu_{n,\rho,p_1}(r)}{\operatorname{ess\,sup}_{r<s<\infty} \omega_1(s)} \frac{dr}{r} \right)^{\theta_2} \times \right.$$

$$\left. \times \omega_2^{\theta_2}(t) \mu_{n,\rho,p_1}^{\theta_2}(t)) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{1}{\theta_2}} < \infty.$$

(j) *If* $\theta_1 = \theta_2 = \infty$, *then*

$$B^{10} := \operatorname*{ess\,sup}_{t>0} \omega_2(t) t^{\frac{n}{p_2}} \int_t^\infty \frac{\mu_{n,\rho,p_1}(r)}{\operatorname{ess\,sup}_{r<s<\infty} \omega_1(s)} \frac{dr}{r} < \infty.$$

2. *Let* $p_1 = 1, 0 < p_2 < \infty$ *and* $\rho \in \tilde{S}_{n,1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{1,\theta_1,\omega_1(.)}$ *to* $LM_{p_2,\theta_2,\omega_2(.)}$ *if and only if conditions* (a) - (j) *with* $p_1 = 1$ *are satisfied.*

3. *Let* $p_1 = 1, 1 < p_2 < \infty$ *and* $\rho \in S_{n,1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{1,\theta_1,\omega_1(.)}$ *to* $WLM_{p_2,\theta_2,\omega_2(.)}$ *if and only if conditions* (a) - (j) *with* $p_1 = 1$ *are satisfied.*

Remark 3. For $\rho(t) = t^{\alpha - n}$, $0 < \alpha < n$, hence for the Riesz potential I_α, Theorems $2.1, 3.1, 3.2$ and 3.3 were proved in [4].

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